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Lagrange Duality in Multiobjective Fractional Programming Problems with n-Set Functions

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In this paper, Lagrange multiplier theorems are developed for the multiobjective fractional programming problem involving n-set functions. A Lagrange dual is introduced and duality results in terms of efficient solutions are established.

1. INTRODUCTION

Duality theory for multiobjective fractional programming was developed by several authors. Among them, Egudo [7] studied Mond–Weir type and Schaible type duals for a multiobjective fractional programming problem. Lagrangian duality for multiobjective fractional problem was studied by Weir and Jeyakumar [18] under preinvexity assumptions and by Suneja and Aggarwal [16] under cone convexity assumptions. Relations between a Lagrange multiplier of a multiobjective programming problem and a weak saddle point of its vector-valued Lagrangian function were obtained by many authors [5, 11, 15, 17], under different assumptions on the functions involved.

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Recently, Coladas et al. [4] introduced two types of duals to a more general multiobjective programming problem. They further suggested that their results could be extended for efficient solutions instead of weakly efficient solutions.

The purpose of this paper is to explore this possible extension suggested by Coladas et al. [4]. Moreover, the present results have been obtained in a more general setting, that is, for the multiobjective fractional programming problem involving n-set functions.

Morris [13] was the first one to introduce the general theory for optimizing set functions. These results were subsequently generalized by Bector et al. [2], Corley [6], Hsia and Lee [9], Lin [12, 13], and Zalmai [19, 20] to the multiobjective optimization problems involving n-set functions. Hsia and Lee [8] obtained necessary conditions for a properly efficient solutions through a vector-valued Lagrangian function and its associated saddle points. Hsia et al. [10] developed Lagrange multiplier theorems for the cases of single objective and multiobjective programming problems with set functions. Recently, Bector et al. [3] introduced Wolfe type and Mond−Weir type duals for a multiobjective fractional programming problem involving differentiable n-set functions and established duality results in terms of properly efficient solutions. They also related the problem to a certain vector-valued saddle point of a corresponding vector-valued Lagrangian function.

This paper has been divided into four sections. Section 2 includes preliminaries and a statement of the problem while in Section 3, necessary and sufficient conditions for the existence of a saddle point of a vector-valued Lagrangian are obtained. Section 4 is devoted to the construction of the Lagrangian dual and to establish duality results.

2. PRELIMINARIES

The following notations, definitions, and results are used in the sequel.

The m-dimensional Euclidean space of m-tuples is denoted by $\mathbb{R}^m$ and the interior of a subset K of $\mathbb{R}^m$ is denoted by int K.

A subset K of $\mathbb{R}^m$ is a cone if $\alpha x \in K$ whenever $x \in K$ and $\alpha \geqq 0$.

Let K be a cone in $\mathbb{R}^n$. Then K is said to be pointed if $K \cap (-K) = \{0\}$.

A positive dual cone K^* of K is defined as

$$K^* = \left\{ y^* \in \mathbb{R}^m \mid y^t y^* \geqq 0, \text{ for all } y \in K \right\}.$$

LEMMA 2.1 [15]. (a) *If $K \subset \mathbb{R}^m$ is a pointed convex cone, then $K + K \setminus \{0\} = K \setminus \{0\}$.*

(b) *If K is a cone with* $\operatorname{int} K \neq \phi$ *then* $y^t y^* > 0$ *for any* $y \in \operatorname{int} K$ *and* $y^* \in K^* \setminus \{0\}$.

Let K be a pointed convex cone in $\mathbb{R}^m$ with $\operatorname{int} K \neq \phi$ and let E be a non-empty subset of $\mathbb{R}^m$. For $x, y \in \mathbb{R}^m$, we define cone orders with respect to K as follows:

$$x <_K y \quad \text{iff } y - x \in \operatorname{int} K,$$

$$x \leq_K y \quad \text{iff } y - x \in K \setminus \{0\},$$

$$x \leqq_K y \quad \text{iff } y - x \in K.$$

The set of K-minimal points and the set of the K-maximal points of are defined as

$$\operatorname{Min}_K E = \{y \in E \mid \text{there is no } \bar{y} \in E \text{ such that } \bar{y} \leq_K y\},$$

$$\operatorname{Max}_K E = \{y \in E \mid \text{there is no } \bar{y} \in E \text{ such that } y \leq_K \bar{y}\},$$

respectively.

Let $(X, \mathscr{A}, \mu)$ be a finite atomless measure space with $L_1(X, \mathscr{A}, \mu)$ separable. A pseudometric d on $\mathscr{A}^n$, the n-fold product of σ algebra $\mathscr{A}$ of subsets of a given set X, is defined as

$$d(S, T) = \left[\sum_{r=1}^{n} \left[\mu(S_r \Delta T_r) \right]^2 \right]^{1/2},$$

where $S = (S_1, \ldots, S_n)$, $T = (T_1, \ldots, T_n) \in \mathscr{A}^n$, and $S_r \Delta T_r$ denotes the symmetric difference of S_r and T_r. For $f \in L_1(X, \mathscr{A}, \mu)$ and $S_r \in \mathscr{A}$, the integral $\int_{S_r} f \, d\mu$ is denoted by $\langle f, I_{S_r} \rangle$, where I_{S_r} denotes the characteristic function of S_r.

The multiobjective fractional programming problem considered in this paper is

$$\text{(VFP)} \qquad \begin{aligned} \operatorname{Min}_C Q(S) &= (Q_1(S), \ldots, Q_p(S)) \\ &= (F_1(S)/H_1(S), \ldots, F_p(S)/H_p(S)) \\ \text{subject to } & G(S) \leqq_K 0, \\ & S = (S_1, \ldots, S_n) \in \mathscr{L}', \end{aligned}$$

let $\mathscr{L} = \{S \in \mathscr{L}' \mid G(S) \leqq_K 0\}$ be the set of feasible solutions of (VFP), where

(i) C is the nonnegative orthant of n-dimensional Euclidean space of reals, i.e., $C = \mathbb{R}^n_+$, and K is a pointed convex cone in $\mathbb{R}^m$ with a non-empty interior,

(ii) $\mathscr{L}'$ is a convex subfamily of $\mathscr{A}^n$,

(iii) For $i = 1, \dots, p$ and for $j = 1, \dots, m$, $F_i \geqq 0$, $H_i > 0$ and G_j are real valued w^* continuous functions defined on $\mathscr{L}$,

(iv) F and $-H$ are C-convex functions and G is a K-convex function on $\mathscr{L}$.

DEFINITION 2.1. $\bar{S} \in \mathscr{L}$ is called an efficient solution of (VFP) if $Q(\bar{S}) \in \operatorname{Min}_C \{Q(S) \mid S \in \mathscr{L}\}$.

DEFINITION 2.2. An efficient solution $\bar{S} \in \mathscr{L}$ is said to be a properly efficient solution of (VFP) if there exists a scalar $M > 0$ such that

$$Q_i(S) < Q_i(\bar{S})$$

for some index i and a feasible solution S of (VFP) implies

$$\frac{Q_i(\bar{S}) - Q_i(S)}{Q_j(S) - Q_j(\bar{S})} \leqq M$$

for all j such that $Q_j(S) > Q_j(\bar{S})$.

For a convex set function, Hsia et al. [10] have proved the following alternative theorem.

LEMMA 2.2. *Let τ be a pointed closed convex cone in $\mathbb{R}^n$ with $\operatorname{int} \tau \neq \phi$. Let P be a τ-convex, w^* continuous n-set function defined on a convex subfamily $\mathscr{L}'$ of $\mathscr{A}^n$. If the system $P(S) <_\tau 0$ has no solution over $\mathscr{L}'$, then there exists a non-zero $\mu^* \in \tau^*$ such that $\mu^{*t} P(S) \geqq 0$ for all $S \in \mathscr{L}'$.*

3. LAGRANGIAN FUNCTION AND SADDLE POINT

Define the vector-valued Lagrangian $L: \mathscr{L} \times \Gamma \to \mathbb{R}^p$ of (VFP) by

$$L(S, U) = Q(S) + \left[\operatorname{diag}\left(H_1(S), \dots, H_p(S)\right)\right]^{-1} UG(S)$$

where $[\operatorname{diag}(H_1(S), \dots, H_p(S))]$ is the diagonal matrix of order p consisting of $H_1(S), \dots, H_p(S)$ as its diagonal entries and Γ is the set of all $p \times m$ matrices U satisfying $UK \subset C$.

Remark 3.1. Since $H_1(S) > 0$ for $i = 1, \dots, p$, $[\operatorname{diag}(H_1(S), \dots, H_p(S))]$ is therefore a full row rank matrix and is hence invertible.

Following the notations of Coladas et al. [4], we write

$$U \circ G(S) = \left[\operatorname{diag}\left(H_1(S), \dots, H_p(S)\right)\right]^{-1} UG(S).$$

DEFINITION 3.1. A point $(\bar{S}, \bar{U}) \in \mathscr{L} \times \Gamma$ is called a C-saddle point of the vector valued Lagrange function L if

$$L(\bar{S}, \bar{U}) \in \mathrm{Min}_C\{L(S, \bar{U}) \mid S \in \mathscr{L}\} \cap \mathrm{Max}_C\{L(\bar{S}, U) \mid U \in \Gamma\}.$$

The following theorem provides necessary and sufficient conditions for the existence of a saddle point.

THEOREM 3.1. *$(\bar{S}, \bar{U})$ is a C-saddle point of L if and only if*

(1) $L(\bar{S}, \bar{U}) \in \mathrm{Min}_C\{L(\bar{S}, \bar{U}) \mid S \in \mathscr{L}\}$,

(2) $G(\bar{S}) \leqq_K 0$,

(3) $\bar{U} \circ G(\bar{S}) = 0$.

Proof. First we suppose that $(\bar{S}, \bar{U})$ is a C-saddle point of L. By the definition of a C-saddle point of L, condition (1) is satisfied and

$$L(\bar{S}, \bar{U}) \in \mathrm{Max}_C\{L(\bar{S}, U) \mid U \in \Gamma\}.$$

The above expression implies

$$L(\bar{S}, \bar{U}) \nleq_C L(\bar{S}, U), \quad \text{for all } U \in \Gamma$$

which further implies

$$U \circ G(\bar{S}) - \bar{U} \circ G(\bar{S}) \notin C \setminus \{0\}, \quad \text{for all } U \in \Gamma. \tag{3.1}$$

Let

$$B = \{U \circ G(\bar{S}) - \bar{U} \circ G(\bar{S}) \mid U \in \Gamma\}.$$

Clearly, B is a non-empty convex subset of $\mathbb{R}^p$ and by (3.1)

$$B \cap C = \{0\}. \tag{3.2}$$

Also,

$$-B^* = \{\mu \in \mathbb{R}^p \mid \langle \mu, U \circ G(\bar{S}) - \bar{U} \circ G(\bar{S}) \rangle \leqq 0, \forall U \in \Gamma\}$$

is a non-empty, closed, convex cone in $\mathbb{R}^p$. Hence, (3.2) implies

$$-B^* \cap \mathrm{int}\, C^* \neq \phi.$$

Thus, there exists $\bar{\mu} \in \mathrm{int}\, C^*$ such that

$$\bar{\mu}^t\big(U \circ G(\bar{S}) - \bar{U} \circ G(\bar{S})\big) \leqq 0, \quad \text{for all } U \in \Gamma. \tag{3.3}$$

We assert that $G(\bar{S}) \leqq_K 0$; that is, condition (2) is satisfied. Let, if possible, $G(\bar{S}) \nleqq_K 0$. Then there would exist $\bar{\lambda} \in K^*$ such that

$$\bar{\lambda}^t G(\bar{S}) > 0. \tag{3.4}$$

As $\bar{\mu} \in \operatorname{int} C^*$, we choose a vector $\xi \in \operatorname{int} C$ such that

$$\bar{\mu}^t \xi = 1. \tag{3.5}$$

Set $\hat{U} = [\operatorname{diag}(H_1(\bar{S}), \dots, H_p(\bar{S}))]\bar{\lambda}^t \xi$. Then $\hat{U} \in \Gamma$ and (3.5) yields

$$\bar{\mu}^t \big(\hat{U} \circ G(S)\big) = \bar{\lambda}^t G(\bar{S}). \tag{3.6}$$

Making the norm $\|\bar{\lambda}\|$ of $\bar{\lambda}$ sufficiently large, we get

$$\bar{\mu}^t \big(\hat{U} \circ G(\bar{S}) - \bar{U} \circ G(\bar{S})\big) = \bar{\lambda}^t G(\bar{S}) - \bar{\mu}^t \big(\bar{U} \circ G(\bar{S})\big) > 0$$

which contradicts (3.3) for $U = \hat{U}$. Therefore, (2) is satisfied. Since $\bar{U} \in \Gamma$ and $G(\bar{S}) \leqq_K 0$, hence

$$\bar{U} G(\bar{S}) \leqq_C 0. \tag{3.7}$$

Further, since C is a cone and $H_i(\bar{S}) > 0$ for $i = 1, \dots, p$, (3.7) gives

$$\bar{U} \circ G(\bar{S}) \leqq_C 0. \tag{3.8}$$

Letting $U = 0$ in (3.3), we get

$$\bar{\mu}^t \big(\bar{U} \circ G(\bar{S})\big) \geqq 0,$$

which, in view of the fact that $\bar{\mu} \in \operatorname{int} C^*$, implies

$$\bar{U} \circ G(\bar{S}) \nleqq_C 0. \tag{3.9}$$

(3.8) and (3.9) together yield

$$\bar{U} \circ G(\bar{S}) = 0.$$

That is, (3) is satisfied.

Next, we show that $(\bar{S}, \bar{U})$ is a C-saddle point of L if (1)–(3) are satisfied.

Suppose that (1)–(3) are satisfied. Let, if possible,

$$L(\bar{S}, \bar{U}) \notin \operatorname{Max}_C \{L(\bar{S}, U) \mid U \in \Gamma\}.$$

Then there would exist $\tilde{U} \in \Gamma$ such that

$$L(\bar{S}, \bar{U}) \leq_C L(\bar{S}, \tilde{U}).$$

By condition (3), we get

$$0 \leq_C \tilde{U} \circ G(\bar{S}). \tag{3.10}$$

Also, for $\tilde{U} \in \Gamma$ and $G(\bar{S}) \leqq_K 0$, we have

$$\tilde{U} \circ G(\bar{S}) \leqq_C 0,$$

which contradicts (3.10). Hence, we have

$$L(\bar{S}, \bar{U}) = \operatorname{Max}_C \{ L(\bar{S}, U) \mid U \in \Gamma \}.$$

The above expression together with (1) implies that $(\bar{S}, \bar{U})$ is a C-saddle point of L. The proof is thus completed. ∎

The next result establishes a relation between a C-saddle point of L and an efficient solution of (VFP).

THEOREM 3.2. *If* $(\bar{S}, \bar{U})$ *is a C-saddle point of L, then $\bar{S}$ is an efficient solution of* (VFP).

Proof. Suppose that $(\bar{S}, \bar{U})$ is a C-saddle point of L. Then by Theorem 3.1, $\bar{S} \in \mathscr{L}$ and $\bar{U} \circ G(\bar{S}) = 0$.

Let, if possible, $\bar{S}$ not be an efficient solution of (VFP). Then there would exist $\hat{S} \in \mathscr{L}$ such that

$$Q(\hat{S}) \leq_C Q(\bar{S}). \tag{3.11}$$

Also, $\hat{S} \in \mathscr{L}$ and $\bar{U} \in \Gamma$ implies

$$\bar{U} \circ G(\hat{S}) \leqq_C 0. \tag{3.12}$$

(3.12) together with $\bar{U} \circ G(\bar{S}) = 0$ implies

$$\bar{U} \circ G(\hat{S}) \leqq_C \bar{U} \circ G(\bar{S}). \tag{3.13}$$

(3.12) and (3.13), along with Lemma 2.1(a), give

$$Q(\hat{S}) + \bar{U} \circ G(\hat{S}) \leqq_c Q(\bar{S}) + \bar{U} \circ G(\bar{S});$$

that is,

$$L(\hat{S}, \bar{U}) \leq_C L(\bar{S}, \bar{U}).$$

This contradicts the fact that

$$L(\bar{S},\bar{U}) \in \mathrm{Min}_C\{L(S,\bar{U}) \mid S \in \mathscr{L}\}.$$

Therefore $\bar{S}$ is an efficient solution of (VFP). This completes the proof. ∎

Let $w = (w_1,\dots,w_p)^t \in \mathbb{R}^p$, $w \geqq 0$ and set

$$\Omega(S,w) = \big(F_1(S) - w_1 H_1(S),\dots,F_p(S) - w_p H_p(S)\big).$$

Remark 3.2. In case H is C-linear, then w need not be nonnegative.

DEFINITION 3.2. The problem (VFP) is said to satisfy the Slater constraint qualification if there exists $\hat{S} \in \mathscr{L}'$ such that $G(\hat{S}) <_K 0$.

Under the assumption of the Slater constraint qualification, we establish a relation between a C-saddle point of L and a properly efficient solution of (VFP).

THEOREM 3.3. *Suppose that the Slater constraint qualification is satisfied for the problem* (VFP). *If* $\bar{S}$ *is a properly efficient solution of* (VFP), *then there exists* $\bar{U} \in \Gamma$ *such that* $(\bar{S},\bar{U})$ *is a* C*-saddle point of* L.

Proof. Let $\bar{w} = Q(\bar{S}) = \frac{F(\bar{S})}{H(\bar{S})}$. Thus $\Omega(\bar{S},\bar{w}) = 0$. Since $\bar{S}$ is a properly efficient solution of (VFP), we therefore have that $\bar{S}$ is also a properly efficient solution of the following programming problem:

$$\begin{aligned}&\mathrm{Min}_C\ \Omega(S,\bar{w})\\ &\text{subject to } S \in \mathscr{L}.\end{aligned}$$

Now, F and $-H$ are C-convex, w^* continuous functions on $\mathscr{L}$; therefore, it follows that $\Omega(S,\bar{w})$ is C-convex, w^* continuous on $\mathscr{L}$. Also G is a K-convex, w^* continuous function on $\mathscr{L}$.

Hence, it follows from Lemma 4.1 of [10] that there exists $\bar{\mu} \in \operatorname{int} C^*$ such that $\bar{S}$ is an optimal solution of the following scalar convex program:

$$\begin{aligned}&\mathrm{Min}\,\langle \bar{\mu}, \Omega(S,\bar{w})\rangle\\ &\text{subject to } S \in \mathscr{L}.\end{aligned}$$

Therefore, the system

$$\left.\begin{aligned}\langle \bar{\mu}, \Omega(S,\bar{w})\rangle &< 0\\ G(S) &<_K 0\end{aligned}\right] \tag{3.14}$$

has no solution $S \in \mathscr{L}'$.

Then by Lemma 2.2, there exists $\bar{\lambda} \in K^*$ such that

$$\bar{\mu}^t \Omega(S, \bar{w}) + \bar{\lambda}^t G(S) \geqq 0, \text{ for all } S \in \mathscr{L}'. \tag{3.15}$$

Letting $S = \bar{S}$ in (3.15) and using $\Omega(\bar{S}, \bar{w}) = 0$, we get

$$\bar{\lambda}^t G(\bar{S}) \geqq 0. \tag{3.16}$$

Further, since $G(\bar{S}) \leqq_K 0$ and $\bar{\lambda} \in K^*$, we have

$$\bar{\lambda}^t G(\bar{S}) \leqq 0. \tag{3.17}$$

(3.16) and (3.17) imply

$$\bar{\lambda}^t G(\bar{S}) = 0. \tag{3.18}$$

Choose $\xi \in \operatorname{int} C$ such that $\bar{\mu}^t \xi = 1$ and set $\bar{U} = \xi \bar{\lambda}^t$. Then, clearly $\bar{U} \in \Gamma$ and $\bar{\mu}^t \bar{U} = \bar{\lambda}^t$. By using (3.18), we obtain

$$\begin{aligned} \bar{U} \circ G(\bar{S}) &= \left[\operatorname{diag}\left(H_1(S), \dots, H_p(S)\right)\right]^{-1} \bar{U} G(\bar{S}) \\ &= \left[\operatorname{diag}\left(H_1(S), \dots, H_p(S)\right)\right]^{-1} \xi \bar{\lambda}^t G(\bar{S}) \\ &= 0. \end{aligned}$$

By Theorem 3.1, if $(\bar{S}, \bar{U})$ is not a C-saddle point of L, then

$$L(\bar{S}, \bar{U}) \notin \operatorname{Min}_C \left\{ L(S, \bar{U}) \mid S \in \mathscr{L} \right\};$$

that is, there exists $\tilde{S} \in \mathscr{L}$ such that

$$L(\tilde{S}, \bar{U}) \leq_C L(\bar{S}, \bar{U})$$

which, in view of the fact that $\bar{U} \circ G(\bar{S}) = 0$, can be rewritten as

$$Q(\tilde{S}) + \bar{U} \circ G(\tilde{S}) \leq_C Q(\tilde{S}) = \bar{w}. \tag{3.19}$$

Since $C \setminus \{0\}$ is a cone and $H(\tilde{S}) > 0$, (3.19) therefore yields

$$\Omega(\tilde{S}, \bar{w}) + \bar{U} G(\tilde{S}) \leq_C 0.$$

Using Lemma 2.1(b), we obtain

$$\bar{\mu}^t \left(\Omega(\tilde{S}, \bar{w}) + \bar{U} G(\tilde{S}) \right) < 0$$

on account of $\bar{\mu} \in \operatorname{int} C^*$, which further, on using the definition of $\bar{U}$, implies

$$\bar{\mu}^t \Omega(\tilde{S}, \bar{w}) + \bar{\lambda}^t G(\tilde{S}) < 0.$$

This contradicts (3.15) for $S = \tilde{S}$. Hence, $(\bar{S}, \bar{U})$ is a C-saddle point of L. The proof is thus completed. ∎

4. DUAL MAP AND DUALITY THEORY

We now derive the duality for (VFP) under the Slater constraint qualification.

Define the dual function by

$$\psi(U) = \mathrm{Min}_C\{L(S,U) \mid S \in \mathscr{L}\}$$

and write

$$\psi(\Gamma) = \bigcup\{\psi(U) \mid U \in \Gamma\}.$$

For the multiobjective fractional programming problem (VFP), the corresponding Lagrange dual problem is defined as follows:

$$\text{(DFP)} \qquad \mathrm{Max}_C\ \psi(\Gamma).$$

DEFINITION 4.1. $\bar{y} \in \mathbb{R}^p$ is said to be a feasible value (efficient value) of (DFP) if there exists $\bar{U} \in \Gamma$ such that $\bar{y} \in \psi(\bar{U})$ (and $\bar{y} \in \mathrm{Max}_C\ \psi(\Gamma)$).

THEOREM 4.1 (weak duality). *Let $\bar{S}$ be a feasible solution of* (VFP) *and $\bar{y}$ be a feasible value of* (DFP). *Then we cannot have $Q(\bar{S}) \leq_C \bar{y}$.*

Proof. Since $\bar{y}$ is a feasible value of (DFP), therefore there exists $\bar{U} \in \Gamma$ such that $\bar{y} \in \psi(\bar{U})$. That is, $\bar{y} \in \mathrm{Min}_C\{L(S,\bar{U}) \mid S \in \mathscr{L}\}$. Therefore, there does not exist any $S \in \mathscr{L}$ for which

$$L(S,\bar{U}) \leq_C \bar{y}. \tag{3.20}$$

Let, if possible,

$$Q(\bar{S}) \leq_C \bar{y}. \tag{3.21}$$

Since $\bar{S}$ is feasible for (VFP), $\bar{U} \in \Gamma$, $H(\bar{S}) > 0$ and C is a cone, it therefore follows that

$$\bar{U} \circ G(\bar{S}) \leqq_C 0. \tag{3.22}$$

(3.21) and (3.22), along with Lemma 2.1(a), give

$$Q(\bar{S}) + \bar{U} \circ G(\bar{S}) \leq_C \bar{y};$$

that is,

$$L(\bar{S},\bar{U}) \leq_C \bar{y}.$$

This contradicts (3.20) for $S = \bar{S}$. Hence, we cannot have $Q(\bar{S}) \leq_C \bar{y}$. This completes the proof. ∎

Remark 4.1. In Theorem 4.1, the set $K \subset \mathbb{R}^m$ need not be a cone. Moreover, no convexity restrictions are required on the n-set functions F, $-H$, and G.

THEOREM 4.2 (strong duality). *Suppose that the Slater constraint qualification is satisfied for the problem* (VFP). *If $\bar{S}$ is a properly efficient solution of* (VFP), *then $Q(\bar{S})$ is an efficient value of* (DFP).

Proof. Let $\bar{S}$ be an efficient solution of (VFP). By Theorem 3.3, there exists $\bar{U} \in \Gamma$ such that $(\bar{S}, \bar{U})$ is a C-saddle point of L and hence, on account of Theorem 3.1 (3), $\bar{U} \circ G(\bar{S}) = 0$. Thus, $L(\bar{S}, \bar{U}) = Q(\bar{S})$. Also

$$L(\bar{S}, \bar{U}) \in \mathrm{Min}_C \{ L(S, \bar{U}) \mid S \in \mathscr{L} \};$$

that is,

$$Q(\bar{S}) \in \psi(\bar{U}).$$

This implies that $Q(\bar{S})$ is a feasible value for (DFP).

If $Q(\bar{S})$ is not an efficient value of (DFP), then there would exist a $\tilde{U} \in \Gamma$ such that for some $\tilde{y} \in \psi(\tilde{U})$, we have $Q(\bar{S}) \leq_C \tilde{y}$. This contradicts Theorem 4.1 for a feasible $\bar{S}$ of (VFP) and a feasible value $\bar{y}$ of (DFP). Hence, $Q(\bar{S})$ is an efficient value of (DFP). The proof is thus completed. ∎

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REFERENCES

1. M. S. Bazaraa and C. M. Shetty, "Nonlinear Programming: Theory and Algorithms," Wiley, New York, 1979.
2. C. R. Bector, D. Bhatia, and S. Pandey, Efficiency and duality for nonlinear multiobjective programs involving n-set functions, *J. Math. Anal. Appl.* **182** (1994), 486–500.
3. C. R. Bector, D. Bhatia, and S. Pandey, Duality for multiobjective fractional programming involving n-set functions, *J. Math. Anal. Appl.* **186** (1994), 747–768.
4. L. Coladas, Z. Li, and S. Wang, Two types of duality in multiobjective fractional programming, *Bull. Austral. Math. Soc.* **54** (1996), 99–114.
5. H. W. Corley, Existence and Lagrangian duality for maximization of set-valued functions, *J. Optim. Theory Appl.* **54** (1987), 489–501.
6. H. W. Corley, Optimization theory for n-set functions, *J. Math. Anal. Appl.* **127** (1987), 193–205.

7. R. R. Egudo, Multiobjective fractional duality, *Bull. Austral. Math. Soc.* **37** (1988), 367–376.
8. W. S. Hsia and T. Y. Lee, Proper *D*-solutions of multiobjective programming with set functions, *J. Optim. Theory Appl.* **53** (1987), 247–258.
9. W. S. Hsia and T. Y. Lee, Lagrangian functions and duality theory in multiobjective programming with set functions, *J. Optim. Theory Appl.* **57** (1988), 239–241.
10. W. S. Hsia, T. Y. Lee, and J. Y. Lee, Lagrange multiplier theorem of multiobjective programming problems with set function, *J. Optim. Theory Appl.* **70** (1991), 137–155.
11. Z. F. Li and S. Y. Wang, Lagrange multipliers and saddle points in multiobjective programming, *J. Optim. Theory Appl.* **83** (1994), 63–81.
12. L. J. Lin, Optimality of differentiable vector-valued *n*-set function, *J. Math. Anal. Appl.* **149** (1990), 255–270.
13. L. J. Lin, On the optimality conditions of vector valued *n*-set functions, *J. Math. Anal. Appl.* **161** (1991), 367–387.
14. R. J. T. Morris, Optimal constrained selection of a measurable subset, *J. Math. Anal. Appl.* **70** (1979), 546–562.
15. Y. Sawaragi, H. Nakayama, and T. Tanino, "Theory of Multiobjective Optimization," Academic Press, New York, 1985.
16. S. K. Suneja and S. Aggarwal, Lagrange duality in multiobjective fractional programming, *Opsearch* **32** (1995), 239–252.
17. S. Y. Wang and Z. F. Li, Scalarization and Lagrange duality in multiobjective optimization, *Optimization* **26** (1992), 315–324.
18. T. Weir and V. Jeyakumar, A class of non convex functions and mathematical programming, *Bull. Austral. Math. Soc.* **38** (1988), 177–189.
19. G. J. Zalmai, Sufficiency criteria and duality in nonlinear programs involving *n*-set function, *J. Math. Anal. Appl.* **149** (1990), 322–338.
20. G. J. Zalmai, Optimality conditions and duality for multiobjective measurable subset selection problems, *Optimization* **22** (1991), 221–238.