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Z-cyclic whist tournaments with a patterned starter initial round

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Abstract

If $v \equiv 1 \pmod 4$ we define a Z-cyclic patterned starter whist tournament on v players, $ZCPS\text{-}Wh(v)$, to be a cyclic whist tournament for which the set of initial round partner pairs form a patterned starter for Z_v. Via orbits, we present all $ZCPS\text{-}Wh(v)$ for $v \in \{5, 13, 17, 25, 29, 37, 41\}$. Watson (1954), Bose and Cameron (1965), and Baker (1975) each introduced constructions which, under certain conditions, yield $ZCPS\text{-}Wh(v)$. Comparisons betwen these three constructions are made and it is demonstrated that beginning with $v = 25$ there exist $ZCPS\text{-}Wh(v)$ other than those obtained by these three constructions.

1. Introduction

Whist tournaments for v players, $Wh(v)$, are known to exist [1,9] for all $v \equiv 0, 1 \pmod 4$. For some history related to this problem see [1,3,6,11]. In this paper we consider only $v \equiv 1 \pmod 4$, specifically $v \in \{5, 13, 17, 25, 29, 37, 41\}$.

Definition [1]. A whist tournament, $Wh(4n+1)$, for $4n+1$ players is a schedule of games each involving two players playing against two others, such that

(i) the games are arranged in $4n+1$ rounds, each of n games,
(ii) each player plays in one game in all but one of the rounds,
(iii) each player partners every other players exactly once,
(iv) each player opposes every other player exactly twice.

We consider that each game (alternatively whist table) is represented by a 4-tuple (a, b, c, d) which denotes that the partnership a, c opposes that of b, d. Any cyclic permutation of a whist table does not destroy the partnerships. A $Wh(4n+1)$ is said to be Z-cyclic if every a, b, c, d belongs to Z_{4n+1} and round $j+1$ can be generated by adding 1 (mod $4n+1$) to all elements of round j. We adopt the convention that the initial round of a Z-cyclic $Wh(4n+1)$ is that which omits 0. In the case that

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$4n + 1 = p^k$ where p is a prime, $p \equiv 1 \pmod 4$, and k is a positive integer, Bose and Cameron [5] provide a solution, $Wh(p^k)$, whose initial round consists of the following n tables.

$$(1, x^{\alpha}, x^{2n}, x^{2n+\alpha}) \text{ times } 1, x^2, x^4, \ldots, x^{2n-2}. \tag{1.1}$$

Here x is a primitive element for $GF(v)$, $v = 4n + 1$, and α is an odd integer, $1 \leqslant \alpha \leqslant 4n - 1$, for which there exists an odd integer β, $1 \leqslant \beta \leqslant 4n - 1$, such that $x^{\alpha} + 1 = x^{\beta}(x^{\alpha} - 1)$. The existence of at least one such pair (α, β) is guaranteed by Mann's Lemma [10]. Oftentimes there are a number of such pairs (α, β) for a given v. Indeed if (α, β) is a pair and $\alpha \neq \beta$ then the pair (β, α) yields, in general, a different $Wh(v)$. Baker's construction [3] also applies to $GF(v)$. His initial round consists of the following n tables:

$$(1, x^n, x^{2n}, x^{3n}) \text{ times } 1, x, x^2, \ldots, x^{n-1}. \tag{1.2}$$

Again x denotes a primitive element for $GF(v)$. Since in $GF(4n + 1)$, $x^{2n} = -1$, both (1.1) and (1.2) are such that each partner pair is an additive inverse pair in $GF(v)$. Consequently in each case the set of partner pairs in the initial round form a patterned starter for the additive group in $GF(v)$. In general, a *starter* is defined as follows. Let G be an additive abelian group of odd order g. Let S be a set of unordered pairs $S = \{(x_i, y_i)\colon\ x_i, y_i \in G,\ 1 \leqslant i \leqslant (g-1)/2\}$. S is called a starter for G if (1) $\{x_i\colon\ 1 \leqslant i \leqslant (g-1)/2\} \cup \{y_i\colon\ 1 \leqslant i \leqslant (g-1)/2\} = G \backslash \{0\}$ and (2) $\{\pm (x_i - y_i)\colon\ 1 \leqslant i \leqslant (g-1)/2\} = G \backslash \{0\}$. If $y_i = -x_i$, $i = 1, 2, \ldots, (g-1)/2$, S is called the patterned starter for G. In the special case $k = 1$, $GF(p) = Z_p$, and we say that both constructions yield *Z-cyclic patterned starter whist tournaments*, $ZCPS$-$Wh(v)$. At first glance constructions (1.1) and (1.2) appear to be quite different and they are if n is even. On the other hand if n is odd, the equality $x^n + 1 = x^{3n}(x^n - 1)$ shows that at least one of the Bose–Cameron solutions produces the same set of initial round tables as the Baker solution although the tables are generated in a different order.

For the case $v = \prod_{i=1}^{l} p_i$ where each p_i is a prime, $p_i \equiv 1 \pmod 4$, p_i not necessarily distinct from p_j, Watson [12] makes the following construction. Take the $v - 1$ tables

$$(x, 1, -x, -1) \text{ times } 1, 2, 3, \ldots, v - 1. \tag{1.3}$$

where $x \in Z_v$ is such that $x^2 \equiv -1 \pmod v$ (for v as described, -1 is always a quadratic residue [8]). Each *distinct* whist table in the collection (1.3) occurs exactly four (4) times corresponding to the four cyclic arrangements of the 4-tuple $(xu, u, -xu, -u)$, $u \in Z_v$, $u \neq 0$. The collection of distinct whist tables in (1.3) forms an initial round of a $ZCPS$-$Wh(v)$. It is not difficult to show that when $l = 1$ and $p_1 = 4n + 1$, then $x = \pm y^n$ where y is a primitive element for Z_{p_1}. Hence in the special case $l = 1$, Watson's construction coincides with Baker's. If, in addition, n is odd then for at least one Bose–Cameron solution, all three constructions coincide. If $l \geqslant 2$ then Watson's construction is independent of the other two. If $l \geqslant 2$ and there is at least one pair (i, j) such that $p_i \neq p_j$ then Watson's construction is meaningful whereas the other two are not.

For each $v \in \{5, 13, 17, 25, 29, 37, 41\}$ we present, via orbits, all $ZCPS\text{-}Wh(v)$. These results are presented in Section 2. In Section 3 we summarize the properties of these orbits related to the specializations of Triplewhist, Directedwhist, and Threeperson whist tournaments.

An interesting fact is that for $v = 5$, 13, and 17 the only $ZCPS\text{-}Wh(v)$ are those given by one or the other of Constructions (1.1)–(1.3) but for $v = 25$, 29, 37, 41 there exist $ZCPS\text{-}Wh(v)$ that do not correspond to any of these three. An empirical observation is that as v increases the totality of orbits given by (1.1)–(1.3) becomes a lesser percentage of the totality of all orbits.

For Z-cyclic $Wh(4n)$ it is conventional to take the v-set as $Z_{4n-1} \cup \{\infty\}$ and as initial round that for which ∞ and 0 are partners. To say that such a $Wh(v)$ is a $ZCPS\text{-}Wh(v)$ would require that the set of all initial round partner pairs, excluding $(\infty, 0)$, constitutes a patterned starter for the additive group Z_{4n-1}. Other than $n = 1$ there are no known examples of $ZCPS\text{-}Wh(4n)$, although Anderson and Finizio [2] come close. Likewise there are no known examples of $ZCPS\text{-}Wh(4n+1)$ when $4n+1$ contains a prime factor $\equiv 3 \pmod 4$. It is a fact, see [7], that there are no $ZCPS\text{-}Wh(9)$ and $ZCPS\text{-}Wh(21)$. Via exhaustive computer search it is also the case that there is no $ZCPS\text{-}Wh(33)$.

2. Orbits of *ZCPS-Wh(v)*

Each additive generator, g, of Z_v induces a mapping Φ_g of Z_v onto itself defined by $\Phi_g(x) = m$ whenever $xg \equiv m \pmod v$. The set $G(v) = \{\Phi_g\colon g \text{ is an additive generator of } Z_v\}$ is a group. For a given v we denote by $ZCPS\text{-}WHIST(v)$ the set *all* $ZCPS\text{-}Wh(v)$. In Table 2 we list the orbits produced when $G(v)$ acts on $ZCPS\text{-}WHIST(v)$ for $v \in \{5, 13, 17, 25, 29, 37, 41\}$. In [7] a similar analysis was carried out for *all* Z-cyclic $Wh(v)$ for each $v \in \{4, 5, 8, 12, 13, 16, 17, 20, 21\}$. Table 1 summarizes the results of the present study. Recall that g is an additive generator for Z_v if and only if g is coprime to v.

The notation $m(n)$ in the column 'order of orbits' means that there are n orbits of order m. In the listing of the orbit it is enough to provide the initial round of a representative element from the orbit. The remaining elements in the orbit can be

Table 1

v	$\lvert G(v)\rvert$	$\lvert ZCPS\text{-}WHIST(v)\rvert$	# of orbits	order of orbits
5	4	1	1	1
13	12	1	1	1
17	16	3	2	2(1), 1(1)
25	20	15	5	5(2), 2(2), 1(1)
29	28	19	5	7(2), 2(2), 1(1)
37	36	272	22	18(12), 9(5), 3(2), 2(2), 1(1)
41	40	718	50	20(28), 10(12), 5(5), 4(2), 1(1)

Table 2

v	Orbit	Order	Representative element
5	1	1	(1, 2, 4, 3)
13	1	1	(1, 5, 12, 8), (2, 3, 11, 10), (4, 6, 9, 7)
17	1	2	(1, 5, 16, 12), (2, 7, 15, 10), (13, 14, 4, 3), (11, 9, 6, 8)
	2	1	(1, 13, 16, 4), (9, 2, 8, 15), (5, 14, 12, 3), (7, 11, 10, 6)
25	1	5	(1, 2), (3, 9), (4, 12), (5, 10), (6, 8), (7, 11)
	2	2	(1, 2), (3, 11), (4, 8), (5, 10), (6, 12), (7, 9)
	3	2	(1, 3), (2, 9), (4, 12), (5, 10), (6, 7), (8, 11)
	4	5	(1, 4), (2, 11), (3, 7), (5, 12), (6, 8), (9, 10)
	5	1	(1, 7), (2, 11), (3, 4), (5, 10), (6, 8), (9, 12)
29	1	2	(1, 2), (3, 13), (4, 8), (5, 10), (6, 12), (7, 14), (9, 11)
	2	2	(1, 3), (2, 9), (4, 12), (5, 14), (6, 11), (7, 8), (10, 13)
	3	7	(1, 4), (2, 13), (3, 7), (5, 11), (6, 14), (8, 9), (10, 12)
	4	7	(1, 5), (2, 12), (3, 10), (4, 7), (6, 14), (8, 9), (11, 13)
	5	1	(1, 12), (2, 5), (3, 7), (4, 10), (6, 14), (8, 9), (11, 13)
37	1	18	(1, 2), (3, 7), (4, 16), (5, 14), (6, 17), (8, 13), (9, 15), (10, 12), (11, 18)
	2	18	(1, 2), (3, 7), (4, 15), (5, 18), (6, 14), (8, 13), (9, 16), (10, 12), (11, 17)
	3	18	(1, 2), (3, 9), (4, 13), (5, 18), (6, 16), (7, 14), (8, 10), (11, 15), (12, 17)
	4	18	(1, 2), (3, 11), (4, 15), (5, 12), (6, 16), (7, 9), (8, 17), (10, 14), (13, 18)
	5	18	(1, 2), (3, 11), (4, 13), (5, 17), (6, 10), (7, 12), (8, 18), (9, 15), (14, 16)
	6	18	(1, 2), (3, 12), (4, 10), (5, 15), (6, 18), (7, 11), (8, 13), (9, 17), (14, 16)
	7	18	(1, 2), (3, 13), (4, 9), (5, 17), (6, 14), (7, 16), (8, 10), (11, 15), (12, 18)
	8	9	(1, 2), (3, 14), (4, 17), (5, 7), (6, 12), (8, 15), (9, 13), (10, 18), (11, 16)
	9	18	(1, 2), (3, 15), (4, 9), (5, 11), (6, 14), (7, 16), (8, 18), (10, 12), (13, 17)
	10	3	(1, 2), (3, 18), (4, 13), (5, 7), (6, 12), (8, 16), (9, 14), (10, 17), (11, 15)
	11	9	(1, 2), (3, 18), (4, 13), (5, 7), (6, 12), (8, 15), (9, 17), (10, 14), (11, 16)
	12	3	(1, 3), (2, 12), (4, 11), (5, 14), (6, 18), (7, 10), (8, 13), (9, 17), (15, 16)
	13	18	(1, 3), (2, 16), (4, 13), (5, 6), (7, 17), (8, 14), (9, 12), (10, 15), (11, 18)
	14	18	(1, 3), (2, 7), (4, 16), (5, 18), (6, 13), (8, 14), (9, 17), (10, 11), (12, 15)
	15	18	(1, 4), (2, 16), (3, 12), (5, 7), (6, 14), (8, 18), (9, 15), (10, 11), (13, 17)
	16	18	(1, 4), (2, 12), (3, 10), (5, 13), (6, 15), (7, 18), (8, 14), (9, 11), (16, 17)
	17	9	(1, 5), (2, 12), (3, 18), (4, 13), (6, 7), (8, 11), (9, 17), (10, 15), (14, 16)
	18	9	(1, 5), (2, 17), (3, 11), (4, 13), (6, 7), (8, 18), (9, 12), (10, 15), (14, 16)
	19	2	(1, 5), (2, 7), (3, 15), (4, 17), (6, 16), (8, 9), (10, 13), (11, 18), (12, 14)
	20	9	(1, 6), (2, 14), (3, 7), (4, 13), (5, 18), (8, 11), (9, 17), (10, 12), (15, 16)
	21	1	(1, 6), (2, 12), (3, 18), (4, 13), (5, 7), (8, 11), (9, 17), (10, 14), (15, 16)
	22	2	(1, 13), (2, 3), (4, 15), (5, 11), (6, 9), (7, 17), (8, 12), (10, 18), (14, 16)
41	1	20	(1, 2), (3, 8), (4, 16), (5, 13), (6, 19), (7, 17), (9, 18), (10, 12), (11, 15), (14, 20)
	2	20	(1, 2), (3, 8), (4, 20), (5, 17), (6, 14), (7, 11), (9, 18), (10, 16), (12, 19), (13, 15)
	3	20	(1, 2), (3, 8), (4, 16), (5, 20), (6, 12), (7, 15), (9, 19), (10, 17), (11, 13), (14, 18)
	4	20	(1, 2), (3, 9), (4, 17), (5, 19), (6, 13), (7, 11), (8, 18), (10, 15), (12, 20), (14, 16)
	5	20	(1, 2), (3, 9), (4, 14), (5, 16), (6, 20), (7, 12), (8, 17), (10, 18), (11, 13), (15, 19)
	6	10	(1, 2), (3, 10), (4, 12), (5, 15), (6, 17), (7, 19), (8, 14), (9, 18), (11, 13), (16, 20)
	7	20	(1, 2), (3, 10), (4, 16), (5, 19), (6, 17), (7, 9), (8, 14), (11, 15), (12, 20), (13, 18)
	8	20	(1, 2), (3, 10), (4, 19), (5, 15), (6, 18), (7, 9), (8, 14), (11, 16), (12, 20), (13, 17)
	9	20	(1, 2), (3, 11), (4, 14), (5, 12), (6, 15), (7, 18), (8, 20), (9, 13), (10, 16), (17, 19)
	10	20	(1, 2), (3, 12), (4, 17), (5, 13), (6, 16), (7, 9), (8, 19), (10, 14), (11, 18), (15, 20)
	11	20	(1, 2), (3, 12), (4, 17), (5, 13), (6, 18), (7, 9), (8, 14), (10, 20), (11, 16), (15, 19)
	12	20	(1, 2), (3, 12), (4, 14), (5, 19), (6, 10), (7, 15), (8, 13), (9, 20), (11, 17), (16, 18)

Table 2 (continued)

v	Orbit	Order	Representative element
41	13	20	(1, 2), (3, 12), (4, 10), (5, 17), (6, 14), (7, 18), (8, 15), (9, 19), (11, 13), (16, 20)
	14	20	(1, 2), (3, 13), (4, 16), (5, 14), (6, 11), (7, 20), (8, 10), (9, 17), (12, 18), (15, 19)
	15	10	(1, 2), (3, 13), (4, 15), (5, 12), (6, 14), (7, 11), (8, 20), (9, 18), (10, 16), (17, 19)
	16	10	(1, 2), (3, 13), (4, 15), (5, 12), (6, 14), (7, 19), (8, 10), (9, 18), (11, 17), (16, 20)
	17	10	(1, 2), (3, 14), (4, 12), (5, 15), (6, 13), (7, 19), (8, 10), (9, 18), (11, 17), (16, 20)
	18	10	(1, 2), (3, 14), (4, 12), (5, 15), (6, 13), (7, 11), (8, 20), (9, 18), (10, 16), (17, 19)
	19	10	(1, 2), (3, 14), (4, 19), (5, 7), (6, 16), (8, 12), (9, 18), (10, 15), (11, 17), (13, 20)
	20	20	(1, 2), (3, 15), (4, 17), (5, 12), (6, 8), (7, 18), (9, 13), (10, 16), (11, 20), (14, 19)
	21	10	(1, 2), (3, 16), (4, 12), (5, 15), (6, 11), (7, 19), (8, 10), (9, 18), (13, 17), (14, 20)
	22	4	(1, 2), (3, 17), (4, 8), (5, 18), (6, 11), (7, 15), (9, 16), (10, 20), (12, 14), (13, 19)
	23	20	(1, 2), (3, 17), (4, 15), (5, 13), (6, 11), (7, 9), (8, 18), (10, 19), (12, 16), (14, 20)
	24	20	(1, 2), (3, 17), (4, 15), (5, 13), (6, 11), (7, 9), (8, 20), (10, 16), (12, 19), (14, 18)
	25	20	(1, 2), (3, 18), (4, 15), (5, 7), (6, 12), (8, 17), (9, 19), (10, 14), (11, 16), (13, 20)
	26	20	(1, 2), (3, 18), (4, 15), (5, 11), (6, 8), (7, 16), (9, 19), (10, 14), (12, 17), (13, 20)
	27	20	(1, 2), (3, 18), (4, 10), (5, 12), (6, 16), (7, 11), (8, 17), (9, 20), (13, 15), (14, 19)
	28	20	(1, 2), (3, 20), (4, 6), (5, 14), (7, 13), (8, 19), (9, 17), (10, 15), (11, 18), (12, 16)
	29	20	(1, 3), (2, 17), (4, 5), (6, 14), (7, 18), (8, 20), (9, 15), (10, 13), (11, 16), (12, 19)
	30	20	(1, 3), (2, 16), (4, 7), (5, 14), (6, 19), (8, 18), (9, 15), (10, 11), (12, 17), (13, 20)
	31	4	(1, 3), (2, 15), (4, 12), (5, 17), (6, 9), (7, 16), (8, 19), (10, 11), (13, 18), (14, 20)
	32	20	(1, 3), (2, 9), (4, 13), (5, 17), (6, 14), (7, 20), (8, 18), (10, 15), (11, 12), (16, 19)
	33	20	(1, 3), (2, 10), (4, 13), (5, 16), (6, 20), (7, 12), (8, 15), (9, 19), (11, 14), (17, 18)
	34	20	(1, 3), (2, 10), (4, 15), (5, 12), (6, 20), (7, 16), (8, 13), (9, 19), (11, 14), (17, 18)
	35	20	(1, 3), (2, 7), (4, 16), (5, 20), (6, 13), (8, 19), (9, 15), (10, 18), (11, 12), (14, 17)
	36	20	(1, 3), (2, 16), (4, 17), (5, 6), (7, 12), (8, 18), (9, 15), (10, 19), (11, 14), (13, 20)
	37	10	(1, 3), (2, 12), (4, 20), (5, 16), (6, 13), (7, 8), (9, 14), (10, 19), (11, 17), (15, 18)
	38	10	(1, 4), (2, 18), (3, 10), (5, 9), (6, 17), (7, 19), (8, 14), (11, 13), (12, 20), (15, 16)
	39	10	(1, 4), (2, 18), (3, 14), (5, 9), (6, 13), (7, 19), (8, 10), (11, 17), (12, 20), (15, 16)
	40	5	(1, 4), (2, 19), (3, 12), (5, 9), (6, 16), (7, 18), (8, 10), (11, 17), (13, 20), (14, 15)
	41	20	(1, 4), (2, 10), (3, 16), (5, 19), (6, 15), (7, 9), (8, 18), (11, 12), (13, 17), (14, 20)
	42	10	(1, 5), (2, 18), (3, 14), (4, 9), (6, 13), (7, 19), (8, 10), (11, 20), (12, 15), (16, 17)
	43	5	(1, 5), (2, 18), (3, 15), (4, 9), (6, 13), (7, 10), (8, 19), (11, 20), (12, 14), (16, 17)
	44	10	(1, 5), (2, 14), (3, 18), (4, 9), (6, 16), (7, 10), (8, 19), (11, 12), (13, 20), (15, 17)
	45	2	(1, 6), (2, 12), (3, 20), (4, 17), (5, 11), (7, 8), (9, 13), (10, 19), (14, 16), (15, 18)
	46	5	(1, 6), (2, 14), (3, 18), (4, 7), (5, 19), (8, 10), (9, 13), (11, 17), (12, 20), (15, 16)
	47	5	(1, 9), (2, 5), (3, 14), (4, 18), (6, 15), (7, 11), (8, 20), (10, 16), (12, 13), (17, 19)
	48	5	(1, 9), (2, 18), (3, 14), (4, 5), (6, 13), (7, 11), (8, 20), (10, 16), (12, 15), (17, 19)
	49	1	(1, 9), (2, 18), (3, 14), (4, 5), (6, 13), (7, 19), (8, 10), (11, 17), (12, 15), (16, 20)
	50	2	(1, 11), (2, 19), (3, 4), (5, 14), (6, 8), (7, 18), (9, 17), (10, 13), (12, 16), (15, 20)

obtained by acting on the representative element by $G(v)$. It is clear that any two elements belonging to the same orbit are equivalent (isomorphic) as block designs. Beginning with $v = 25$ in Table 2 we provide only the first two entries of the 4-tuple (a, b, c, d) since $c = -a = v - a$ and $d = -b = v - b$.

The orbits associated with the three special constructions are given in Table 3. The specific values of the (α, β) associated with the Bose–Cameron orbits are functions of the particular primitive element chosen for $GF(v)$. Here the smallest primitive element was chosen in each case.

Table 3

v	Baker	Watson	Bose–Cameron (α, β)	New orbits
5	1	1	1 (1, 3), (3, 1)	ϕ
13	1	1	1 (3, 9), (9, 3)	ϕ
17	2	2	1 (3, 5), (5, 3), (11, 11), (13, 13)	ϕ
25	ϕ	1	ϕ	2–5
29	5	5	1 (1, 5), (15, 23), (13, 9), (27, 19)	
			2 (5, 1), (19, 27), (9, 13), (23, 15)	3, 4
			5 (7, 21), (21, 7)	
37	21	21	19 (5, 11), (13, 7), (23, 25), (31, 29)	
			21 (9, 27), (27, 9)	1–18, 20
			22 (11, 5), (7, 13), (29, 31), (25, 23)	
41	49	49	45 (1, 17), (19, 3), (21, 23), (39, 37)	1–44, 46–48
			50 (3, 19), (17, 1), (23, 21), (37, 39)	

3. Special features of the orbits

The whist table (a, b, c, d) is said to consist of the ordered pairs ab, bc, cd, and da. Furthermore a, b are called *opponents of the first kind* as are c, d and a, d are called *opponents of the second kind* as are b, c. The following specializations of whist tournaments appear in the literature.

(1) A *triplewhist tournament*, $TWh(v)$, is a $Wh(v)$ that has the additional property that each player is an opponent of the first kind and an opponent of the second kind exactly once with every other player [11].

(2) A *directedwhist tournament*, $DWh(v)$, is a $Wh(v)$ that has the additional property that each of the $v(v-1)$ possible ordered pairs formed from the v-players occurs exactly once in the tournment [3, 4].

(3) A *three person whist tournament*, $3PWh(v)$, is a $Wh(v)$ that has the additional property that no two games of the tournament have three (3) or more players in common [6].

For each of the values of v presented in Section 2 we list those orbits which qualify for one or more of the above specializations. We note first of all that *none* of the solutions contained in Section 2 can be rearranged (i.e. change the location of a set of partner pairs, say $(a, b, c, d) \to (a, d, c, b)$) as a $TWh(v)$ and that *all* of the solutions contained in Section 2 can be rearranged as $DWh(v)$. The $DWh(v)$ is precisely the $ZCPS\text{-}Wh(v)$ that is written as the representative element for the orbit. Using symmetric differences one can verify that the solution offered is a $DWh(v)$. We illustrate this for the case $v = 13$. One obtains the following collection of ordered pair differences: $4, 7, -4, -7, 1, 8, -1, -8, 2, 3, -2, -3$. Replacing each $-a$ by $13 - a$ this list covers the non-zero elements of Z_{13} exactly once. To determine $3PWh(v)$ one must construct the entire tournament and test each pair of games. This is best carried out on a computer. In Table 4 we list only the orbits that are $3PWh(v)$.

Table 4

v	orbits
5	ϕ
13	$\{1\}$
17	$\{1,2\}$
25	$\{4\}$
29	$\{1,3,4,5\}$
37	$\{4,5,10,15,16,17,18,19,20,21,22\}$
41	$\{1,2,10,12,13,19,22,28,38,39,40,41,42,43,44,45,46,47,48,49,50\}$

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