Global Existence of Positive Periodic Solutions of Periodic Predator–Prey System with Infinite Delays

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A set of easily verifiable sufficient conditions is derived for the global existence of periodic solutions with strictly positive components for a periodic predator–prey system with infinite delays by using the method of coincidence degree.

Key Words: positive periodic solutions; global existence; predator–prey system; infinite delays; coincidence degree.

1. INTRODUCTION

The predator–prey system, due to its theoretical and practical significance, has been studied extensively [1–3, 5–13, 16–23]. There have been many good results on stability [1, 2, 5, 7–12, 17, 18], oscillation [2, 5, 6, 12], persistence [21–23], and the existence of periodic solutions through Hopf-type bifurcation [3, 7, 13, 19]. In most of the models considered so far, it has been assumed that all biological and environmental parameters are constants in time. Any biological or environmental parameter, however, is naturally subject to fluctuation in time and if a model is to take into account such fluctuation then the models must be nonautonomous.

We will confine ourselves here to the case that the biological or environmental parameters are periodic at some common period. A very basic and important ecological problem associated with the study of multispecies population interactions in periodic environments is the global existence of a periodic solution with strictly positive components which plays the role

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played by the equilibrium of the autonomous models. It is natural to ask for conditions under which the resulting periodic nonautonomous system would have a periodic solution.

The purpose of this paper is to derive a set of easily verifiable sufficient conditions for the global existence of periodic solutions with strictly positive components for a predator–prey system of one predator and one prey with infinite delays of the form

\[
\frac{dN_1(t)}{dt} = N_1(t) \left[ b_1(t) - \int_{-\infty}^{t} D_1(t, u)N_1(u)du - \int_{-\infty}^{t} \alpha(t, u)N_2(u)du \right]
\]

\[
\frac{dN_2(t)}{dt} = N_2(t) \left[ -b_2(t) + \int_{-\infty}^{t} D_2(t, u)N_1(u)du \right],
\]

where \(N_1(t), N_2(t)\) denote the densities (per square unit of the habitat) of the prey and the predator population at time \(t\), respectively, \(b_i \in C(R, R)\) are \(\omega\)-periodic functions with \(\int_0^{\omega} b_i(t)dt > 0\), \(D_i, \alpha: R \times R \to (0, +\infty)\) satisfy \(D_i(t + \omega, s + \omega) = D_i(t, s)\) and \(\alpha(t + \omega, s + \omega) = \alpha(t, s)\), and \(\int_{-\infty}^{t} D_i(t, u)du\) and \(\int_{-\infty}^{t} \alpha(t, u)du\) are continuous in \(t\). The periodic oscillation of the parameters seems reasonable in view of any seasonal phenomena to which they might be subjected, e.g., mating habits, availability of food, weather conditions, harvesting and hunting, etc. We have assumed in (1.1) that when the predator is absent, the prey species is governed by the well known delay logistic equation which has been studied extensively in the literature (see, for instance, [2, 12, 17]) and in which each individual competes with all others of the system for common resources and intraspecific competition involves response delays to resource limitations extending over the entire past as denoted by \([-\int_{-\infty}^{t} D_1(t, u)N_1(u)du]\) (for the details of the biological significance of this delay one can refer to [2, 17]).
point out that some special autonomous cases of (1.1) have been studied in the literature [2, 3].

The method used here will be the coincidence degree theory proposed by Gaines and Mawhin [4], which has been widely used in the study of ordinary differential equations. Recently, some authors also applied the coincidence degree method to study the existence of periodic solutions for scalar periodic differential equations with finite discrete delays [14–16]. Some good results are obtained. However, as far as we know, it has not been applied to vector system of differential equations with infinite delays, particularly to a system with infinite distributed delays.

2. GLOBAL EXISTENCE OF POSITIVE PERIODIC SOLUTIONS

In order to obtain the existence of positive periodic solutions of (1.1), for the reader’s convenience, we shall summarize in the following a few concepts and results from [4] that will be basic for this section.

Let $X, Z$ be normed vector spaces, let $L: \text{Dom} L \subset X \to Z$ be a linear mapping, and $N: X \to Z$ is a continuous mapping. The mapping $L$ will be called a Fredholm mapping of index zero if $\dim \ker L = \text{codim} \text{Im} L < +\infty$ and $\text{Im} L$ is closed in $Z$. If $L$ is a Fredholm mapping of index zero there exist continuous projectors $P: X \to X$ and $Q: Z \to Z$ such that $\text{Im} P = \ker L, \text{Im} L = \ker Q = \text{Im}(I - Q)$. It follows that $L | \text{Dom} L \cap \ker P: (I - P)X \to \text{Im} L$ is invertible. We denote the inverse of that map by $K_P$. If $\Omega$ is an open bounded subset of $X$, the mapping $N$ will be called $L$-compact on $\Omega$ if $QN(\Omega)$ is bounded and $K_P(I - Q)N: \Omega \to X$ is compact. Since $\text{Im} Q$ is isomorphic to $\ker L$, there exist isomorphisms $J: \text{Im} Q \to \ker L$.

In the proof of our existence theorem below, we will use the continuation theorem of Gaines and Mawhin [4, p. 40].

**Lemma 2.1** (continuation theorem). Let $L$ be a Fredholm mapping of index zero and let $N$ be $L$-compact on $\Omega$. Suppose

(a) For each $\lambda \in (0, 1)$, every solution $x$ of $Lx = \lambda Nx$ is such that $x \not\in \partial \Omega$;

(b) $QN x \neq 0$ for each $x \in \partial \Omega \cap \ker L$ and

\[ \text{deg}\{JQN, \Omega \cap \ker L, 0\} \neq 0. \]

Then the equation $Lx = Nx$ has at least one solution lying in $\text{Dom} L \cap \Omega$.

**Lemma 2.2.** The domain $R_+^2 = \{(y_1, y_2) \mid y_1 \geq 0, y_2 \geq 0\}$ is invariant with respect to (1.1).
In order to use the continuation theorem for (2.2), we first define

\[ N_1(t) = N_1(0) \exp \left\{ \int_0^t \left[ b_1(s) - \int_{-\infty}^s D_1(s, u)N_1(u)du \right. \right. \\
- \int_{-\infty}^s \alpha(s, u)N_2(u)du \left. \right] ds \right\} \\
N_2(t) = N_2(0) \exp \left\{ \int_0^t \left[ -b_2(s) + \int_{-\infty}^s D_2(s, u)N_1(u)du \right] ds \right\} ,

the conclusion follows immediately for all \( t \in [0, +\infty) \). The proof is complete.

Considering the biological significance of (1.1), we specify \( N_i(0) > 0 \), \( i = 1, 2 \). For convenience, in this paper we use the following notations:

\[ \bar{b}_i = \frac{1}{\omega} \int_0^\omega b_i(t)dt, \quad \overline{B}_i = \frac{1}{\omega} \int_0^\omega |b_i(t)|dt, \]
\[ \bar{\alpha} = \frac{1}{\omega} \int_0^\omega \int_{-\infty}^t \alpha(t, u)du dt, \quad \overline{D}_i = \frac{1}{\omega} \int_0^\omega \int_{-\infty}^t D_i(t, u)du dt. \]

Our main result on the global existence of a positive periodic solution of (1.1) is the following theorem.

**Theorem 2.1.** Suppose that \( \bar{b}_1/\bar{b}_2 > \overline{D}_1/\overline{D}_2 \exp(\{\overline{B}_1 + \bar{b}_1\} \omega) \). Then (1.1) has at least one \( \omega \)-periodic solution with strictly positive components.

**Proof.** Let

\[ N_i(t) = \exp\{x_i(t)\}, \quad i = 1, 2. \quad (2.1) \]

On substituting (2.1) into (1.1), we may rewrite (1.1) in the form

\[ \frac{dx_1(t)}{dt} = b_1(t) - \int_{-\infty}^t D_1(t, u)\exp\{x_1(u)\}du - \int_{-\infty}^t \alpha(t, u)\exp\{x_2(u)\}du \\
\frac{dx_2(t)}{dt} = -b_2(t) + \int_{-\infty}^t D_2(t, u)\exp\{x_1(u)\}du. \quad (2.2) \]

In order to use the continuation theorem for (2.2), we first define

\[ X = Z = \{ x(t) = (x_1(t), x_2(t))^T \in C(R, R^2), x(t + \omega) = x(t) \} \]
\[ \|x\| = \|x_1, x_2\| = \max_{t \in [0, \omega]} |x_1(t)| + \max_{t \in [0, \omega]} |x_2(t)| \quad \text{for any } x \in X \text{ (or } Z). \]
Then $X$ and $Z$ are Banach spaces when they are endowed with the norm $\|\cdot\|$. Let

$$
N_x = \begin{pmatrix}
    b_1(t) - \int_{-\infty}^{t} D_1(t, u) \exp\{x_1(u)\} du - \int_{-\infty}^{t} a(t, u) \exp\{x_2(u)\} du \\
    -b_2(t) + \int_{-\infty}^{t} D_2(t, u) \exp\{x_1(u)\} du
\end{pmatrix},
$$

$x \in X$, 

$$
L_x = \frac{dx(t)}{dt}, \quad P_x = \frac{1}{\omega} \int_{0}^{\omega} x(t) dt, \quad x \in X; \quad Qz = \frac{1}{\omega} \int_{0}^{\omega} z(t) dt, \quad z \in Z.
$$

It is not difficult to show that

$$
\ker L = \mathbb{R}^2, \quad \im L = \left\{ z \in Z : \int_{0}^{\omega} z(t) dt = 0 \right\}
$$

is closed in $Z$, 

$$
\dim \ker L = 2 = \text{codim} \im L,
$$

and $P, Q$ are continuous projectors such that

$$
\im P = \ker L, \quad \ker Q = \im L = \im(I - Q).
$$

It follows that $L$ is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to $L$) $K_P : \im L \to \ker P \cap \text{Dom} L$ is given by

$$
K_P(z) = \int_{0}^{t} z(s) ds - \frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{t} z(s) ds dt.
$$

Thus

$$
QN_x = \begin{pmatrix}
    \frac{1}{\omega} \int_{0}^{\omega} \left[ b_1(s) - \int_{-\infty}^{s} D_1(s, u) \exp\{x_1(u)\} du \right] ds \\
    -\int_{-\infty}^{s} a(s, u) \exp\{x_2(u)\} du \right] ds \\
    \frac{1}{\omega} \int_{0}^{\omega} \left[ -b_2(s) + \int_{-\infty}^{s} D_2(s, u) \exp\{x_1(u)\} du \right] ds
\end{pmatrix}
$$
Assume that \( x \) is a solution of (2.3) for a certain \( \lambda \in (0, 1) \).

Integrating (2.3) over the interval \([0, \omega]\), we obtain

\[
\int_0^\omega \left[ b_1(t) - \int_{-\infty}^t D_1(t, u) \exp\{x_1(u)\} du \right] dt = 0,
\]

\[
\int_0^\omega \left[ -b_2(t) + \int_{-\infty}^t D_2(t, u) \exp\{x_1(u)\} du \right] dt = 0.
\]
That is,
\[
\int_0^\omega \left\{ \int_{-\infty}^t D_2(t, u) \exp\{x_1(u)\} du + \int_{-\infty}^t \alpha(t, u) \exp\{x_2(u)\} du \right\} \, dt
= \int_0^\omega b_1(t) \, dt = \bar{b}_1 \omega, \quad (2.4)
\]
\[
\int_0^\omega \int_{-\infty}^t D_2(t, u) \exp\{x_1(u)\} \, du \, dt = \int_0^\omega b_2(t) \, dt = \bar{b}_2 \omega. \quad (2.5)
\]
It follows from (2.3)–(2.5) that
\[
\int_0^\omega |\dot{x}_1(t)| \, dt = \lambda \int_0^\omega \left| b_1(t) - \int_{-\infty}^t D_1(t, u) \exp\{x_1(u)\} du \right| \, dt
- \int_{-\infty}^t \alpha(t, u) \exp\{x_2(u)\} du \right| \, dt
< \int_0^\omega |b_1(t)| \, dt + \int_0^\omega \left\{ \int_{-\infty}^t D_1(t, u) \exp\{x_1(u)\} du 
+ \int_{-\infty}^t \alpha(t, u) \exp\{x_2(u)\} du \right\} \, dt
= (\bar{B}_1 + \bar{b}_1) \omega \quad (2.6)
\]
and
\[
\int_0^\omega |\dot{x}_2(t)| \, dt = \lambda \int_0^\omega \left| b_2(t) + \int_{-\infty}^t D_2(t, u) \exp\{x_1(u)\} du \right| \, dt
\leq (\bar{B}_2 + \bar{b}_2) \omega. \quad (2.7)
\]
Since \( x(t) = (x_1(t), x_2(t))^T \in X \), there exist \( \xi_i, \eta_i \in [0, \omega] \) such that
\[
x_i(\xi_i) = \min_{t \in [0, \omega]} x_i(t), \quad x_i(\eta_i) = \max_{t \in [0, \omega]} x_i(t), \quad i = 1, 2. \quad (2.8)
\]
From (2.5) we can see
\[
\bar{b}_2 \omega \geq \int_0^\omega \int_{-\infty}^t D_2(t, u) \exp\{x_1(\xi_1)\} \, du \, dt = \exp\{x_1(\xi_1)\} \bar{D}_2 \omega;
\]
that is,
\[
x_1(\xi_1) \leq \ln \left\{ \frac{\bar{b}_2}{\bar{D}_2} \right\}.
\]
Then
\[
x_1(t) \leq x_1(\xi_1) + \int_0^\omega |\dot{x}_1(t)| \, dt \leq \ln \left\{ \frac{\bar{b}_2}{\bar{D}_2} \right\} + (\bar{B}_1 + \bar{b}_1) \omega. \quad (2.9)
\]
On the other hand, by (2.5) we also have
\[ \tilde{b}_2 \omega \leq \int_0^\omega \int_{-\infty}^t D_2(t, u) \exp\{x_1(\eta_1)\} du \, dt = \exp\{x_1(\eta_1)\} \bar{D}_2 \omega; \]
that is
\[ x_1(\eta_1) \geq \ln \left\{ \frac{\tilde{b}_2}{\bar{D}_2} \right\}. \]

Thus
\[ x_1(t) \geq x_1(\eta_1) - \int_0^\omega |\dot{x}_1(t)| \, dt \geq \ln \left\{ \frac{\tilde{b}_2}{\bar{D}_2} \right\} - (\bar{B}_1 + \tilde{b}_1) \omega. \quad (2.10) \]

Equations (2.9) and (2.10) imply
\[ \max_{t \in [0, \omega]} |x_1(t)| \leq \max \left\{ \ln \left\{ \frac{\tilde{b}_2}{\bar{D}_2} \right\} + (\bar{B}_1 + \tilde{b}_1) \omega, \right. \]
\[ \left. \ln \left\{ \frac{\tilde{b}_2}{\bar{D}_2} \right\} - (\bar{B}_1 + \tilde{b}_1) \omega \right\} := M_1. \quad (2.11) \]

By (2.4), one obtains
\[ \tilde{b}_1 \omega \geq \int_0^\omega \int_{-\infty}^t \alpha(t, u) \exp\{x_2(u)\} du \, dt \]
\[ \geq \int_0^\omega \int_{-\infty}^t \alpha(t, u) \exp\{x_2(\xi_2)\} du \, dt = \tilde{\alpha} \omega \exp\{x_2(\xi_2)\}; \]
that is,
\[ x_2(\xi_2) \leq \ln \left\{ \frac{\tilde{b}_1}{\tilde{\alpha}} \right\}. \]

Therefore
\[ x_2(t) \leq x_2(\xi_2) + \int_0^\omega |\dot{x}_2(t)| \, dt \leq \ln \left\{ \frac{\tilde{b}_1}{\tilde{\alpha}} \right\} + (\bar{B}_2 + \tilde{b}_2) \omega := M_2. \quad (2.12) \]

From (2.4) and (2.9) we find
\[ \exp\{x_2(\eta_2)\} \tilde{\alpha} \omega \]
\[ \geq \tilde{b}_1 \omega - \int_0^\omega \int_{-\infty}^t D_1(t, u) \exp\{x_1(u)\} du \, dt \]
\[ \geq \tilde{b}_1 \omega - \int_0^\omega \int_{-\infty}^t D_1(t, u) \exp \left\{ \ln \left\{ \frac{\tilde{b}_2}{\bar{D}_2} \right\} + (\bar{B}_1 + \tilde{b}_1) \omega \right\} du \, dt \]
\[ = \tilde{b}_1 \omega - \frac{\bar{D}_1 \tilde{b}_2 \omega}{\bar{D}_2} \exp\{ (\bar{B}_1 + \tilde{b}_1) \omega \}; \]
then

\[ x_2(\eta_2) \geq \ln \left\{ \frac{\tilde{b}_1 - (\overline{D}_1 \tilde{b}_2/D_2) \exp((\overline{B}_1 + \tilde{b}_1) \omega)}{\tilde{a}} \right\} := M_3. \]

Hence

\[ x_2(t) \geq x_2(\eta_2) - \int_0^\omega |\dot{x}_2(t)| \, dt \geq M_3 - (\overline{B}_2 + \tilde{b}_2) \omega. \quad (2.13) \]

From (2.12) and (2.13) we obtain

\[ \max_{t \in [0, \omega]} |x_2(t)| \leq \max\{|M_2|, |M_3 - (\overline{B}_2 + \tilde{b}_2) \omega|\} := M_4. \quad (2.14) \]

Clearly, \( M_i (i = 1, 2, 3, 4) \) are independent of \( \lambda \). Under the assumption in Theorem 2.1, the system of algebraic equations

\[
\begin{align*}
\tilde{b}_1 - \overline{D}_1 v_1 - \tilde{a} v_2 &= 0 \\
\tilde{b}_2 - \overline{D}_2 v_1 &= 0
\end{align*}
\]

has a unique solution \((v_1^*, v_2^*)^T \in R^2\) with \( v_1^* > 0 \). Denote \( M = M_1 + M_4 + M_5 \), where \( M_5 > 0 \) is taken sufficiently large such that \( \|([\ln \{v_1^*\}, \ln \{v_2^*\}]^T) = |\ln \{v_1^*\}| + |\ln \{v_2^*\}| \leq M_5 \), and take \( \Omega = \{x(t) = (x_1(t), x_2(t))^T \in X : \|x\| < M\} \). It is clear that \( \Omega \) verifies the requirement (a) in Lemma 2.1. When \( x \in \partial \Omega \cap \text{Ker} L = \partial \Omega \cap R^2 \), \( x \) is a constant vector in \( R^2 \) with \( \|x\| = M \). Then

\[
QN_x = \begin{pmatrix}
\tilde{b}_1 - \overline{D}_1 \exp\{x_1\} - \tilde{a} \exp\{x_2\} \\
-\tilde{b}_2 + \overline{D}_2 \exp\{x_1\}
\end{pmatrix} \neq 0.
\]

Furthermore, in view of the assumption in Theorem 2.1, it is easy to see that

\[
\deg\{JQN_x, \Omega \cap \text{Ker} L, 0\} \neq 0.
\]

Here \( J \) can be the identity mapping since \( \text{Im} P = \text{Ker} L \). By now we have proved that \( \Omega \) verifies all the requirements in Lemma 2.1. Hence (2.2) has at least one solution \( x^*(t) = (x_1^*(t), x_2^*(t))^T \) in \( \text{Dom} L \cap \overline{\Omega} \). Set \( N_i^*(t) = \exp\{x_i^*(t)\}, \ i = 1, 2 \); then by the medium of (2.1) we know that \( N(t) = (N_1^*(t), N_2^*(t))^T \) is an \( \omega \)-periodic solution of (1.1) with strictly positive components. The proof of Theorem 2.1 is complete.
In system (1.1), we neglect the instantaneous negative feedbacks in prey species. If we take those into account, then (1.1) can be modified as

\[
\frac{dN_1(t)}{dt} = N_1(t) \left[ b_1(t) - c(t)N_1(t) - \int_{-\infty}^{t} D_1(t, u)N_1(u)du - \beta(t)N_2(t) \right] - \int_{-\infty}^{t} \alpha(t, u)N_2(u)du
\]

\[
\frac{dN_2(t)}{dt} = N_2(t) \left[ -b_2(t) + \int_{-\infty}^{t} D_2(t, u)N_1(u)du \right],
\]

(2.15)

where \( c(t), \beta \in C(R, [0, +\infty)) \) are \( \omega \)-periodic functions; \( b_i, D_i, \alpha \) are the same as defined in Theorem 2.1.

**Theorem 2.2.** Suppose that \( \bar{b}_1/\bar{b}_2 > (\bar{D}_1 + \bar{c})/\bar{D}_2 \exp\{ (\bar{B}_1 + \bar{b}_1)\omega \} \). Then (2.15) has at least one \( \omega \)-periodic solution with strictly positive components where \( \bar{c} = \frac{1}{\omega} \int_{0}^{\omega} c(t)dt \).

The proof is exactly the same as that of Theorem 2.1, so we omit it.

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