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Combining a monad and a comonad

John Power[a,*,1], Hiroshi Watanabe[b,2]

[a] *Laboratory for the Foundations of Computer Science, Division of Informatics, University of Edinburgh, King's Buildings, Edinburgh EH9 3JZ, UK*

[b] *Semantics Group, Osaka LERC, Electrotechnical Laboratory, Amagasaki 661-0974, Japan*

Abstract

We give a systematic treatment of distributivity for a monad and a comonad as arises in giving category theoretic accounts of operational and denotational semantics, and in giving an intensional denotational semantics. We do this axiomatically, in terms of a monad and a comonad in a 2-category, giving accounts of the Eilenberg–Moore and Kleisli constructions. We analyse the eight possible relationships, deducing that two pairs are isomorphic, but that the other pairs are all distinct. We develop those 2-categorical definitions necessary to support this analysis.

Keywords: 2-Category; Monad; Comonad; Algebra; Coalgebra; Kleisli construction; Distributive law; Bialgebra

1. Introduction

In recent years, there has been an ongoing attempt to incorporate operational semantics into a category theoretic treatment of denotational semantics. The denotational semantics is given by starting with a signature Σ for a language without variable binding, and considering the category Σ-*Alg* of Σ-algebras [4]. The programs of the language form the initial Σ-algebra. For operational semantics, one starts with a behaviour functor B and considers the category B-*Coalg* of B-coalgebras [5, 7]. By combining these two, one can consider the combination of denotational and operational semantics [14, 16]. Under size conditions, the functor Σ gives rise to a free monad T on it, the functor B gives rise to a cofree comonad D on it, and the fundamental structure one needs to consider is a distributive law of T over D, i.e., a natural transformation

* Corresponding author.
E-mail addresses: ajp@dcs.ed.ac.uk (J. Power), hirowata@etl.go.jp (H. Watanabe).
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$\lambda : TD \Rightarrow DT$ subject to four axioms; and one builds the category λ-*Bialg* from it, a λ-bialgebra being an object X of the base category together with a T-structure and a D-structure on X, subject to one evident coherence axiom. This phenomenon was the subject of the Turi and Plotkin's [16], with leading example given by an idealised parallel language, with operational semantics given by labelled transition systems. In fact, the work of this paper sprang from discussions between one of the authors and Plotkin, whom we acknowledge gratefully.

As a separate piece of work, Brookes and Geva [2] have also proposed the study of a monad and a comonad in combination. For them, the Kleisli category for the comonad gives an intensional semantics, with maps to be regarded as algorithms. They add a monad in the spirit of Moggi to model what has been called a notion of computation [11]. They then propose to study the category for which an arrow is a map of the form $DX \to TY$ in the base category, where T is the monad and D is the comonad. In order for this to form a category, one needs a distributive law of D over T, i.e., a natural transformation $\lambda : DT \Rightarrow TD$ subject to four coherence axioms. Observe that this distributive law allowing one to make a two-sided version of a Kleisli construction is in the opposite direction to that required to build a category of bialgebras.

Motivated by these two examples, in particular the former, we seek an account of the various combinations of a monad and a comonad, with a treatment of Eilenberg–Moore and Kleisli constructions. That is the topic of this paper. The answer is not trivial. It is not just a matter of considering the situation for a distributive law between two monads and taking a dual of one of them, as there are fundamental differences. For instance, to give a pair of monads T and T' and a distributive law of T over T' is equivalent to giving a monad structure on $T'T$ [1] with appropriate coherence, but nothing like that is the case for a distributive law of a monad T over a comonad D. To give a distributive law of T over T' is also equivalent to giving a lifting of the monad T to T'-*Alg*, but not a lifting of T' to T-*Alg*. However, to give a distributive law of a monad T over a comonad D is equivalent to lifting T to D-*Coalg* and also to lifting D to T-*Alg*. Dual remarks, with the Kleisli construction replacing the Eilenberg–Moore construction, apply to distributive laws of comonads over monads. So we need an analysis specifically of distributive laws between a monad and a comonad, and that does not amount to a mild variant of the situation for two monads.

In principle, when one includes an analysis of maps between distributive laws, one has eight choices here: given (T, D, λ) on a category C and (T', D', λ') on C' and a functor $J : C \to C'$, one could consider natural transformations $t : T'J \Rightarrow JT$ and $d : JD \Rightarrow D'J$, or the other three alternatives given by dualisation; and one could dualise by reversing the directions of λ and λ'. But not all of these possibilities have equal status. Two of them each arise in two different ways, reflecting the fact that a category λ-Bialg of bialgebras for a monad T and a comonad D may be seen as both the category of algebras for a monad on D-*Coalg* and as a category of coalgebras for a comonad on T-*Alg*. And two of the eight possibilities do not correspond to applying an Eilenberg–Moore or Kleisli construction to an Eilenberg–Moore or Kleisli construction at all. We investigate the possibilities in Sections 6–8.

As an application of morphisms of distributive laws, consider the Turi and Plotkin work [16]. Suppose we have two languages, each specified by a distributive law for a syntax monad over a behaviour comonad. To give translations of both syntax and behaviour, i.e., a monad morphism and a comonad morphism, that respect the operational semantics, is equivalent to giving a morphism of distributive laws. So this framework provides a consistent and comprehensive translation of languages both in syntax and semantics. Similar remarks apply to the other combinations of monads and comonads.

We make our investigations in terms of an arbitrary 2-category K. The reason is that although the study of operational and denotational semantics in [16] was done in terms of ordinary categories, i.e., modulo size, in the 2-category *Cat*, it was done without a direct analysis of recursion, for which one would pass to the 2-category of O-categories, i.e., categories for which the homsets are equipped with ω-cpo structure, with maps respecting such structure. More generally, that work should and probably soon will be incorporated into axiomatic domain theory, requiring study of the 2-category V-*Cat* for a symmetric monoidal closed V subject to some domain-theoretic conditions [3]. Moreover, our definitions and analysis naturally live at the level of 2-categories, so that level of generality makes the choices clearest and the proofs simplest. Mathematically, this puts our analysis exactly at the level of generality of the study of monads by Street in [15], but see also Johnstone's [6] for an analysis of adjoint lifting that extends to this setting. The 2-categorical treatment clarifies the conditions needed for adjoint lifting. The topic of our study, distributivity for monads and comonads, agrees with that of MacDonald and Stone [9, 10] when restricted to *Cat*. Mulry [12] has also done some investigation into liftings to Kleisli categories.

Much of the abstract work of the first four technical sections of this paper is already in print, primarily in Street's paper [15]. But that is an old paper that was directed towards a mathematical readership; it contains no computational examples or analysis; and the material relevant to us is interspersed with other work that is not relevant. We happily acknowledge Street's contribution, but thought it worthwhile to repeat the relevant part before reaching the substantial new work of this paper, which appears in Sections 6–8.

Formally, we recall the definition of 2-category in Section 2, define the notion of a monad in a 2-category, and introduce the 2-categories $\mathsf{Mnd}(K)$ and $\mathsf{Mnd}^*(K)$. We characterise the Eilenberg–Moore construction and the liftings to those constructions in Section 3. We also explain a dual, yielding the Kleisli construction and the liftings to those constructions in Section 4. This is all essentially in Street's paper [15]. In Section 5, we give another dual, yielding accounts of the Eilenberg–Moore and Kleisli constructions for comonads, and the liftings to them. Then lies the heart of the paper, in which we consider the eight possible combinations of monads and comonads, characterising all of them. For a given 2-category K, we first consider the 2-category $\mathsf{CmdMnd}(K)$ in Section 6. We characterise the category of bialgebras using this 2-category. It also yields a characterisation of functors between categories of bialgebras. In Section 7, we consider $\mathsf{Mnd}^*\mathsf{Cmd}^*(K)$, characterising the Kleisli category of a monad and a comonad and functors between them. We consider the other possibilities in Section 8, which consists of four cases, i.e., four 2-categories of distributive laws. We give explanations of the constructions of 0-, 1- and 2-cells from the 2-categories of distributive laws. We also give some examples of categories constructed in this way when $K = Cat$.

2. Monads in 2-categories

In this section, we define the notion of 2-category and supplementary notions. We then define the notion of a monad in a 2-category K and we define two 2-categories, $\mathsf{Mnd}(K)$ and $\mathsf{Mnd}^*(K)$, of monads in K.

Definition 2.1. A 2-*category* K consists of
- a set of 0-*cells* or *objects*,
- for each pair of 0-cells X and Y, a category $K(X,Y)$ called the *homcategory from X to Y*,
- for each triple of 0-cells, X, Y and Z, a *composition* functor

  $$\circ : K(Y,Z) \times K(X,Y) \to K(X,Z),$$

- for each 0-cell X, an object id_X of $K(X,X)$, or equivalently, a functor $id_X : 1 \to K(X,X)$, called the *identity on X*,

such that the following diagrams of functors commute:

$$\begin{tikzcd}
K(Z,W)\times K(Y,Z)\times K(X,Y) \arrow[r, "\circ\times K(X,Y)"] \arrow[d, "K(Z,W)\times\circ"'] & K(Y,W)\times K(X,Y) \arrow[d, "\circ"] \\
K(Z,W)\times K(X,Z) \arrow[r, "\circ"'] & K(X,W)
\end{tikzcd}$$

$$\begin{tikzcd}
K(X,Y)\times K(X,X) \arrow[dr, "\circ"'] & K(X,Y) \arrow[l, "K(X,Y)\times id_X"'] \arrow[r, "id_Y\times K(X,Y)"] \arrow[d, "="] & K(Y,Y)\times K(X,Y) \arrow[dl, "\circ"] \\
& K(X,Y) &
\end{tikzcd}$$

In the definition of a 2-category, the objects of each $K(X,Y)$ are often called 1-*cells* and the arrows of each $K(X,Y)$ are often called 2-*cells*. We typically abbreviate the composition functors by juxtaposition and use $\cdot$ to represent composition within a homcategory.

Obviously, the definition of 2-category is reminiscent of the definition of category: if one takes the definition of category and replaces homsets by homcategories, composition functions by composition functors, and the axioms by essentially the same axioms but asserting that pairs of functors rather than functions are equal, then one has exactly the definition of a 2-category.

Example 2.2. The leading example of a 2-category is *Cat*, in which the 0-cells are small categories and $Cat(C,D)$ is defined to be the functor category $[C,D]$. In this paper, we sometimes treat *Cat* as though *Set* is a 0-cell of *Cat*. Technically, the existence of two strongly inaccessible cardinals together with a careful variation in the use of the term small allows that.

Example 2.3. For any symmetric monoidal closed category V, one has a 2-category V-*Cat*, whose objects are small V-categories, and with homcategories given by V-functors and V-natural transformations. Two specific examples of this are

- the 2-category *LocOrd* of small locally ordered categories, locally ordered functors, and natural transformations, where V is the category Poset of posets and order-preserving functions.
- the 2-category of small O-categories, O-functors, and natural transformations, where O is the cartesian closed category of ω-cpo's.

Each 2-category K has an underlying ordinary category K_0 given by the 0- and 1-cells of K. A 2-*functor* between 2-categories K and L is a functor from K_0 to L_0 that respects the 2-cell structure. A 2-*natural transformation* between 2-functors is an ordinary natural transformation that respects the 2-cell structure. Given a 2-functor $U : K \to L$, these definitions give rise to the notion of a left 2-*adjoint*, which is a left adjoint that respects the 2-cells. More details and equivalent versions of these definitions appear and are analysed in [8].

Now, we have the definition of 2-category, we can define the notion of a monad in any 2-category K, generalising the definition of monad on a small category, which amounts to the case of $K = Cat$.

Definition 2.4. A *monad* in a 2-category K consists of a 0-cell C, a 1-cell $T : C \to C$, and 2-cells $\mu : T^2 \Rightarrow T$ and $\eta : Id \Rightarrow T$ subject to commutativity of the following diagrams in the homcategory $K(C,C)$:

$$
\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2 \arrow[dr, "\mu"'] & T \arrow[l, "T\eta"'] \arrow[r, "\eta T"] \arrow[d, "="] & T^2 \arrow[dl, "\mu"] \\
 & T &
\end{tikzcd}
$$

For example, if one lets $K = Cat$, then a monad in K as we have just defined it amounts exactly to a small category with a monad on it. More generally, if $K = V$-Cat, then a monad in K amounts exactly to a small V-category together with a V-monad on it. So, for instance, a monad in O-*Cat* amounts to a small O-category together with a monad on it, such that the monad respects the ω-cpo structure of the homs.

For any 2-category K, one can construct a 2-category of monads in K.

Definition 2.5. For any 2-category K, the following data forms a 2-category $\mathsf{Mnd}(K)$:

- 0-cells are monads in K.
- A 1-cell in $\mathsf{Mnd}(K)$ from (C, T, μ, η) to (C', T', μ', η') is a 1-cell $J : C \to C'$ in K, together with a 2-cell $j : T'J \Rightarrow JT$ in K, subject to commutativity in $K(C, C')$ of

$$
\begin{tikzcd}
T'^2J \arrow[r, "T'j"] \arrow[dr, "\mu' J"'] & T'JT \arrow[r, "jT"] & JT^2 \arrow[d, "J\mu \text{ and } J\eta"] & J \arrow[r, "\eta' J"] \arrow[d] & T'J \arrow[dl, "j"] \\
 & T'J \arrow[r, "j"'] & JT & JT &
\end{tikzcd}
$$

- A 2-cell in $\mathsf{Mnd}(K)$ from (J, j) to (H, h) is a 2-cell $\alpha : J \Rightarrow H$ in K subject to the evident axiom expressing coherence with respect to j and h, i.e., the following diagram commutes:

$$
\begin{tikzcd}
T'J \arrow[r, "j"] \arrow[d, "T'\alpha"] & JT \arrow[d, "\alpha T"] \\
T'H \arrow[r, "h"'] & HT
\end{tikzcd}
$$

Example 2.6 (*Turi and Plotkin* [16]). Suppose we are given a language (without variable binding) generated by a signature. The denotational models of this language are given by Σ-algebras on *Set*, where Σ is functor defined by $\Sigma X = \coprod X^{\mathrm{arity}(\sigma)}$ where σ varies over signature. A Σ-algebra is a set X together with a map $h : \Sigma X \to X$, equivalently an interpretation of each σ on the set X. In general, each polynomial functor on *Set* freely generates a monad on *Set*, so there exists a monad (T, μ, η) on *Set* such that Σ-*alg* is isomorphic to T-Alg, the category of Eilenberg–Moore algebras for the monad (T, μ, η). In this case, the set TX for a set X is the set of terms freely generated by the signature applied to X.

Next, suppose we are given Σ and Σ'. The endofunctors freely generate monads (T, μ, η) and (T', μ', η'), respectively. Every natural transformation $\Sigma \Rightarrow T'$ lifts uniquely to a natural transformation $t : T \Rightarrow T'$ such that (Id, t) is a morphism from (Set, T', μ', η') to (Set, T, μ, η) in $\mathsf{Mnd}(Cat)$. The X component of t is a map from TX to $T'X$, i.e., a map which sends each term generated by Σ to a term generated by Σ' respecting the term structure. So translation of languages can sometimes be captured as a morphism of monads.

Example 2.7 (*Turi and Plotkin* [16]). Consider

$$\Sigma_1 X = 1 + A \times X.$$

A Σ_1-algebra consists of a set X together with a constant $nil: 1 \to X$ and for each element $a \in A$ an atomic action $a.: X \to X$.

Now, consider a second language Σ_2 by adding parallel operator $\|$ to the signature of Σ_1. The corresponding polynomial functor is given by

$$\Sigma_2 X = 1 + A \times X + X \times X.$$

For these two languages Σ_1 and Σ_2, we can give an example of a natural transformation $\Sigma_1 \to \Sigma_2$ by defining the X component to be the inclusion of $\Sigma_1 X$ into the first and second components of $\Sigma_2 X$.

Both endofunctors Σ_1, Σ_2 freely generate monads (T_1, μ_1, η_1), (T_2, μ_2, η_2), respectively. The natural transformation $T_1 \Rightarrow T_2$ induced by the above natural transformation from Σ_1 to Σ_2 is the inclusion of $T_1 X$ in $T_2 X$.

Finally in this section, we mention a dual construction. For any 2-category K, one may consider the opposite 2-category K^{op}, which has the same 0-cells as K but $K^{\mathrm{op}}(X, Y) = K(Y, X)$, with composition induced by that of K. This allows us to make a different construction of a 2-category of monads in K, as we could say

Definition 2.8. For a 2-category K, define $\mathsf{Mnd}^*(K) = \mathsf{Mnd}(K^{\mathrm{op}})^{\mathrm{op}}$.

Analysing the definition, a 0-cell of $\mathsf{Mnd}^*(K)$ is a monad in K; a 1-cell from (C, T, μ, η) to (C', T', μ', η') is a 1-cell $J: C \to C'$ in K, together with a 2-cell $j: JT \Rightarrow T'J$ in K, subject to two coherence axioms, expressing coherence between μ and μ' and between η and η'; and a 2-cell from (J, j) to (H, h) is a 2-cell in K from J to H subject to one axiom expressing coherence with respect to j and h. The central difference between $\mathsf{Mnd}(K)$ and $\mathsf{Mnd}^*(K)$ is in the 1-cells, because j is in the opposite direction.

3. Eilenberg–Moore constructions

In this section, we develop our definitions of the previous section, in particular that of $\mathsf{Mnd}(K)$, by characterising the Eilenberg–Moore constructions in terms of the existence of an adjoint to a inclusion 2-functor [15].

For each 2-category K, there is a forgetful 2-functor $U: \mathsf{Mnd}(K) \to K$ sending a monad (C, T, μ, η) in K to its underlying object C. This 2-functor has a right 2-adjoint given by the 2-functor $Inc: K \to \mathsf{Mnd}(K)$ sending an object X of K to (X, id, id, id), i.e., to X together with the identity monad on it. The definition of $\mathsf{Mnd}(K)$ and analysis of it are the central topics of study of [15], a summary of which appears in [8].

Definition 3.1. A 2-category K *admits Eilenberg–Moore constructions for monads* if the 2-functor $Inc: K \to \mathsf{Mnd}(K)$ has a right 2-adjoint.

Remark 3.2. Note in general what the above 2-adjunction means. There is an isomorphism between two categories for each monad $\mathbb{T}=(C,T,\mu,\eta)$ and 0-cell X in K:

$$\mathsf{Mnd}(K)(Inc(X),\mathbb{T}) \cong K(X,\mathbb{T}\text{-Alg}).$$

We denote the $\mathbb{T}$-component $\varepsilon_{\mathbb{T}}: Inc(\mathbb{T}\text{-Alg}) \to \mathbb{T}$ of the counit by a pair $(U^{\mathbb{T}}, u^{\mathbb{T}})$.

$$\begin{tikzcd}
 & C \arrow[dd, "T"] \\
\mathbb{T}\text{-Alg} \arrow[ur, "U^{\mathbb{T}}"] \arrow[dr, "U^{\mathbb{T}}"'] \arrow[r, phantom, "\Downarrow u^{\mathbb{T}}"] & \\
 & C
\end{tikzcd}$$

Then the universality for 1-cells means that for each 1-cell $(J,j): Inc(X) \to (C,T,\mu,\eta)$ in $\mathsf{Mnd}(K)$, i.e., for each 1-cell $J: X \to C$ and each 2-cell $j: TJ \Rightarrow J$ satisfying coherence conditions, there exists a unique 1-cell $J': X \to \mathbb{T}\text{-Alg}$ in K such that $U^{\mathbb{T}}J' = J$ and $u^{\mathbb{T}}J' = j$.

Next, the universality for 2-cells means that for each 2-cell $\alpha: (J,j) \Rightarrow (H,h): Inc(X) \to (C,T,\mu,\eta)$, i.e., for each 2-cell $\alpha: J \Rightarrow H$ subject to a coherence condition, there exists a unique 2-cell $\alpha': J' \Rightarrow H': X \to \mathbb{T}\text{-Alg}$ in K such that $U^{\mathbb{T}}\alpha' = \alpha$, where both 1-cells J', H' are implied by the universality for 1-cells.

Proposition 3.3. *If $K = Cat$, then $Inc: Cat \to \mathsf{Mnd}(Cat)$ has a right 2-adjoint given by the Eilenberg–Moore construction for a monad on a small category.*

Proof. Let $\mathbb{T}=(C,T,\mu,\eta)$ be a monad in Cat. We have a forgetful functor $U^{\mathbb{T}}: \mathbb{T}\text{-Alg} \to C$ as usual. Let $u^{\mathbb{T}}: TU^{\mathbb{T}} \Rightarrow U^{\mathbb{T}}$ be a natural transformation given by $u^{\mathbb{T}}_k = k: TX \to X$ for each $\mathbb{T}$-algebra $k: TX \to X$. Then we have a 1-cell $(U^{\mathbb{T}}, u^{\mathbb{T}}): Inc(\mathbb{T}\text{-Alg}) \to \mathbb{T}$ in $\mathsf{Mnd}(Cat)$. We show that this 1-cell satisfies the universal property.

Given a category X and given a map $(J,j): Inc(X) \to (C,T,\mu,\eta)$, define a functor $\widehat{(J,j)}: X \to \mathbb{T}\text{-Alg}$ on objects by putting $\widehat{(J,j)}a = j_a: TJa \to Ja$, and arrows by sending $f: a \to b$ to $Jf: Ja \to Jb$. Then we have $(U^{\mathbb{T}}, u^{\mathbb{T}}) \circ Inc(\widehat{(J,j)}) = (J,j)$ in $\mathsf{Mnd}(Cat)$. The unicity of $\widehat{(J,j)}$ is obvious.

For two-dimensional property, let $\alpha: (J,j) \Rightarrow (H,h)$ be a 2-cell in $\mathsf{Mnd}(Cat)$, define $\hat{\alpha}_a = \alpha_a: Ja \to Ha$ for each object a in X, then $\hat{\alpha}: \widehat{(J,j)} \Rightarrow \widehat{(H,h)}$ turns out to be a natural transformation by coherence condition of α. It is easy to show that this $\hat{\alpha}$ is the unique natural transformation which satisfies $(U^{\mathbb{T}}, u^{\mathbb{T}})\hat{\alpha} = \alpha$. $\square$

Remark 3.4. Note here what the universal property says: it says that for any small category X and any small category C with a monad T on it, there is a natural isomorphism of categories between $[X, T\text{-}Alg]$ and the category for which an object is a functor $J : X \to C$ together with a natural transformation $TJ \Rightarrow J$ subject to two coherence conditions generalising those in the definition of T-algebra. This is a stronger condition than the assertion that every adjunction gives rise to a unique functor into the category of algebras of the induced monad.

Example 3.5. If V has equalisers, then $V\text{-}Cat$ admits Eilenberg–Moore constructions for monads, and again, the construction is exactly as one expects. This is a fundamental observation underlying [15].

Proposition 3.6. *Suppose K admits Eilenberg–Moore constructions, i.e., the 2-functor Inc has a right 2-adjoint* $(-)\text{-Alg} : \mathsf{Mnd}(K) \to K$. *Then for any 0-cell* $\mathbb{T} = (C, T, \mu, \eta)$ *of* $\mathsf{Mnd}(K)$, *there exists an adjunction* $\langle F^{\mathbb{T}}, U^{\mathbb{T}}, \eta^{\mathbb{T}}, \varepsilon^{\mathbb{T}} \rangle : C \to \mathbb{T}\text{-Alg}$ *in the 2-category K that generates the monad* $\mathbb{T}$.

Proof. The proof is written in [15]. $\square$

Consider the 1-cell $(T, \mu) : \mathbb{T} \to \mathbb{T}$ in $\mathsf{Mnd}(K)$. By using the universality for 1-cells, we have a unique 1-cell $F^{\mathbb{T}} : C \to \mathbb{T}\text{-Alg}$ such that $u^{\mathbb{T}} F^{\mathbb{T}} = \mu$ and $U^{\mathbb{T}} F^{\mathbb{T}} = T$. Next, let $\eta^{\mathbb{T}}$ be the unit $\eta : Id \Rightarrow T = U^{\mathbb{T}} F^{\mathbb{T}}$ of the monad $\mathbb{T} = (C, T, \mu, \eta)$. Since the 2-cell $u^{\mathbb{T}} : TU^{\mathbb{T}} \Rightarrow U^{\mathbb{T}}$ in K is a 2-cell from $(U^{\mathbb{T}} F^{\mathbb{T}} U^{\mathbb{T}}, \mu U^{\mathbb{T}})$ to $(U^{\mathbb{T}}, u^{\mathbb{T}})$ in $\mathsf{Mnd}(K)$, by using the universality for 2-cells, there exists a unique 2-cell $\varepsilon^{\mathbb{T}} : F^{\mathbb{T}} U^{\mathbb{T}} \Rightarrow Id$ such that

$$U^{\mathbb{T}} \varepsilon^{\mathbb{T}} = u^{\mathbb{T}}. \tag{1}$$

Again by the universal property, $U^{\mathbb{T}}(\varepsilon^{\mathbb{T}} F^{\mathbb{T}} \cdot F^{\mathbb{T}} \eta^{\mathbb{T}}) = U^{\mathbb{T}} \varepsilon^{\mathbb{T}} F^{\mathbb{T}} \cdot U^{\mathbb{T}} F^{\mathbb{T}} \eta^{\mathbb{T}} = \mu \cdot T\eta = Id$ implies that $\varepsilon^{\mathbb{T}} F^{\mathbb{T}} \cdot F^{\mathbb{T}} \eta^{\mathbb{T}} = Id$. By using Eq. (1) and the coherence condition, $U^{\mathbb{T}} \varepsilon^{\mathbb{T}} \cdot \eta^{\mathbb{T}} U^{\mathbb{T}} = u^{\mathbb{T}} \cdot \eta U^{\mathbb{T}} = Id$ follows. Hence we can show the existence of an adjunction in the 2-category K. $\square$

3.1. Liftings to Eilenberg–Moore constructions

Now assume K admits Eilenberg–Moore constructions for monads. For each monad $\mathbb{T} = (C, T, \mu, \eta)$ in K, we call the 0-cell $\mathbb{T}$-Alg in K an *Eilenberg–Moore construction* for the monad $\mathbb{T}$. Here, we investigate the existence and nature of liftings of 1-cells to Eilenberg–Moore constructions at the level of generality we have been developing.

Definition 3.7. Let $\mathbb{T} = (C, T, \mu, \eta)$ and $\mathbb{T}' = (C', T', \mu', \eta')$ be 0-cells in $\mathsf{Mnd}(K)$. A 1-cell $J : C \to C'$ in K *lifts to a 1-cell* $\bar{J}$ *on Eilenberg–Moore constructions* if the following diagram commutes in K:

$$\begin{tikzcd} \mathbb{T}\text{-Alg} \arrow[r, "\bar{J}"] \arrow[d, "U^{\mathbb{T}}"'] & \mathbb{T}'\text{-Alg} \arrow[d, "U^{\mathbb{T}'}"] \\ C \arrow[r, "J"'] & C' \end{tikzcd}$$

Definition 3.8. Suppose both 1-cells $J, H : C \to C'$ lift to $\bar{J}, \bar{H}$, respectively on Eilenberg–Moore constructions. A 2-cell $\alpha : J \Rightarrow H : C \to C'$ *lifts to* a 2-cell $\bar{\alpha} : \bar{J} \Rightarrow \bar{H}$ *on Eilenberg–Moore constructions* if the equation $U^{\mathbb{T}'}\bar{\alpha} = \alpha U^{\mathbb{T}}$ holds.

Lemma 3.9. *The right adjoint* 2*-functor* $(-)$-Alg : $\mathsf{Mnd}(K) \to K$ *sends each* 1*-cell* $(J, j) : (C, T, \mu, \eta) \to (C', T', \mu', \eta')$ *in* $\mathsf{Mnd}(K)$ *to a lifting of* J, *and each* 2*-cell* $\alpha : (J, j) \Rightarrow (H, h)$ *to a lifting of* $\alpha : J \Rightarrow H$.

Proof. By using the 2-naturality of the counit, the following diagram commutes for 1-cells $(J, j) : \mathbb{T} \to \mathbb{T}'$ in $\mathsf{Mnd}(K)$:

$$\begin{tikzcd} Inc\,(\mathbb{T}\text{-Alg}) \arrow[r, "\varepsilon_{\mathbb{T}}"] \arrow[d, "Inc\,((J,j)\text{-Alg})"'] & \mathbb{T} \arrow[d, "(J,j)"] \\ Inc\,(\mathbb{T}'\text{-Alg}) \arrow[r, "\varepsilon_{\mathbb{T}'}"'] & \mathbb{T}' \end{tikzcd}$$

Hence we have $U^{\mathbb{T}'}(J, j)\text{-Alg} = JU^{\mathbb{T}}$.

Similarly, naturality for a 2-cell $\alpha : (J, j) \Rightarrow (H, h)$ implies the equation $U^{\mathbb{T}'}\alpha\text{-Alg} = \alpha U^{\mathbb{T}}$. $\square$

Conversely, every lifting arises uniquely from $\mathsf{Mnd}(K)$.

Theorem 3.10. *Suppose a* 1*-cell* $J : C \to C'$ *lifts to* $\bar{J} : \mathbb{T}\text{-Alg} \to \mathbb{T}'\text{-Alg}$ *on Eilenberg–Moore constructions for monads* $\mathbb{T} = (C, T, \mu, \eta)$ *and* $\mathbb{T}' = (C', T', \mu', \eta')$. *Then there exists a unique* 1*-cell* $(J, j) : \mathbb{T} \to \mathbb{T}'$ *in* $\mathsf{Mnd}(K)$ *such that* $(J, j)\text{-Alg} = \bar{J}$.

Suppose both 1*-cells* $J, H : C \to C'$ *lift to* $\bar{J}, \bar{H} : \mathbb{T}\text{-Alg} \to \mathbb{T}'\text{-Alg}$, *respectively, arising from* 1*-cells* $(J, j), (H, h) : \mathbb{T} \to \mathbb{T}'$, *respectively, i.e.,* $(J, j)\text{-Alg} = \bar{J}$ *and* $(H, h)\text{-Alg} = \bar{H}$. *If a* 2*-cell* $\alpha : J \Rightarrow H$ *lifts to* $\bar{\alpha} : \bar{J} \Rightarrow \bar{H}$ *on Eilenberg–Moore constructions, then* α *is a* 2*-cell in* $\mathsf{Mnd}(K)$ *from* (J, j) *to* (H, h) *such that* $\alpha\text{-Alg} = \bar{\alpha}$.

Proof. Given $\bar{J}: \mathbb{T}\text{-Alg} \to \mathbb{T}'\text{-Alg}$, define the 2-cell $j: T'J \Rightarrow JT$ as follows:

$$\begin{array}{lll}
U^{\mathbb{T}'}F^{\mathbb{T}'}J & & \\
\| & & \\
U^{\mathbb{T}'}F^{\mathbb{T}'}J\mathrm{id} \overset{U^{\mathbb{T}'}F^{\mathbb{T}'}J\eta^{\mathbb{T}}}{\Longrightarrow} & U^{\mathbb{T}'}F^{\mathbb{T}'}JU^{\mathbb{T}}F^{\mathbb{T}} & \\
& \| & \\
& U^{\mathbb{T}'}F^{\mathbb{T}'}U^{\mathbb{T}'}\bar{J}F^{\mathbb{T}} \overset{U^{\mathbb{T}'}\varepsilon^{\mathbb{T}'}\bar{J}F^{\mathbb{T}}}{\Longrightarrow} & U^{\mathbb{T}'}\mathrm{id}\bar{J}F^{\mathbb{T}} \\
& & \| \\
& & JU^{\mathbb{T}}F^{\mathbb{T}}
\end{array}$$

For this 2-cell j, note that $j \cdot \eta'J = J\eta$ and $J\mu \cdot jT \cdot T'j = j \cdot \mu'J$. So (J, j) is a 1-cell from the $\mathbb{T}$ to $\mathbb{T}'$. Since $(J, j)\text{-Alg}: \mathbb{T}\text{-Alg} \to \mathbb{T}'\text{-Alg}$ is the unique 1-cell such that $u^{\mathbb{T}'}(J, j)\text{-Alg} = Ju^{\mathbb{T}} \cdot jU^{\mathbb{T}}$, we need only show that $Ju^{\mathbb{T}} \cdot jU^{\mathbb{T}} = u^{\mathbb{T}'}\bar{J}$. But equation (1) implies $Ju^{\mathbb{T}} \cdot jU^{\mathbb{T}} = JU^{\mathbb{T}}\varepsilon^{\mathbb{T}} \cdot jU^{\mathbb{T}} = U^{\mathbb{T}'}\varepsilon^{\mathbb{T}'}\bar{J} = u^{\mathbb{T}'}\bar{J}$. So by universality, we have $(J, j)\text{-Alg} = \bar{J}$.

By definition of $(-)$-Alg, the 2-cell $\alpha\text{-Alg}: (J, j)\text{-Alg} \Rightarrow (H, h)\text{-Alg}$ is the unique one such that $U^{\mathbb{T}'}\alpha\text{-Alg} = \alpha U^{\mathbb{T}}$. So universality for 2-cells implies $\alpha\text{-Alg} = \bar{\alpha}$. □

Corollary 3.11. *Liftings of* 1*-cells to Eilenberg–Moore constructions are equivalent to* 1*-cells in* $\mathsf{Mnd}(K)$. *Liftings of* 2*-cells to Eilenberg–Moore constructions are equivalent to* 2*-cells in* $\mathsf{Mnd}(K)$.

Given an arbitrary 2-category K, we have constructed the 2-category $\mathsf{Mnd}(K)$ of monads in K. Modulo size, this construction can itself be made 2-functorial, yielding a 2-functor $\mathsf{Mnd}: 2\text{-}Cat \to 2\text{-}Cat$, taking a small 2-category K to $\mathsf{Mnd}(K)$, with a 2-functor $G: K \to L$ sent to a 2-functor $\mathsf{Mnd}(G): \mathsf{Mnd}(K) \to \mathsf{Mnd}(L)$, and similarly for a 2-natural transformation. In fact, the 2-category 2-*Cat* forms a 3-category, and the 2-functor Mnd extends to a 3-functor, but we do not use those facts further in this paper, so we do not give the definitions here. It follows that, given a 2-adjunction $F \dashv U: K \to L$, one obtains another 2-adjunction $\mathsf{Mnd}(F) \dashv \mathsf{Mnd}(U): \mathsf{Mnd}(K) \to \mathsf{Mnd}(L)$. We shall use this fact later.

4. Kleisli construction

In this section, we consider a dual to the work of the previous section. This is not just a matter of reversing the direction of every arrow in sight. But by putting $L = K^{\mathrm{op}}$, we can deduce results about $\mathsf{Mnd}^*(K)$ from results about $\mathsf{Mnd}(L)$. In particular, we have

Proposition 4.1. (1) *The construction* $\mathsf{Mnd}^*(K)$ *yields a* 2*-functor* $\mathsf{Mnd}^*: 2\text{-}Cat \to 2\text{-}Cat$.

(2) *The forgetful* 2*-functor* $U: \mathsf{Mnd}^*(K) \to K$ *has a left* 2*-adjoint given by* $Inc: K \to \mathsf{Mnd}^*(K)$, *sending an object* X *of* K *to the identity monad on* X.

We can characterise Kleisli constructions by using the 2-category $\mathsf{Mnd}^*(K)$. We can show the following by the dual argument to Proposition 3.3.

Proposition 4.2. *If $K = Cat$, then $Inc : Cat \to \mathsf{Mnd}^*(Cat)$ has a left 2-adjoint given by the Kleisli construction for a monad on a small category.*

Spelling out the action of the 2-functor $(-)\text{-Kl} : \mathsf{Mnd}^*(Cat) \to Cat$ on 1-cells and 2-cells, a 1-cell $(J, j) : (C, T, \mu, \eta) \to (C', T', \mu', \eta')$ is sent to the functor $(J, j)\text{-Kl} : \mathbb{T}\text{-Kl} \to \mathbb{T}'\text{-Kl}$, which sends an object a of $\mathbb{T}$-Kl to the object Ja of $\mathbb{T}'$-Kl, and an arrow $f : a \to b$ of $\mathbb{T}$-Kl, i.e., an arrow $\hat{f} : a \to Tb$ of C, to the arrow of $\mathbb{T}'$-Kl given by $j_b \circ J\hat{f} : Ja \to T'Jb$. A 2-cell $\alpha : (J, j) \Rightarrow (H, h)$ is sent to the natural transformation $\alpha\text{-Kl} : (J, j)\text{-Kl} \Rightarrow (H, h)\text{-Kl}$ whose a component is given by $\eta'_{Ha} \circ \alpha_a : Ja \to T'Ha$.

The above construction and proof extend readily to the case of $K = V\text{-}Cat$.

In light of this result, we say

Definition 4.3. A 2-category K *admits Kleisli constructions for monads* if the 2-functor $Inc : K \to \mathsf{Mnd}^*(K)$ has a left 2-adjoint.

Proposition 4.4. *Suppose a 2-category K admits Kleisli constructions for monads with the left 2-adjoint to Inc given by $(-)\text{-Kl} : \mathsf{Mnd}^*(K) \to K$. For any 0-cell $\mathbb{T} = (C, T, \mu, \eta)$ of $\mathsf{Mnd}^*(K)$, there is an adjunction $\langle F_{\mathbb{T}}, G_{\mathbb{T}}, \eta_{\mathbb{T}}, \varepsilon_{\mathbb{T}} \rangle : C \to \mathbb{T}\text{-Kl}$ in K that generates the monad $\mathbb{T}$.*

Proof. Dual to the proof of Proposition 3.6. $\square$

4.1. Liftings to Kleisli constructions

Now, we assume a 2-category K admits Kleisli constructions for monads. For each monad $\mathbb{T} = (C, T, \mu, \eta)$ in K we call $\mathbb{T}$-Kl a *Kleisli construction* for the monad $\mathbb{T}$.

We can define the liftings to Kleisli constructions as follows:

Definition 4.5. Let $\mathbb{T} = (C, T, \mu, \eta)$ and $\mathbb{T}' = (C', T', \mu', \eta')$ be 0-cells in $\mathsf{Mnd}^*(K)$. A 1-cell $J : C \to C'$ in K *lifts to a* 1-cell $\bar{J} : \mathbb{T}\text{-Kl} \to \mathbb{T}'\text{-Kl}$ *on Kleisli constructions* if the following diagram commutes in K:

$$\begin{tikzcd} \mathbb{T}\text{-Kl} \arrow[r, "\bar{J}"] & \mathbb{T}'\text{-Kl} \\ C \arrow[u, "F_{\mathbb{T}}"] \arrow[r, "J"'] & C' \arrow[u, "F_{\mathbb{T}'}"'] \end{tikzcd}$$

We can also define the notion of a lifting of a 2-cell.

Definition 4.6. Suppose 1-cells $J, H : C \to C'$ lift to $\bar{J}, \bar{H} : \mathbb{T}\text{-Kl} \to \mathbb{T}'\text{-Kl}$ on Kleisli constructions. A 2-cell $\alpha : J \Rightarrow H$ *lifts to a* 2-cell $\bar{\alpha} : \bar{J} \Rightarrow \bar{H}$ *on Kleisli constructions* if the equation $\bar{\alpha} F_{\mathbb{T}} = F_{\mathbb{T}'} \alpha$ holds.

Since $\mathsf{Mnd}^*(K) = \mathsf{Mnd}(K^{\mathrm{op}})^{\mathrm{op}}$, we remark that

Lemma 4.7. *The following two conditions are equivalent*:
1. K *admits Kleisli constructions for monads.*
2. K^{op} *admits Eilenberg–Moore constructions for monads.*

So, dualising Theorem 3.10, we have

Theorem 4.8. *Suppose a* 1-*cell* $J : C \to C'$ *lifts to* $\bar{J} : \mathbb{T}\text{-Kl} \to \mathbb{T}'\text{-Kl}$ *on Kleisli constructions for monads* $\mathbb{T} = (C, T, \mu, \eta)$ *and* $\mathbb{T}' = (C', T', \mu', \eta')$. *Then there exists a unique* 1-*cell* $(J, j) : \mathbb{T} \to \mathbb{T}'$ *in* $\mathsf{Mnd}^*(K)$ *such that* $(J, j)\text{-Kl} = \bar{J}$.

Suppose both 1-*cells* $J, H : C \to C'$ *lift to* $\bar{J}, \bar{H} : \mathbb{T}\text{-Kl} \to \mathbb{T}'\text{-Kl}$, *respectively*, *arising from* 1-*cells* $(J, j), (H, h) : \mathbb{T} \to \mathbb{T}'$, *respectively*, *i.e.*, $(J, j)\text{-Kl} = \bar{J}$ *and* $(H, h)\text{-Kl} = \bar{H}$. *If a* 2-*cell* $\alpha : J \Rightarrow H$ *lifts to* $\bar{\alpha} : \bar{J} \Rightarrow \bar{H}$ *on Kleisli constructions*, *then* α *is a* 2-*cell in* $\mathsf{Mnd}^*(K)$ *from* (J, j) *to* (H, h) *such that* $\alpha\text{-Kl} = \bar{\alpha}$.

Corollary 4.9. *Liftings of* 1-*cells to Kleisli constructions are equivalent to* 1-*cells in* $\mathsf{Mnd}^*(K)$. *Liftings of* 2-*cells to Kleisli constructions are equivalent to* 2-*cells in* $\mathsf{Mnd}^*(K)$.

5. Comonads in 2-categories

We now turn from monads to comonads. The results we seek about comonads follow from those about monads by consideration of another duality applied to an arbitrary 2-category. Given a 2-category K, one may consider two distinct duals: K^{op} as in the previous section and K^{co}. The 2-category K^{co} is defined to have the same 0-cells as K but with $K^{\mathrm{co}}(X, Y)$ defined to be $K(X, Y)^{\mathrm{op}}$.

In K^{op}, the 1-cells are reversed, but the 2-cells are not, whereas in K^{co}, the 2-cells are reversed but the 1-cells are not. One can of course reverse both 1-cells and 2-cells, yielding K^{coop}, or isomorphically, K^{opco}.

Definition 5.1. A *comonad* in K is defined to be a monad in K^{co}, i.e., a 0-cell C, a 1-cell $D : C \to C$, and 2-cells $\delta : D \Rightarrow D^2$ and $\varepsilon : D \Rightarrow Id$, subject to the duals of the three coherence conditions in the definition of monad.

Taking $K = Cat$, a comonad in K as we have just defined it is exactly a small category together with a comonad on it.

One requires a little care in defining $\mathsf{Cmd}(K)$, the 2-category of comonads in K. If one tries to define $\mathsf{Cmd}(K)$ to be $\mathsf{Mnd}(K^{\mathrm{co}})$, then there is no forgetful 2-functor from $\mathsf{Cmd}(K)$ to K.

Definition 5.2. For a 2-category K, define $\mathsf{Cmd}(K)$ to be $\mathsf{Mnd}(K^{\mathrm{co}})^{\mathrm{co}}$.

Explicitly, a 0-cell in $\mathsf{Cmd}(K)$ is a comonad in K. A 1-cell in $\mathsf{Cmd}(K)$ from (C,D,δ,ε) to $(C',D',\delta',\varepsilon')$ is a 1-cell $J: C \to C'$ in K together with a 2-cell $j: JD \Rightarrow D'J$ subject to two coherence conditions, one relating δ and δ', the other relating ε and ε'. A 2-cell from (J,j) to (H,h) is a 2-cell in K from J to H subject to one coherence condition relating j and h.

Note carefully the definition of a 1-cell in $\mathsf{Cmd}(K)$. It consists of a 1-cell and a 2-cell in K; of those, the 1-cell goes in the same direction as that in the definition of $\mathsf{Mnd}(K)$, but the 2-cell goes in the opposite direction.

Example 5.3. In [16], categories of coalgebras for behaviour endofunctors on Set are used. Examples are $B_1X = 1 + A \times X$ and $B_2X = P_\omega(A \times X)$, where P_ω is the finite powerset functor. A B_1-coalgebra is a set X together with a map $X \to B_1X$, i.e., a deterministic A-labelled transition system. A B_2-coalgebra is a finitely branching A-labelled transition system. B_1-coalgebras are used for deterministic processes and B_2-coalgebras are used for non-deterministic processes.

Similar to the algebras for endofunctors, endofunctors like B_1, B_2 on *Set* cofreely generate comonads, i.e., there exist comonads $(D_1,\delta_1,\varepsilon_1)$, $(D_2,\delta_2,\varepsilon_2)$, respectively, on *Set* such that $B_1\text{-}coalg \cong \mathbb{D}_1\text{-Coalg}$ and $B_2\text{-}coalg \cong \mathbb{D}_2\text{-Coalg}$.

Suppose endofunctors B and B' cofreely generate comonads $\mathbb{D}$ and $\mathbb{D}'$, respectively. Then every natural transformation $B \Rightarrow B'$ between two behaviour functors generates a natural transformation $d: D \Rightarrow D'$ such that (Id, d) is a morphism from $\mathbb{D}$ to $\mathbb{D}'$ in $\mathsf{Cmd}(Cat)$. This analysis can be extended to consider natural transformations from D to B', but we do not have examples at that full level of generality.

For the above endofunctors B_1 and B_2, we can consider the natural transformation $B_1 \Rightarrow B_2$ whose X component sends $*$ to $\emptyset$ and (a,x) to $\{(a,x)\}$. It generates a comonad morphism from $\mathbb{D}_1$ to $\mathbb{D}_2$.

Also, one may define $\mathsf{Cmd}^*(K) = \mathsf{Cmd}(K^{\mathrm{op}})^{\mathrm{op}}$. Since the operations $(-)^{\mathrm{op}}$ and $(-)^{\mathrm{co}}$ commute, we have

Proposition 5.4. *For any* 2-*category* K, $\mathsf{Cmd}^*(K) = \mathsf{Mnd}^*(K^{\mathrm{co}})^{\mathrm{co}}$.

5.1. Eilenberg–Moore constructions for comonads

Just as in the situation for monads, there is an underlying 2-functor $U: \mathsf{Cmd}(K) \to K$, which has a right 2-adjoint given by $Inc: K \to \mathsf{Cmd}(K)$, sending an object X to the identity comonad on X; and again, one may say

Definition 5.5. A 2-category K *admits Eilenberg–Moore constructions for comonads* if $Inc: K \to \mathsf{Cmd}(K)$ has a right 2-adjoint.

Although not stated explicitly in [15], it follows routinely that the 2-category *Cat* admits Eilenberg–Moore constructions for comonads, and they are given by the usual Eilenberg–Moore construction. Again here, the construction $\mathsf{Cmd}(K)$ yields a 2-functor $\mathsf{Cmd} : 2\text{-}Cat \to 2\text{-}Cat$.

Proposition 5.6. *Suppose a* 2*-category* K *admits Eilenberg–Moore constructions for comonads. We denote the right* 2*-adjoint by* $(-)\text{-Coalg} : \mathsf{Cmd}(K) \to K$. *For any* 0*-cell* $\mathbb{D} = (C, D, \delta, \varepsilon)$ *of* $\mathsf{Cmd}(K)$, *there is an adjunction* $\langle U^{\mathbb{D}}, G^{\mathbb{D}}, \eta^{\mathbb{D}}, \varepsilon^{\mathbb{D}} \rangle : \mathbb{D}\text{-Coalg} \to C$ *in* K *that generates the comonad* $\mathbb{D}$.

Proof. Dual to the proof of Proposition 3.6. □

5.1.1. Liftings to Eilenberg–Moore constructions

Now, dually to the case for monads, assume K admits Eilenberg–Moore constructions for comonads. For each comonad $\mathbb{D} = (C, D, \delta, \varepsilon)$ in K, we call $\mathbb{D}$-Coalg an *Eilenberg–Moore construction* for $\mathbb{D}$.

Definition 5.7. Let $\mathbb{D} = (C, D, \delta, \varepsilon)$ and $\mathbb{D}' = (C', D', \delta', \varepsilon')$ be 0-cells in $\mathsf{Cmd}(K)$. A 1-cell $J : C \to C'$ in K *lifts to a* 1-cell $\bar{J} : \mathbb{D}\text{-Coalg} \to \mathbb{D}'\text{-Coalg}$ *on Eilenberg–Moore constructions* if the following diagram commutes in K:

$$\begin{tikzcd} \mathbb{D}\text{-Coalg} \arrow[r, "\bar{J}"] \arrow[d, "U^{\mathbb{D}}"'] & \mathbb{D}'\text{-Coalg} \arrow[d, "U^{\mathbb{D}'}"] \\ C \arrow[r, "J"'] & C' \end{tikzcd}$$

Definition 5.8. Suppose 1-cells $J, H : C \to C'$ lift to $\bar{J}, \bar{H} : \mathbb{D}\text{-Coalg} \to \mathbb{D}'\text{-Coalg}$ on Eilenberg–Moore constructions. A 2-cell $\alpha : J \Rightarrow H$ *lifts to a* 2-cell $\bar{\alpha} : \bar{J} \Rightarrow \bar{H}$ *on Eilenberg–Moore constructions* if $U^{\mathbb{D}'}\bar{\alpha} = \alpha U^{\mathbb{D}}$.

Since $\mathsf{Cmd}(K) = \mathsf{Mnd}(K^{\mathrm{co}})^{\mathrm{co}}$, we remark that

Lemma 5.9. *The following two conditions are equivalent*:
1. *K admits Eilenberg–Moore constructions for comonads.*
2. *K^{co} admits Eilenberg–Moore constructions for monads.*

So, dualising Theorem 3.10, we have

Theorem 5.10. *If* $\bar{J} : \mathbb{D}\text{-Coalg} \to \mathbb{D}'\text{-Coalg}$ *is a lifting of a* 1*-cell* $J : C \to C'$ *to Eilenberg–Moore constructions*, *then there is a unique* 1*-cell* $(J, j) : (C, D, \delta, \varepsilon) \to (C', D', \delta', \varepsilon')$ *in* $\mathsf{Cmd}(K)$ *such that* $(J, j)\text{-Coalg} = \bar{J}$.

Suppose 1*-cells* $J, H : C \to C'$ *lift to* (J, j)-Coalg, (H, h)-Coalg $: \mathbb{D}$-Coalg $\to \mathbb{D}'$-Coalg *for* 1*-cells* $(J, j), (H, h) : \mathbb{D} \to \mathbb{D}'$ *in* $\mathsf{Cmd}(K)$, *respectively. If a* 2*-cell* $\bar{\alpha} : (J, j)$-Coalg $\Rightarrow$ (H, h)-Coalg *in* K *is a lifting of a* 2*-cell* $\alpha : J \Rightarrow H$, *then* α *is a* 2*-cell in* $\mathsf{Cmd}(K)$ *from* (J, j) *to* (H, h) *such that* α-Coalg $= \bar{\alpha}$.

Corollary 5.11. *Liftings of* 1*-cells to Eilenberg–Moore constructions are equivalent to* 1*-cells in* $\mathsf{Cmd}(K)$. *Liftings of* 2*-cells to Eilenberg–Moore constructions are equivalent to* 2*-cells in* $\mathsf{Cmd}(K)$.

5.1.2. Liftings to Kleisli constructions

Now assume K admits Kleisli constructions for comonads. For each comonad $\mathbb{D} = (C, D, \delta, \varepsilon)$ in K, we call $\mathbb{D}$-CoKl a *Kleisli construction* for the comonad $\mathbb{D}$.

Definition 5.12. Let $\mathbb{D} = (C, D, \delta, \varepsilon)$ and $\mathbb{D}' = (C', D', \delta', \varepsilon')$ be 0-cells in $\mathsf{Cmd}^*(K)$. A 1-cell $J : C \to C'$ in K *lifts to a* 1-cell $\bar{J} : \mathbb{D}$-CoKl $\to \mathbb{D}'$-CoKl *on Kleisli constructions* if the following diagram commutes in K:

$$\begin{tikzcd}
\mathbb{D}\text{-CoKl} \arrow[r, "\bar{J}"] & \mathbb{D}'\text{-CoKl} \\
C \arrow[u, "F_{\mathbb{D}}"] \arrow[r, "J"'] & C' \arrow[u, "F_{\mathbb{D}'}"']
\end{tikzcd}$$

Definition 5.13. Suppose 1-cells, $J, H : C \to C'$ lift to $\bar{J}, \bar{H} : \mathbb{D}$-CoKl $\to \mathbb{D}'$-CoKl, respectively, on Kleisli constructions. A 2-cell $\alpha : J \Rightarrow H$ *lifts to a* 2-cell $\bar{\alpha} : \bar{J} \Rightarrow \bar{H}$ *on Kleisli constructions* if $\bar{\alpha} F_{\mathbb{D}} = F_{\mathbb{D}'} \alpha$.

Similarly to Lemma 4.7,

Lemma 5.14. *The following two conditions are equivalent*:
1. K *admits Kleisli constructions for comonads.*
2. K^{op} *admits Eilenberg–Moore constructions for comonads.*

Once again by dualising Theorem 3.10, we have

Theorem 5.15. *If* $\bar{J} : \mathbb{D}$-CoKl $\to \mathbb{D}'$-CoKl *is a lifting of a* 1*-cell* $J : C \to C'$ *to Kleisli constructions for comonads, then there is a unique* 1*-cell* $(J, j) : (C, D, \delta, \varepsilon) \to (C', D', \delta', \varepsilon')$ *in* $\mathsf{Cmd}^*(K)$ *such that* (J, j)-CoKl $= \bar{J}$.

Suppose 1*-cells* $J, H : C \to C'$ *lift to* (J, j)-CoKl, (H, h)-CoKl $: \mathbb{D}$-CoKl $\to \mathbb{D}'$-CoKl *for* 1*-cells* $(J, j), (H, h) : \mathbb{D} \to \mathbb{D}'$, *respectively. If a* 2*-cell* $\bar{\alpha} : (J, j)$-CoKl $\Rightarrow (H, h)$-CoKl *in* K *is a lifting of a* 2*-cell* $\alpha : J \Rightarrow H$, *then* α *is a* 2*-cell in* $\mathsf{Cmd}^*(K)$ *from* (J, j) *to* (H, h) *such that* α-CoKl $= \bar{\alpha}$.

Corollary 5.16. *Liftings of* 1*-cells to Kleisli constructions for comonads are equivalent to* 1*-cells in* $\mathsf{Cmd}^*(K)$. *Similarly, liftings of* 2*-cells to Kleisli constructions for comonads are equivalent to* 2*-cells in* $\mathsf{Cmd}^*(K)$.

6. $\mathsf{CmdMnd}(K)$

In previous sections, we have defined 2-functors Mnd, Mnd^*, Cmd and Cmd^*. So in principle, one might guess that there are eight possible ways of combining a monad and a comonad as there are three dualities: start with the monad or start with the comonad; taking $(\ )^*$ on the monad or not; and likewise for the comonad. In fact, as we shall see, there are precisely six. First, we analyse the 2-functor CmdMnd. In order to do that, we give the definition of a distributive law of a monad over a comonad in a 2-category.

Definition 6.1. Given a monad (T,μ,η) and a comonad (D,δ,ε) on an object C of a 2-category K, a *distributive law of T over D* is a 2-cell $\lambda : TD \Rightarrow DT$ which satisfies laws involving each of μ, η, δ and ε:

$$\lambda \cdot \mu D = D\mu \cdot \lambda T \cdot T\lambda, \quad \lambda \cdot \eta D = D\eta,$$

$$D\lambda \cdot \lambda D \cdot T\delta = \delta T \cdot \lambda, \quad T\varepsilon = \varepsilon T \cdot \lambda.$$

Definition 6.2. For any 2-category K, the following data forms a 2*-category* $Dist(K)$ *of distributive laws*:

- A 0-cell consists of a 0-cell C of K, a monad $\mathbb{T}$ on it, a comonad $\mathbb{D}$ on it, and a distributive law $\lambda : TD \Rightarrow DT$.
- A 1-cell $(J, j_t, j_d) : (C, T, D, \lambda) \to (C', T', D', \lambda')$ consists of a 1-cell $J : C \to C'$ in K together with a 2-cell $j_t : T'J \Rightarrow JT$ subject to the monad laws, together with a 2-cell in K of the form $j_d : JD \Rightarrow D'J$ subject to the comonad laws, all subject to one coherence condition given by a hexagon

$$\begin{tikzcd}
 & JTD \arrow[r, "J\lambda"] & JDT \arrow[dr, "j_dT"] & \\
T'JD \arrow[ur, "j_tD"] \arrow[dr, "T'j_d"'] & & & D'JT \\
 & T'D'J \arrow[r, "\lambda' J"'] & D'T'J \arrow[ur, "D'j_t"'] &
\end{tikzcd}$$

- A 2-cell from (J, j_t, j_d) to (H, h_t, h_d) consists of a 2-cell from J to H in K subject to two conditions expressing coherence with respect to j_t and h_t and coherence with respect to j_d and h_d.

Proposition 6.3. *For any* 2-*category* K, *the* 2-*category* $\mathsf{CmdMnd}(K)$ *is isomorphic to* $Dist(K)$.

Thus $Dist(Cat)$ gives as 0-cells exactly the data considered by Turi and Plotkin [16]. Turi and Plotkin did not, in that paper, address the 1-cells of $Dist(Cat)$, but they propose to do so in future. The 0-cells provide them with a combined operational and denotational semantics for a language; the 1-cells allow them to account for the interpretation of one language presented in such a way into another language thus presented. In fact, it was in response to Plotkin's specific proposal about how to do that much of the work of this paper was done. For a simple example, one might have a monad and comonad on the category *Set*, and embed it into the category of ω-cpo's in order to add an account of recursion.

Example 6.4. We give an example of a distributive law for a monad over a comonad. Let (T,μ,η) be the monad on *Set* sending a set X to the set X^* of finite lists, and let (D,δ,ε) be the comonad that sends a set X to the set of streams X^ω. Consider the natural transformation $\lambda: TD \Rightarrow DT$ whose X component sends a finite list of streams $\bar{a}_1\bar{a}_2\cdots\bar{a}_n$ with $\bar{a}_i = a_{i1}a_{i2}a_{i3}\cdots$, $(1 \leqslant i \leqslant n)$ to the stream of finite lists $(a_{11}a_{21}\cdots a_{n1})(a_{12}a_{22}\cdots a_{n2})(a_{13}a_{23}\cdots a_{n3})\cdots$. This natural transformation satisfies the axioms for a distributive law of a monad over a comonad. Hence these data give an example of a 0-cell of $\mathsf{CmdMnd}(Cat)$. It also becomes a 0-cell of both $\mathsf{Cmd}^*\mathsf{Mnd}(Cat)$ and $\mathsf{Mnd}^*\mathsf{Cmd}(Cat)$ later.

Example 6.5. The distributive laws in [16] are given in the following manner. For a given language Σ and a suitable behaviour B, Turi and Plotkin model a GSOS rule by a natural transformation $\Sigma(Id \times B) \Rightarrow BT$, where (T,μ,η) is the monad freely generated by the endofunctor Σ. They then show that the monad (T,μ,η) lifts to B-$Coalg$ the category of B-coalgebras for the endofunctor B, which means T,μ and η lift.

$$
\begin{tikzcd}
B\text{-}coalg \arrow[r, "\tilde{T}"] \arrow[d, "U_B"'] & B\text{-}coalg \arrow[d, "U_B"] \\
Set \arrow[r, "T"'] & Set
\end{tikzcd}
$$

Since $B\text{-}coalg \cong \mathbb{D}$-Coalg for the comonad (D,δ,ε) cofreely generated by B, this diagram is equivalent to the lifting diagram for the monad (T,μ,η) to the category of Eilenberg–Moore coalgebras for the comonad $\mathbb{D}$. By Theorem 3.10, this is equivalent to one datum and two conditions:

- A natural transformation $\lambda: TD \Rightarrow DT$ such that $(T,\lambda):(Set,D,\delta,\varepsilon)\to(Set,D,\delta,\varepsilon)$ is a 1-cell of $\mathsf{Cmd}(Cat)$.

- The natural transformation $\mu : T^2 \Rightarrow T$ becomes a 2-cell from $(T, \lambda)^2$ to (T, λ) in $\mathsf{Cmd}(\mathit{Cat})$.
- The natural transformation $\eta : \mathit{Id} \Rightarrow T$ gives a 2-cell from Id to (T, λ) in $\mathsf{Cmd}(\mathit{Cat})$.

Hence it is equivalent to give a distributive law $\lambda : TD \Rightarrow DT$.

A corollary of Proposition 6.3, which although easily proved, is conceptually fundamental, is

Corollary 6.6. $\mathsf{CmdMnd}(K)$ *is isomorphic to* $\mathsf{MndCmd}(K)$.

Proof. It is easily to check that $\mathit{Dist}(K)$ is isomorphic to $\mathit{Dist}(K^{\mathrm{co}})^{\mathrm{co}}$. Since $\mathsf{Cmd}(K) = \mathsf{Mnd}(K^{\mathrm{co}})^{\mathrm{co}}$ and $\mathit{Mnd}(K) = \mathsf{Cmd}(K^{\mathrm{co}})^{\mathrm{co}}$, we have

$$\begin{aligned}\mathsf{CmdMnd}(K) &= \mathit{Dist}(K)\\ &= \mathit{Dist}(K^{\mathrm{co}})^{\mathrm{co}}\\ &= \mathsf{Cmd}(\mathsf{Mnd}(K^{\mathrm{co}})^{\mathrm{co}}\\ &= \mathsf{Cmd}(\mathsf{Mnd}(K^{\mathrm{co}})^{\mathrm{coco}})^{\mathrm{co}}\\ &= \mathsf{Cmd}(\mathsf{Cmd}(K)^{\mathrm{co}})^{\mathrm{co}}\\ &= \mathsf{MndCmd}(K).\end{aligned}$$

Theorem 6.7. *Suppose K admits Eilenberg–Moore constructions for monads and comonads. Then, $\mathit{Inc} : K \to \mathsf{CmdMnd}(K)$ has a right 2-adjoint.*

Proof. Since K admits Eilenberg–Moore constructions for monads, $\mathit{Inc} : K \to \mathsf{Mnd}(K)$ has a right 2-adjoint. Since $\mathsf{Cmd} : 2\text{-}\mathit{Cat} \to 2\text{-}\mathit{Cat}$ is a 2-functor, it sends adjunctions to adjunctions, so $\mathsf{Cmd}(\mathit{Inc}) : \mathsf{Cmd}(K) \to \mathsf{CmdMnd}(K)$ has a right 2-adjoint. Since K admits Eilenberg–Moore constructions for comonads, $\mathit{Inc} : K \to \mathsf{Cmd}(K)$ has a right adjoint. Composing the right adjoints gives the result. □

This result gives us a universal property for the construction of the category of λ-Bialgebras, given a monad $\mathbb{T}$, a comonad $\mathbb{D}$, and a distributive law of T over D. In this precise sense, one may see the construction of a category of bialgebras as a generalised Eilenberg–Moore construction.

Using Proposition 6.3 and Corollary 6.6, we may characterise the right 2-adjoint in three ways, giving

Corollary 6.8. *If K admits Eilenberg–Moore constructions for monads and comonads, then given a distributive law of a monad (T, μ, η) over a comonad (D, δ, ε), the following are equivalent*:

- *λ-Bialg determined directly by the universal property of a right 2-adjoint to the inclusion $\mathit{Inc} : K \to \mathit{Dist}(K)$ sending X to the identity distributive law on X*;
- *the Eilenberg–Moore object for the lifting of $\mathbb{T}$ to $\mathbb{D}$*-Coalg;
- *the Eilenberg–Moore object for the lifting of $\mathbb{D}$ to $\mathbb{T}$*-Alg.

By the universal property, the right 2-adjoint ( )-*Bialg* inherits an action on 1- and 2-cells. The behaviour of the right 2-adjoint on 0-cells gives exactly the construction ( )-*Bialg* studied by Turi and Plotkin [16]. Its behaviour on 1-cells will be fundamental to their later development as outlined above.

More concretely, the right 2-adjoint sends each 1-cell $(J, j_t, j_d):(C, \mathbb{T}, \mathbb{D}, \lambda) \to (C', \mathbb{T}, \mathbb{D}, \lambda)$ to a 1-cell J-Bialg : λ-Bialg $\to \lambda'$-Bialg such that the following diagram commutes, where $U_\lambda : \lambda\text{-Bialg} \to C$ is the canonical 1-cell:

$$\begin{tikzcd} \lambda\text{-Bialg} \arrow[r, "J\text{-Bialg}"] \arrow[d, "U_\lambda"'] & \lambda'\text{-Bialg} \arrow[d, "U'_\lambda"] \\ \mathrm{C} \arrow[r, "J"'] & \mathrm{C}' \end{tikzcd}$$

It also sends each 2-cell $\alpha:(J, j_t, j_d) \Rightarrow (H, h_t, h_d)$ to a 2-cell α-Bialg : J-Bialg $\Rightarrow$ H-Bialg satisfying the equation $U_{\lambda'}\alpha\text{-Bialg} = \alpha U_\lambda$.

Remark 6.9. Although all 1- and 2-cells in $\mathsf{MndCmd}(K)$ give liftings to bialgebras, we do not have a converse as we cannot construct the data for a 1-cell in $\mathsf{MndCmd}(K)$ from a given lifting.

Example 6.10. Consider the Eilenberg–Moore construction, i.e., the category of λ-bialgebras, for the monad, comonad, and distributive law λ of Example 6.4. Since the comonad (D, δ, ε) is cofreely generated by the endofunctor *Id* on *Set*, D-Coalg is isomorphic to *Id*-*coalg*, the category of coalgebras for the endofunctor *Id*; this is the category of deterministic dynamical systems. Hence, every object $k : X \to DX$ of D-Coalg can be seen as a dynamical system (X, α) with state space X and transition function $\alpha : X \to X$. Here, $k(x) = x\alpha(x)\alpha^2(x)\cdots$.

The Eilenberg–Moore construction $\mathbb{T}$-Alg for the monad $\mathbb{T}$ is as follows: each object $h : TX \to X$ is a semigroup X with a structure map h which sends a list of elements $x_1x_2\cdots x_n$ to their composite.

So the category λ-Bialg for the distributive law $\lambda : TD \Rightarrow DT$ is as follows. An object $(h : TX \to X,\ k : X \to DX)$ of λ-Bialg is a dynamical system (X, α) where the state space X is given by a semigroup such that $\alpha h(x_1x_2\cdots x_n) = h(\alpha(x_1)\alpha(x_2)\cdots\alpha(x_n))$ for every finite sequence $x_1x_2\cdots x_n$ of X elements.

An arrow $f : (h : TX \to X,\ k : X \to DX) \to (h' : TX \to X,\ k' : X \to DX)$ is a map $f : X \to Y$ that is a morphism of both semigroups and dynamical systems.

7. $\mathsf{Mnd}^*\mathsf{Cmd}^*(K)$

This section is essentially about Kleisli constructions, considering the complete dual to the previous section. One can deduce the following from Corollary 6.6.

Corollary 7.1. $\mathsf{Mnd}^*\mathsf{Cmd}^*(K)$ *is isomorphic to* $\mathsf{Cmd}^*\mathsf{Mnd}^*(K)$.

Moreover, one can deduce an equivalent result to Proposition 6.3: this yields that the isomorphic 2-categories of Corollary 7.1 amount to giving the opposite distributive law to that given by Cmd and Mnd, and hence give an account of Kleisli constructions lifting along Kleisli constructions. The left 2-adjoint to $Inc : K \to \mathsf{Mnd}^*\mathsf{Cmd}^*(K)$ can again be characterised in three ways:

Corollary 7.2. *If K admits Kleisli constructions for monads and comonads, then given a distributive law of a comonad (D, δ, ε) over a monad (T, μ, η), the following are equivalent*:
- λ-Kl *determined directly by the universal property of the inclusion $Inc : K \to Dist^*(K)$ sending X to the identity distributive law on X*,
- *the Kleisli object for the lifting of $\mathbb{T}$ to $\mathbb{D}$*-Kl,
- *the Kleisli object for the lifting of $\mathbb{D}$ to $\mathbb{T}$*-Kl.

This is the construction proposed by Brookes and Geva [2] for giving intensional denotational semantics.

The fundamental step in the proof here lies in the use of the proof of Theorem 6.7, and that proof relies upon the following: some mild conditions on K hold of all our leading examples, allowing us to deduce that K admits Eilenberg–Moore and Kleisli constructions for monads and comonads; and each of the constructions Mnd, Mnd^*, Cmd and Cmd^* is 2-functorial on 2-*Cat*, so preserves adjunctions.

Spelling out the action of the 2-functor on 1- and 2-cells, a 1-cell $(J, j_t, j_d) : (C, \mathbb{T}, \mathbb{D}, \lambda) \to (C', \mathbb{T}', \mathbb{D}', \lambda')$ is sent to the 1-cell J-Kl : λ-Kl $\to \lambda'$-Kl such that the following diagram commutes:

$$
\begin{tikzcd}
\lambda\text{-Kl} \arrow[r, "J\text{-Kl}"] & \lambda'\text{-Kl} \\
C \arrow[u, "F_\lambda"] \arrow[r, "J"'] & C' \arrow[u, "F'_\lambda"']
\end{tikzcd}
$$

Here F_λ and $F_{\lambda'}$ are canonical 1-cells.

A 2-cell $\alpha : (J, j_t, j_d) \Rightarrow (H, h_t, h_d)$ is sent to the 2-cell α-Kl : J-Kl $\Rightarrow H$-Kl such that $\alpha\text{-KlF}_\lambda = F_{\lambda'}\alpha$.

Remark 7.3. Although all 1- and 2-cells in $\mathsf{Mnd}^*\mathsf{Cmd}^*(K)$ give liftings to Kleisli constructions for monads and comonads, we cannot have a converse as we cannot construct the data for a 1-cell in $\mathsf{Mnd}^*\mathsf{Cmd}^*(K)$ from a lifting.

Proposition 7.4. *When $K = Cat$, the Kleisli construction for monads and comonads exists and is given as follows. Let (D, δ, ε) be a comonad and (T, μ, η) be a monad* *on C and $\lambda : DT \to TD$ be a distributive law on D and T. Then the objects of* λ-Kl *are the those of C. An arrow from x to y in* λ-Kl *is given by an arrow $f : Dx \to Ty$ in C. For each object x, the identity is given by $\eta_x \circ \varepsilon_x$. The composition of arrows $f : x \to y$ and $g : y \to z$ in* λ-Kl *seen as arrows $\hat{f} : Dx \to Ty$ and $\hat{g} : Dy \to Tz$ in C is given by the composite $\mu_z(T\hat{g})\lambda_y(D\hat{f})\delta_x$ in C.*

Proof. We need only write the image under the left 2-adjoint $Inc : Cat \to \mathsf{Cmd}^*\mathsf{Mnd}^*(Cat)$. This left adjoint is given by composing the two left 2-adjoints as in Theorem 6.7: the 2-functor Cmd^* applied to the Kleisli construction of T, and the Kleisli construction of D. $\square$

Now, we give an example of a distributive law of a comonad over a monad, hence a 0-cell of $\mathsf{Cmd}^*\mathsf{Mnd}^*(K)$, and the Kleisli construction for a monad and comonad.

Example 7.5. Let $(P, \bigcup, \{-\})$ be the powerset monad on *Set*, i.e., the powerset functor P and union operation $\bigcup : P^2 \Rightarrow P$ and singleton mapping $\{-\} : Id \Rightarrow P$. Let (D, δ, ε) be a comonad on *Set* where the endofunctor D sends a set X to the product set $A \times X$ for some set A. Consider the natural transformation $\lambda : DT \Rightarrow TD$ whose X component sends a pair (a, α) of an element a of A with $\alpha \in P(X)$ to the set $\{(a, x) \mid x \in \alpha\}$. This satisfies the axioms for a distributive law of a comonad over a monad. Hence this gives an example of a 0-cell in $\mathsf{Cmd}^*\mathsf{Mnd}^*(Cat)$. It also turns out to be a 0-cell in both $\mathsf{CmdMnd}^*(Cat)$ and $\mathsf{MndCmd}^*(Cat)$.

This distributive law is essentially the same as the one in the Power and Turi paper [13]. Their monad is the non-empty powerset monad on *Set*.

Example 7.6. Applying Proposition 7.4, we spell out the Kleisli construction λ-Kl for the monad and comonad given in the above example. The objects of the category λ-Kl are the those of *Set*. An arrow from X to Y in λ-Kl is given by a map $f : A \times X \to P(Y)$. The identity arrow for each object X is given by the map $\eta_X \circ \varepsilon_X : A \times X \to P(X)$ which sends a pair (a, x) to the singleton $\{x\}$. The composition of arrows $f : X \to Y$ and $g : Y \to Z$ in λ-Kl seen as maps $\hat{f} : A \times X \to P(Y)$ and $\hat{g} : A \times Y \to P(Z)$ is given by composite $\bigcup_Z (P\hat{g})\lambda_Y(A \times \hat{f})\delta_X : A \times X \to P(Z)$ which sends (a, x) to the subset $\bigcup\{\hat{g}(a, y) \mid y \in \hat{f}(a, x)\}$ of Z.

8. The other four possibilities

Applying the work of previous sections to the remaining four possible combinations of a monad with a comonad, we can summarise the various 2-categories by Table 1, including the previous 2-categories.

Each 2-category is defined as follows:

- A 0-cell consists of a 0-cell C of K, a monad $\mathbb{T}$ on it, a comonad $\mathbb{D}$ on it, and a distributive law λ whose direction is listed in the second column of Table 1.

Table 1
Distributive laws

	λ	j_t	j_d
Cmd Mnd = Mnd Cmd	$TD \Rightarrow DT$	$T'J \Rightarrow JT$	$JD \Rightarrow D'J$
Cmd* Mnd	$TD \Rightarrow DT$	$T'J \Rightarrow JT$	$D'J \Rightarrow JD$
Mnd* Cmd	$TD \Rightarrow DT$	$JT \Rightarrow T'J$	$JD \Rightarrow D'J$
Cmd Mnd*	$DT \Rightarrow TD$	$JT \Rightarrow T'J$	$JD \Rightarrow D'J$
Mnd Cmd*	$DT \Rightarrow TD$	$T'J \Rightarrow JT$	$D'J \Rightarrow JD$
Cmd* Mnd* = Mnd* Cmd*	$DT \Rightarrow TD$	$JT \Rightarrow T'J$	$D'J \Rightarrow JD$

- A 1-cell $(J, j_t, j_d) : (C, \mathbb{T}, \mathbb{D}, \lambda) \to (C', \mathbb{T}', \mathbb{D}', \lambda')$ consists of a 1-cell $J : C \to C'$ in K together with a 2-cell j_t with direction in the third column, subject to monad laws, and a 2-cell j_d in the fourth column, subject to comonad laws, all subject to one coherence hexagon.
- A 2-cell from (J, j_t, j_d) to (H, h_t, h_d) consists of a 2-cell from J to H in K subject to two conditions expressing coherence with respect to j_t and h_t and coherence with respect to j_d and h_d.

Remark 8.1. As described in Table 1, the 2-categories $\mathsf{CmdMnd}(K)$, $\mathsf{Cmd}^*\mathsf{Mnd}(K)$ and $\mathsf{Mnd}^*\mathsf{Cmd}(K)$ have the same 0-cells, and $\mathsf{CmdMnd}^*(K)$, $\mathsf{MndCmd}^*(K)$ and $\mathsf{Cmd}^*\mathsf{Mnd}^*(K)$ have the same 0-cells.

In considering the possible ways of combining a pair of categories each with a monad and a comonad, there appear three possible independent dualities:
- $TD \Rightarrow DT$ or the dual,
- $JT \Rightarrow T'J$ or the dual,
- $D'J \Rightarrow JD$ or the dual.

This gives eight possibilities, but we can see from above list that two of them do not arise. The two that do not arise are

$$TD \Rightarrow DT \quad JT \Rightarrow T'J \quad D'J \Rightarrow JD$$

and the complete dual, dualising all three items,

$$DT \Rightarrow TD \quad T'J \Rightarrow JT \quad JD \Rightarrow D'J.$$

8.1. $\mathsf{Cmd}^*\mathsf{Mnd}(K)$

Consider $\mathsf{Cmd}^*\mathsf{Mnd}(K)$. When K admits Kleisli constructions for comonads and Eilenberg–Moore constructions for monads, we can consider the Kleisli construction for a comonad lifting to the Eilenberg–Moore object for the monad. In detail, for a 0-cell $(C, \mathbb{T}, \mathbb{D}, \lambda)$ in $\mathsf{Cmd}^*\mathsf{Mnd}(K)$, i.e., a monad $\mathbb{T} = (C, T, \mu, \eta)$ and a comonad $\mathbb{D} = (C, D, \delta, \varepsilon)$ with a distributive law $\lambda : TD \Rightarrow DT$, we first lift the comonad $\mathbb{D}$ on to the Eilenberg–Moore construction for the monad $\mathbb{T}$ by applying the 2-functor $\mathsf{Cmd}^*((-)\text{-Alg}) : \mathsf{Cmd}^*\mathsf{Mnd}(K) \to \mathsf{Cmd}^*(K)$ to obtain the comonad $(\mathbb{T}\text{-Alg}, (D, \lambda)\text{-Alg},$

δ-Alg, ε-Alg) in $\mathsf{Cmd}^*(K)$. Then we apply the 2-functor $(-)$-CoKl to obtain the Kleisli construction for the comonad. Observe that the composition 2-functor $(-)$-CoKlCmd^* ($(-)$-Alg) cannot be characterised as a left or right 2-adjoint functor to *Inc*.

When $K = Cat$, this construction gives the following category for a given 0-cell $(C, \mathbb{T}, \mathbb{D}, \lambda)$ in $\mathsf{Cmd}^*\mathsf{Mnd}(Cat)$. Objects are the Eilenberg–Moore algebras for the monad $\mathbb{T}$. An arrow f from $h: Tx \to x$ to $k: Ty \to y$ is an arrow $\hat{f}: Dx \to y$ in C such that $k \circ Tf = f \circ Dh \circ \lambda_x$. For each $\mathbb{T}$-algebra $h: Tx \to x$, the identity arrow is given by the arrow $\varepsilon_x : x \to Dx$ in C.

Example 8.2. Applying the above construction to the 0-cell given in Example 6.4, we have the following category. An object is a $\mathbb{T}$-algebra for the monad $\mathbb{T}$, hence it is a semigroup $h: X^* \to X$. An arrow f from a semigroup $h: X^* \to X$ to $k: Y^* \to Y$ is a morphism of semigroups from the semigroup $h^\omega : (X^\omega)^* \to X^\omega$ to k, where the multiplication of h^ω is defined by $(x_1 x_2 \cdots)(y_1 y_2 \cdots) = (x_1 y_1)(x_2 y_2) \cdots$ for two streams $x_1 x_2 \cdots, y_1 y_2 \cdots \in X^\omega$.

8.2. $\mathsf{Mnd}^*\mathsf{Cmd}(K)$

When K admits Eilenberg–Moore constructions for comonads and Kleisli construction for monads, we have a composite 2-functor $(-)$-KlMnd^*($(-)$-Coalg)$: \mathsf{Mnd}^*\mathsf{Cmd}(K) \to K$. This functor sends each 0-cell of $\mathsf{Mnd}^*\mathsf{Cmd}(K)$ to the Kleisli construction for the monad lifted to the Eilenberg–Moore construction for the comonad.

Spelling out the above construction when $K = Cat$, the construction sends each 0-cell $(C, \mathbb{T}, \mathbb{D}, \lambda)$ to the following category. Objects are $\mathbb{D}$-coalgebras. An arrow from $h: X \to DX$ to $k: Y \to DY$ is an arrow $f: X \to TY$ in C such that $Df \circ h = \lambda_Y \circ Tk \circ f$.

Example 8.3. Recall Example 6.4, the example of a distributive law of a monad over a comonad and consider the above construction. It yields the following category. Objects are $\mathbb{D}$-coalgebras, hence deterministic dynamical systems. An arrow f from a dynamical system (X, α) to (Y, β) is a morphism of dynamical systems from (X, α) to (Y^*, β^*) where (Y^*, β^*) is the dynamical system whose state space is given by the set of finite lists Y^* of the set Y and with transition function β^* given by $\beta^*(y_1 y_2 \cdots y_n) = \beta(y_1)\beta(y_2) \cdots \beta(y_n)$ for $y_1 y_2 \cdots y_n \in Y^*$.

8.3. $\mathsf{CmdMnd}^*(K)$

When K admits Eilenberg–Moore constructions for comonads and Kleisli constructions for monads, we have a 2-functor $(-)$-CoalgCmd($(-)$-Kl)$: \mathsf{CmdMnd}^*(K) \to K$, which sends each 0-cell to an Eilenberg–Moore construction for the comonad lifted to the Kleisli construction for the monad.

Spelling out the construction when $K = Cat$, it sends each 0-cell $(C, \mathbb{T}, \mathbb{D}, \lambda)$ to the following category. An object is an arrow $h: x \to TDx$ in C such that $T\varepsilon_x \circ h = \eta_x$ and $T\delta_x \circ h = \mu_{D^2x} \circ T\lambda_{Dx} \circ TDh \circ h$. An arrow from $h: x \to TDx$ to $k: y \to TDy$ is an arrow $f: x \to Ty$ in C such that $\mu_{Dy} \circ Tk \circ f = \mu_{Dy} \circ T\lambda_y \circ TDf \circ h$.

These conditions on both objects and arrows are strict. One can consider application of this construction to the distributive law given in Example 7.5. Each object is a map $h: X \to P(A \times X)$, hence a labelled transition system, but the first equation on objects says that every state $x \in X$ can only have transitions to itself with labels in A.

Remark 8.4. In the above example, the 0-cell constructed by the Eilenberg–Moore construction for a comonad lifted to the Kleisli construction for a monad is restrictive. In [13], by forgetting the counit and comultiplication of a given comonad, Power and Turi considered the category of coalgebras for an endofunctor rather than a comonad on the Kleisli category for the monad, where they used only distributivity for the non-empty powerset monad and the A-copower endofunctor. In order to provide a framework for their example, we need to investigate the 2-category of endo-1-cells in K.

8.4. $\mathsf{MndCmd}^*(K)$

When K admits Kleisli constructions for comonads and Eilenberg–Moore constructions for monads, we have a 2-functor $(-)\text{-Alg}\mathsf{Mnd}((-)\text{-CoKl}): \mathsf{MndCmd}^*(K) \to K$ sending each 0-cell to the Eilenberg–Moore construction for the monad lifted to the Kleisli construction for the comonad.

Spelling out the construction when $K = Cat$, it sends each 0-cell $(C, \mathbb{T}, \mathbb{D}, \lambda)$ to the following category. An object is an arrow $h: DTx \to x$ in C such that $h \circ D\eta_x = \varepsilon_x$ and $h \circ D\mu_x = h \circ DTh \circ D\lambda_{Tx} \circ \delta_{T^2x}$. An arrow from $h: DTx \to x$ to $k: DTy \to y$ is an arrow $f: Dx \to y$ in C such that $f \circ Dh \circ \delta_{Tx} = k \circ DTf \circ D\lambda_x \circ \delta_{Tx}$.

We can also apply this construction to the distributive law in Example 7.5, but we cannot see any concrete meaning to the objects and arrows in that category.

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