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On the areas of cyclic and semicyclic polygons

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Abstract

We investigate the "generalized Heron polynomial" that relates the squared area of an n-gon inscribed in a circle to the squares of its side lengths. For a $(2m+1)$-gon or $(2m+2)$-gon, we express it as the defining polynomial of a certain variety derived from the variety of binary $(2m-1)$-forms having $m-1$ double roots. Thus we obtain explicit formulas for the areas of cyclic heptagons and octagons, and illuminate some mysterious features of Robbins' formulas for the areas of cyclic pentagons and hexagons. We also introduce a companion family of polynomials that relate the squared area of an n-gon inscribed in a circle, one of whose sides is a diameter, to the squared lengths of the other sides. By similar algebraic techniques we obtain explicit formulas for these polynomials for all $n \leqslant 7$.

1. Introduction

Heron of Alexandria (c. 60 BC) is credited with the formula that relates the area K of a triangle to its side lengths a, b, and c:

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = (a+b+c)/2$ is the semiperimeter. For polygons with more than three sides, the side lengths do not in general determine the area, but they do if the polygon is convex

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and *cyclic* (inscribed in a circle). Brahmagupta, in the seventh century, gave the analogous formula for a convex cyclic quadrilateral with side lengths a, b, c, and d:

$$K=\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where $s=(a+b+c+d)/2$. See [2] for an elementary proof.

Robbins [7] found a way to generalize these formulas. First, drop the requirement of convexity and consider the square of the (signed) area K of a possibly self-intersecting oriented cyclic polygon. For this purpose we can define the area enclosed by a closed curve C to be $\oint_C x\,\mathrm{d}y$. Second, express the relation between K^2 and the side lengths as a polynomial equation with integer coefficients. Given a cyclic polygon, one can permute its edges within its circumscribed circle without changing its area, so the polynomial will be symmetric in the side lengths, and in fact it can be written in terms of $16K^2$ and the elementary symmetric functions σ_i in the *squares* of the side lengths. For instance, the Heron and Brahmagupta formulas can be written

$$16K^2-4\sigma_2+\sigma_1^2-\varepsilon\cdot 8\sqrt{\sigma_4}=0$$

in which ε is 0 for a triangle, 1 for a convex quadrilateral, and -1 for a nonconvex quadrilateral. Hence all cyclic quadrilaterals satisfy the polynomial equation $(16K^2-4\sigma_2+\sigma_1^2)^2-64\sigma_4=0$. The general result is as follows.

Theorem 1 [7]. *For each $n\geqslant 3$, there is a unique (up to sign) irreducible polynomial α_n with integer coefficients, homogeneous in $n+1$ variables with the first variable having degree 2 and the rest having degree 1, such that $\alpha_n(16K^2,a_1^2,\dots,a_n^2)=0$ whenever $a_1,\dots,a_n$ are the side lengths of a cyclic n-gon and K is its area.*

The polynomials α_n are now known in the literature as *generalized Heron polynomials*. For certain sets of n side lengths, as shown in [7], one can find up to Δ_n distinct squared areas, where

$$\Delta_n=\frac{n}{2}\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}-2^{n-2}.$$

Hence one expects that α_n has degree Δ_n in its first variable. This conjecture of Robbins, and two others made in [7], have recently been established. We summarize them in Theorem 2.

Theorem 2 [1,3,8]. *The polynomial α_n is monic in $16K^2$ and has total degree $2\Delta_n$. If n is even, then $\alpha_n=\beta_n\beta_n^*$ for some polynomial β_n in the variables $16K^2,\sigma_1,\dots,\sigma_{n-1},\sqrt{\sigma_n}$, where $\sqrt{\sigma_n}=a_1\cdots a_n$ and β_n^* is β_n with $\sqrt{\sigma_n}$ replaced by $-\sqrt{\sigma_n}$.*

See [3] or Section 5 for the degree, [8] or [1] for monicity, and [8] for the factorization when n is even. Robbins' main interest, however, and the motivation for our research, was to find reasonably explicit formulas for all α_n and β_n.

In [7], Robbins found formulas for α_5 and β_6 that have a curious form. To present them concisely, we reformulate the definition [7] of the *crossing parity* ε of a cyclic n-gon. Assume that the n-gon has vertices $v_1, \dots, v_n$ in the complex plane and circumcenter 0. For odd n let $\varepsilon = 0$, and for even $n = 2m + 2$ define

$$\varepsilon = (-1)^m \operatorname{sign}\left[\frac{v_1 - v_2}{v_1} \cdot \frac{v_2 - v_3}{v_2} \cdots \frac{v_n - v_1}{v_n}\right] \in \{-1, +1\},$$

which is well defined because the product in brackets is real and nonzero if all edges have positive length. (Observe that the complex conjugate of $1 - v_{j+1}/v_j$ is $1 - v_j/v_{j+1}$, so conjugation just multiplies the product by $(-1)^n$.) The factor $(-1)^m$ ensures that $\varepsilon = 1$ for a convex cyclic $(2m+2)$-gon. Hence this definition of ε agrees with the previous definition for $n \in \{3, 4\}$.

Now assume $n \in \{5, 6\}$. Define $u_2 = -4K^2$, and make the substitutions

$$\begin{aligned}
t_1 &= \sigma_1, \\
t_2 &= -\sigma_2 + \frac{1}{4}t_1^2 - u_2, \\
t_3 &= \sigma_3 + \frac{1}{2}t_1 t_2 - \varepsilon \cdot 2\sqrt{\sigma_6}, \\
t_4 &= -\sigma_4 + \frac{1}{4}t_2^2 + \varepsilon \cdot t_1\sqrt{\sigma_6}, \\
t_5 &= \sigma_5 + \varepsilon \cdot t_2\sqrt{\sigma_6}.
\end{aligned} \tag{1}$$

Then, for any cyclic pentagon or hexagon of the given crossing parity, the cubic polynomial $u_2 + t_3 z + t_4 z^2 + t_5 z^3$ has a double root, so its discriminant vanishes:

$$t_3^2 t_4^2 - 4u_2 t_4^3 - 4t_3^3 t_5 + 18u_2 t_3 t_4 t_5 - 27u_2^2 t_5^2 = 0. \tag{2}$$

When the t_i are expanded, this discriminant is a polynomial of degree $\Delta_5 = 7$ in u_2, and hence in $16K^2$. Multiplying it by 2^{18} makes it monic in $16K^2$ and yields α_5, β_6, or β_6^* according to whether ε is 0, $+1$, or -1. Equations (1) and (2) are the main formulas of [7], slightly simplified.

In Section 3, we generalize this construction. Fix n and the crossing parity ε, and let $m = \lfloor (n-1)/2 \rfloor$. We introduce auxiliary quantities $u_2, \dots, u_m$, with $u_2 = -4K^2$, and inductively define certain polynomial expressions t_i in the σ_j and u_j with $j \leqslant i$. For $n = 5$ or 6, these definitions reduce to (1). Our Corollary 5 then says that the polynomial

$$P_n(z) = u_2 + \cdots + u_m z^{m-2} + t_{m+1} z^{m-1} + \cdots + t_{2m+1} z^{2m-1}$$

is divisible by the square of a polynomial of degree $m - 1$. In other words, for any values of the t_i and u_j coming from a cyclic n-gon, $P_n(z)$ has $m - 1$ double roots over $\mathbb{C}$ (counting with multiplicity, and including roots at infinity). In the projective space $\mathbb{P}^{2m-1}$ of nonzero polynomials of degree $\leqslant 2m - 1$ considered up to scalar multiples, the polynomials with a squared factor of degree $m-1$ form a variety of codimension $m-1$, defined locally by $m-1$ equations. So, if we regard u_2 through u_m as indeterminates and expand each t_i in terms of the σ_j and u_j, we can in principle eliminate the $m-2$ unwanted quantities $u_3, \ldots, u_m$ and recover a single polynomial, which is α_{2m+1}, β_{2m+2}, or β^*_{2m+2} depending on ε. In Section 4 we carry out this program for $m=3$ to obtain formulas for α_7 and α_8, the generalized Heron polynomials for cyclic heptagons and octagons.

There is another family of area polynomials, not previously considered, that is susceptible to the same analysis. Call an $(n+1)$-gon *semicyclic* if it is inscribed in a circle with one of its sides being a diameter. Its squared area satisfies a polynomial relation with the squares of the lengths of the other n sides; the degree in the squared area turns out to be

$$\Delta'_n = \frac{n}{2}\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} = \Delta_n + 2^{n-2}.$$

Theorems 1 and 2 carry over to this setting as follows.

Theorem 3. *For each $n \geqslant 2$, there exists a unique monic irreducible polynomial α'_n with integer coefficients, homogeneous in $n+1$ variables with the first variable having degree 2 and the rest having degree 1, such that $\alpha'_n(16K^2, a_1^2, \ldots, a_n^2) = 0$ whenever $a_1, \ldots, a_n$ are the lengths of the sides of a semicyclic $(n+1)$-gon excluding a diameter, and K is its area. The total degree of α'_n is $2\Delta'_n$.*

The proof that α'_n exists and is unique (without assuming monicity) follows the proof of Theorem 1 in [7] almost verbatim, and the argument in [1] shows that α'_n is monic. We establish the degree in Section 5 by an elementary argument, which is independent of the rest of this paper.

Cyclic and semicyclic polygons are similar in many ways. For instance, just as the polygon of largest area one can make with n given side lengths is convex and cyclic, the polygon of largest area one can make with n given side lengths and one free side is convex and semicyclic. We will adduce many algebraic similarities in the following sections. For now we just observe that the polynomial α'_3, which can be worked out by hand, also takes the form of a discriminant: if $u_2 = -4K^2$, then

$$\alpha'_3 = 16\,\mathrm{Discr}\big(z^3 + \sigma_1 z^2 + (\sigma_2 + u_2)z + \sigma_3\big).$$

2. The main identity

All our area formulas are based on a generating function identity, Theorem 4. This identity relates the elementary symmetric functions σ_i in the squared side lengths $a_1^2, \ldots, a_n^2$ to certain quantities τ_j that arise in Robbins' proofs of the pentagon and hexagon formulas. It holds for both cyclic and semicyclic polygons and for both odd and even n.

Suppose we have a cyclic n-gon or semicyclic $(n+1)$-gon inscribed in a circle of radius r centered at the origin in the complex plane. Let its vertices be $v_1, \ldots, v_n$ and

$v_{n+1} = \delta v_1$, where $\delta = 1$ for a cyclic n-gon and $\delta = -1$ for a semicyclic $(n+1)$-gon. Following [7], we introduce the vertex quotients $q_i = v_{i+1}/v_i$ for $i = 1, \dots, n$, and let $\tau_0, \tau_1, \dots, \tau_n$ be the elementary symmetric functions in the q_i. Then $\tau_0 = 1$ and $\tau_n = q_1 q_2 \cdots q_n = \delta$. Elementary geometry yields the equations

$$a_i^2 = r^2\left(2 - q_i - q_i^{-1}\right), \quad 1 \leqslant i \leqslant n, \tag{3}$$

$$16K^2 = -r^4\left(q_1 + \cdots + q_n - q_1^{-1} - \cdots - q_n^{-1}\right)^2 = -r^4(\tau_1 - \delta\tau_{n-1})^2. \tag{4}$$

Using (3) one can express each σ_i in terms of r and the τ_i. Let $g(y) = (y-1)^2 + xy/r^2$. Observe that x is one of the values a_i^2 exactly when $g(y)$ has one of the vertex quotients q_i as a root, or in other words, when $g(y)$ has a common root with the polynomial $f(y) = \prod_{i=1}^n (y - q_i) = \sum_{i=0}^n (-1)^i \tau_i y^{n-i}$. Hence the resultant of $f(y)$ and $g(y)$ is a constant times

$$h(x) = \prod_{i=1}^n \left(x - a_i^2\right) = \sum_{i=0}^n (-1)^i \sigma_i x^{n-i}.$$

Examining the coefficient of x^n in the resultant

$$\operatorname{Res}(f, g) = \prod_{i=1}^n g(q_i) = \prod_{i=1}^n \left((q_i - 1)^2 + x q_i / r^2\right)$$

reveals that the constant is δr^{-2n}. Furthermore, each σ_i is a homogeneous quadratic polynomial in the $\tau_0, \dots, \tau_n$, which is apparent if one writes the resultant as the determinant of the Sylvester matrix [4, p. 398]. A particularly simple example is

$$\sigma_n = \delta(-1)^n r^{2n} (\tau_0 - \tau_1 + \tau_2 - \cdots \pm \tau_n)^2.$$

If $n = 2m + 2$ and $\delta = 1$, Robbins showed that $\sqrt{\sigma_n}$ is expressible in terms of r, the τ_i, and the crossing parity ε:

$$\begin{aligned} \sqrt{\sigma_n} &= |v_1 - v_2| \cdots |v_n - v_{n+1}| = r^n |1 - q_1| \cdots |1 - q_n| \\ &= (-1)^m \varepsilon r^n (1 - q_1) \cdots (1 - q_n) \\ &= (-1)^m \varepsilon r^n (\tau_0 - \tau_1 + \tau_2 - \cdots + \tau_n). \end{aligned} \tag{5}$$

Until now we have been following [7] except for the inclusion of semicyclic polygons.

Consider now the involution that reflects the polygon across the real axis. This operation preserves the squared area and the side lengths, but it replaces each q_i with $\overline{q_i} = q_i^{-1}$ and hence replaces each τ_i with $\delta\tau_{n-i}$. Because each σ_i is a quadratic form in the τ_j preserved by the involution, it can be uniquely decomposed into two parts: a quadratic form in symmetric linear combinations of the τ_j, and a quadratic form in antisymmetric linear combinations of the τ_j. When we perform this decomposition on the whole generating function $\sum_i (-x)^i \sigma_i$, each part factors in a surprising way, which our main identity records.

To write the identity explicitly, we need the following linear combinations of the τ_j, for $0 \leqslant k \leqslant n/2$:

$$d_k = \sum_{i=0}^{k} (-1)^i \binom{n-2k+i-1}{i} (\tau_{k-i} - \tau_{n-k+i}), \tag{6}$$

$$e_k = \sum_{i=0}^{k} (-1)^i \left[\binom{n-2k+i}{i} + \binom{n-2k+i-1}{i-1} \right] (\tau_{k-i} + \tau_{n-k+i}). \tag{7}$$

We follow the convention that $\binom{l}{k} = 0$ for every l when $k < 0$. Let $D(x) = \sum d_k x^k$ and $E(x) = \sum e_k x^k$ be the generating functions for d_k and e_k.

Theorem 4 (Main Identity). *For a cyclic n-gon or semicyclic $(n+1)$-gon of radius r, with $\delta = 1$ or -1 respectively, the symmetric functions σ_i of the squared side lengths and τ_i of the vertex quotients are related by*

$$\delta \cdot \sum_{i=0}^{n} (-x)^i \sigma_i = \frac{1}{4} E\big(r^2 x\big)^2 + \left(r^2 x - \frac{1}{4}\right) D\big(r^2 x\big)^2. \tag{$*$}$$

Proof. When the σ_i are expanded in terms of the τ_j, both sides of the main identity become polynomials in $r^2 x$, so we may assume $r = 1$. Let $f(y) = \sum_{i=0}^{n} (-1)^i \tau_i y^{n-i}$ as before, and replace x with x^{-1} in the definition of g to yield $g(y) = (y-1)^2 + x^{-1} y$. The left-hand side of $(*)$ is then

$$\delta \cdot \sum_{i=0}^{n} (-x)^i \sigma_i = \delta x^n h\big(x^{-1}\big) = x^n \operatorname{Res}(f, g). \tag{8}$$

We will calculate the resultant using its $PGL(2)$-invariance and other standard properties [4]. Make the change of variable $y = (z-1)/(z+1)$ so that the roots of g are related by $z \mapsto -z$ instead of $y \mapsto y^{-1}$. We obtain the polynomials

$$f^*(z) = (z+1)^n f\left(\frac{z-1}{z+1}\right) = \sum_{i=0}^{n} (-1)^i \tau_i (z-1)^{n-i} (z+1)^i,$$

$$g^*(z) = (z+1)^2 g\left(\frac{z-1}{z+1}\right) = x^{-1}\big(z^2 + 4x - 1\big).$$

The transformation $y = (z-1)/(z+1)$ has determinant 2, so [4, p. 399]

$$x^n \operatorname{Res}(f, g) = 2^{-2n} x^n \operatorname{Res}\big(f^*, g^*\big) = 2^{-2n} f^*\big(\sqrt{1-4x}\big) f^*\big(-\sqrt{1-4x}\big).$$

Write $f^*(z) = f_0(z^2) + z f_1(z^2)$, separating even and odd powers of z. Then

$$x^n \operatorname{Res}(f, g) = 2^{-2n}\big[f_0(1-4x)^2 - (1-4x) f_1(1-4x)^2\big], \tag{9}$$

which, combined with (8), explains the form of the main identity.

It remains to evaluate $f_0(1-4x)$ and $f_1(1-4x)$. We consider only f_0, as f_1 is similar but simpler. To do this, it helps to introduce the Fibonacci polynomials

$$F_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} x^i,$$

which count compositions of n by 1's and 2's. They satisfy the recurrence

$$F_n(x) = F_{n-1}(x) + xF_{n-2}(x), \quad n \geqslant 1, \tag{10}$$

with $F_0(x) = 1$ and $F_n(x) = 0$ for $n < 0$, and have generating function

$$F(x;t) = \sum_n F_n(x)t^n = \left(1 - t - xt^2\right)^{-1}. \tag{11}$$

Using these polynomials, the generating functions $D(x)$ and $E(x)$ can be written

$$D(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} d_k x^k = \sum_{i=0}^{\lfloor n/2 \rfloor} (\tau_i - \tau_{n-i}) x^i F_{n-2i-1}(-x),$$

$$E(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} e_k x^k = \sum_{i=0}^{\lfloor n/2 \rfloor} (\tau_i + \tau_{n-i}) x^i \left(F_{n-2i}(-x) - xF_{n-2i-2}(-x)\right).$$

To evaluate $f_0(1-4x)$, we use the definition of $f^*(z)$ to rewrite $f_0(z^2) = (f^*(z) + f^*(-z))/2$ in terms of the sums $\tau_i + \tau_{n-i}$. If we let $\theta_i = 1/2$ if $2i = n$ and $\theta_i = 1$ otherwise, then

$$f_0\left(z^2\right) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^n \theta_i (\tau_i + \tau_{n-i}) \left(1 - z^2\right)^i \sum_{j \geqslant 0} \binom{n-2i}{2j} z^{2j}. \tag{12}$$

Next, we want to substitute $z^2 = 1 - 4x$ and evaluate the sum over j. Using the identity $\sum_m \binom{m}{k} t^m = t^k (1-t)^{-k-1}$ we obtain

$$\sum_{m \geqslant 0} t^m \sum_{j \geqslant 0} \binom{m}{2j} (1-4x)^j = \frac{1-t}{1-2t+4xt^2} = (1-t)F(-x;2t).$$

We extract the coefficient of t^{n-2i} via (11) to rewrite (12) as

$$f_0(1-4x) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^n \theta_i (\tau_i + \tau_{n-i}) (4x)^i 2^{n-2i} \left(F_{n-2i}(-x) - \frac{1}{2} F_{n-2i-1}(-x)\right).$$

The recurrence (10) shows that

$$\theta_i\left(F_{n-2i}(-x)-\frac{1}{2}F_{n-2i-1}(-x)\right)=\frac{1}{2}\big(F_{n-2i}(-x)-xF_{n-2i-2}(-x)\big),$$

and so $f_0(1-4x)=(-1)^n2^{n-1}E(x)$. Likewise $f_1(1-4x)=(-1)^n2^{n-1}D(x)$. These equations, combined with (8) and (9), yield the main identity $(*)$. □

3. Consequences of the main identity

The main identity tells us how to generalize the definition of the quantities t_i and u_j that were so useful in simplifying the pentagon and hexagon formulas. Cyclic n-gons have $e_0=\tau_0+\tau_n=2$ by (7), so the expansion of $E(r^2x)^2$ includes linear terms in the e_k. The substitutions that replace the σ_i with the t_i exploit these linear terms to isolate and eliminate the variables e_k. This process, in effect, moves $E(r^2x)^2/4$ to the left-hand side of the main identity and rewrites the new left-hand side in terms of the new variables t_i and u_j. The algebraic relationship among the t_i and u_j is then expressed by the factorization of the remaining term on the right-hand side.

Corollary 5. *Given a cyclic n-gon of crossing parity ε and radius r, let $m=\lfloor (n-1)/2\rfloor$ and let $u_j=r^{2j}\sum_{i=1}^{j-1}(d_i/4-d_{i-1})d_{j-i}$ for $j\geqslant 1$. Inductively define $t_0=-2$ and*

$$t_j=(-1)^{j+1}\sigma_j+\sum_{1\leqslant i,j-i\leqslant m}\frac{t_it_{j-i}}{4}+\begin{cases}-u_j, & \text{if } j\leqslant m,\\ \varepsilon t_{j-m-1}\sqrt{\sigma_n}, & \text{if } j>m,\end{cases}\tag{13}$$

for $j=1,\dots,2m+1$. Then $t_j=-e_jr^{2j}$ for $0\leqslant j\leqslant m$, and the polynomial

$$P_n(z)=u_2+u_3z+\cdots+u_mz^{m-2}+t_{m+1}z^{m-1}+\cdots+t_{2m+1}z^{2m-1}$$

factors as $(1/4-r^2z)(z^{-1}D(r^2z))^2$.

By (6) and (4), we have $u_2=r^4d_1^2/4=r^4(\tau_1-\tau_{n-1})^2/4=-4K^2$, and $u_1=0$ by definition. Thus the t_j and u_j in Corollary 5 agree with those defined in Section 1; see (1).

Proof. We have $e_0=\tau_0+\tau_n=2$ by the definition (7), so $t_0=-e_0r^0$. Now, for $1\leqslant j\leqslant m$, we prove by induction on j that $t_j=-e_jr^{2j}$. The coefficient of x^j in $E(r^2x)^2/4$ is

$$r^{2j}\sum_{i=0}^{j}\frac{e_ie_{j-i}}{4}=r^{2j}e_j+\sum_{i=1}^{j-1}\frac{t_it_{j-i}}{4}$$

by the induction hypothesis. The coefficient of x^j in $(r^2x-1/4)D(r^2x)^2$ is $-u_j$, by the definition of u_j, so the equation $t_j=-e_jr^{2j}$ follows by comparing coefficients of x^j in the main identity.

For $j > m$ we must consider the coefficient $r^{2m+2}e_{m+1}$ of x^{m+1} in $E(r^2x)$. If $n = 2m+1$, then E has degree m by definition so this coefficient is zero. But if $n = 2m+2$, then the coefficient is

$$\begin{aligned} r^{2m+2}e_{m+1} &= r^n\left(2\tau_{m+1} + \sum_{i=1}^{m+1}(-1)^i 2(\tau_{m+1-i} + \tau_{m+1+i})\right) \\ &= -2\varepsilon\sqrt{\sigma_n} \end{aligned}$$

by (7) and (5). So for $m < j \leqslant 2m+1$, the coefficient of x^j in $E(r^2x)^2/4$ is

$$r^{2j}\sum_{i=j-m}^{m}\frac{e_i e_{j-i}}{4} + r^{2j}\frac{e_{j-m-1}e_{m+1}}{2} = \sum_{i=j-m}^{m}\frac{t_i t_{j-i}}{4} + t_{j-m-1}\varepsilon\sqrt{\sigma_n},$$

and this equation holds whether n is odd or even because $\varepsilon = 0$ when n is odd. Thus, by the main identity, $-t_j$ is the coefficient of x^j in $(r^2x - 1/4)D(r^2x)^2$ for $j = m+1, \ldots, 2m+1$.

We now see that $(r^2x - 1/4)D(r^2x)^2$, which is a polynomial of degree $2m+1$ whose two lowest terms vanish, is exactly $-x^2P_n(x)$. $\square$

There is a geometric argument that Corollary 5 contains enough information to recover the generalized Heron polynomial α_n. To simplify the explanation, assume $n = 2m+1$ and $m \geqslant 2$. The nonzero polynomials that factor like $P_n(z)$, namely

$$\left\{p(z) = (az+b)q(z)^2 \in \mathbb{C}[z] \;\middle|\; \deg(q) \leqslant m-1,\ p(z) \not\equiv 0\right\},$$

naturally form a projective variety of codimension $m-1$ in $\mathbb{P}^{2m-1}$, the homogeneous coordinates being the coefficients of $p(z)$. This variety is irreducible because it is the image of $\mathbb{P}^1 \times \mathbb{P}^{m-1}$ under a regular map. Hence the affine variety $X_m \subset \mathbb{A}^{2m}$ of such polynomials (now including the zero polynomial), which has the same ideal, is also irreducible.

By the inductive definition (13), each t_j for $1 \leqslant j \leqslant 2m+1$ is a polynomial function of $\sigma_1, \ldots, \sigma_j$ and $u_2, \ldots, u_{\min(j,m)}$. These functions give rise to a morphism $f : \mathbb{A}^{3m} \to \mathbb{A}^{2m}$, namely

$$f : (\sigma_1, \ldots, \sigma_{2m+1}; u_2, \ldots, u_m) \mapsto (t_{m+1}, \ldots, t_{2m+1}; u_2, \ldots, u_m),$$

which we claim is a trivial bundle with fiber $\mathbb{A}^m$. Consider a point $x = (t_{m+1}, \ldots, t_{2m+1}; u_2, \ldots, u_m)$ in the range, and choose arbitrary values for $\sigma_1, \ldots, \sigma_m$. Let $j > m$. The definition of t_j involves σ_j in exactly one term, whose coefficient is $(-1)^{j+1}$, so we can turn it around to express σ_j as a polynomial function of $t_{j-m}, \ldots, t_j$, and thence by further use of (13) as a polynomial in $(\sigma_1, \ldots, \sigma_m; u_2, \ldots, u_m; t_{m+1}, \ldots, t_j)$. Thus there is a unique point in $f^{-1}(x)$ having the chosen values of $(\sigma_1, \ldots, \sigma_m)$. In other words, each fiber of f is naturally parameterized by $(\sigma_1, \ldots, \sigma_m)$.

Putting these facts together, we find that $f^{-1}(X_m) \approx X_m \times \mathbb{A}^m$ is irreducible and of codimension $m-1$. When we apply the projection $\pi : \mathbb{A}^{3m} \to \mathbb{A}^{2m+2}$ that eliminates the

$m-2$ variables $u_3, \dots, u_m$, the closure of the image $\pi(f^{-1}(X_m))$ is an irreducible variety of codimension at least 1 that contains $V(\alpha_n)$, so it must equal $V(\alpha_n)$. The polynomial α_n is determined (up to a normalizing constant) as the defining polynomial of this variety.

Corollary 5 therefore reduces the problem of finding α_n to two subproblems: finding the defining equations of the variety X_m, and then, after expanding $t_{m+1}, \dots, t_{2m+1}$ in terms of the σ_i and u_j, eliminating the $m-2$ variables $u_3, \dots, u_m$.

The application of the main identity to semicyclic polygons is similar but slightly different. In this case $e_0 = \tau_0 + \tau_n = 0$ and $d_0 = \tau_0 - \tau_n = 2$, so the main identity involves linear terms in the d_k but not the e_k. Our definitions of t_i and u_j are therefore designed to extract and eliminate the variables d_k. Again we can distill the relationship among the t_i and u_j to the factorization of a polynomial $P'_n(z)$. This time, due to the factor $(r^2x - 1/4)$ in the main identity, the expression for t_i explicitly includes r^2, so there remains one more unwanted variable to eliminate for a given n.

Corollary 6. *Given a semicyclic $(n+1)$-gon of radius r, let $m = \lfloor (n-1)/2 \rfloor$ and let $u_j = r^{2j} \sum_{i=1}^{j-1} e_i e_{j-i}/4$ for $1 \leqslant j \leqslant m$. Inductively define $t_0 = -2$ and*

$$t_j = (-1)^{j+1}\sigma_j + \sum_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j-i \leqslant m}} \frac{t_i t_{j-i}}{4} - r^2 \sum_{\substack{0 \leqslant i-1 \leqslant m \\ 0 \leqslant j-i \leqslant m}} t_{i-1} t_{j-i} + \begin{cases} -u_j, & \text{if } j \leqslant m, \\ 0, & \text{if } j > m, \end{cases} \tag{14}$$

for $j = 1, \dots, n$. Then $t_j = -d_j r^{2j}$ for $0 \leqslant j \leqslant m$, and the polynomial

$$P'_n(z) = u_2 + u_3 z + \cdots + u_m z^{m-2} + t_{m+1} z^{m-1} + \cdots + t_n z^{n-2}$$

is the square of $E(r^2 z)/2z$. In particular, $t_n = 0$ if n is odd.

Again $u_2 = r^4 e_1^2/4 = r^4(\tau_1 + \tau_{n-1})^2/4 = -4K^2$ by (4), since now $\delta = -1$.

Proof. As in Corollary 5, the claims follow from equating coefficients of x^j in the main identity and inducting on j to evaluate t_j for $0 \leqslant j \leqslant m$. If $n = 2m+1$, the degree of $E(x)$ is just m, so $t_n = 0$. □

The polynomial $P'_n(z)$ contains $m-1$ unwanted variables, namely r^2 and $u_3, \dots, u_m$. If $n = 2m+1$, then $P'_n(z)$ is a polynomial of degree $2m-2$ that is a perfect square, which gives rise to $m-1$ equations in its coefficients, and we have the additional equation $t_n = 0$. If $n = 2m+2$, then $P'_n(z)$ is a square of degree $2m$, which yields m equations. In either case Corollary 6 holds enough information, in principle, to derive the area formula α'_n. As before, one can make this claim precise using some algebraic geometry.

4. Explicit formulas

In this section we apply the results of Section 3 to produce area formulas for cyclic heptagons and octagons, and also semicyclic quadrilaterals, pentagons, hexagons, and heptagons. The formulas are collected in Theorems 8 and 9 below.

Because the degree of the generalized Heron polynomial α_n is exponential in n, and the number of terms could be exponential in n^2, there is some question as to what constitutes an explicit formula. Our formulas have concise descriptions, and if a polygon is given with exact (for instance, rational) side lengths, the polynomial satisfied by its area can be computed exactly using standard operations such as evaluating the determinant of a matrix of univariate polynomials.

We begin by applying Corollary 5 to $n \in \{7, 8\}$. It gives us a binary quintic form

$$x^5 P_n(y/x) = u_2 x^5 + u_3 x^4 y + t_4 x^3 y^2 + t_5 x^2 y^3 + t_6 x y^4 + t_7 y^5$$

whose coefficients are polynomials in $u_2, u_3, \sigma_1, \ldots, \sigma_7$ and perhaps $\sqrt{\sigma_8}$, and which, when its coefficients are evaluated for any cyclic n-gon, has two linear factors over $\mathbb{C}$ of multiplicity two. The condition for a quintic form Q to factor in this way is given by the vanishing of a certain covariant C, which in the notation of transvectants [6] is

$$C = 2Q(H,i)^{(2)} + 25H(Q,i)^{(2)} + 6Qi^2, \quad H = (Q,Q)^{(2)}, \quad i = (Q,Q)^{(4)}.$$

Here $(f,g)^{(d)} = \sum_{i=0}^{d} (-1)^i \binom{d}{i} (\partial^d f/\partial x^i \partial y^{d-i})(\partial^d g/\partial x^{d-i} \partial y^i)$. This fact about quintics is presumably classical, but we have not found a reference.

In any case, C is a form of degree 9 in $\{x, y\}$ whose coefficients are forms of degree 5 in the coefficients of the original quintic, so its coefficients give us ten degree-5 polynomials in $u_2, u_3, t_4, t_5, t_6, t_7$ that must vanish. These same ten polynomials can be obtained as the Gröbner basis, with a graded term ordering, for the ideal of the variety of quintic forms that factor as a linear form times the square of a quadratic. The ten polynomials remain homogeneous when we regard u_j and t_j as having degree j.

To obtain the desired relation between u_2 and the σ_i, we must expand the coefficients of C as polynomials in u_3 and then eliminate u_3. We can do this most explicitly using resultants with respect to u_3. The two simplest coefficients of C are

$$\begin{aligned} F = {} & u_3^2 t_4^3 - 4u_2 t_4^4 - 4u_3^3 t_4 t_5 + 18u_2 u_3 t_4^2 t_5 - 27u_2^2 t_4 t_5^2 \\ & + \left(8u_3^4 - 42u_2 u_3^2 t_4 + 36u_2^2 t_4^2 + 54u_2^2 u_3 t_5 - 80u_2^3 t_6\right) t_6 \\ & + \left(8u_2 u_3^3 - 30u_2^2 u_3 t_4 + 50u_2^3 t_5\right) t_7, \end{aligned} \tag{15}$$

of total degree 18, and

$$\begin{aligned} G = {} & u_3^2 t_4^2 t_5 - 4u_2 t_4^3 t_5 - 4u_3^3 t_5^2 + 18u_2 u_3 t_4 t_5^2 - 27u_2^2 t_5^3 \\ & + \left(2u_3^3 t_4 - 8u_2 u_3 t_4^2 - 6u_2 u_3^2 t_5 + 36u_2^2 t_4 t_5 - 8u_2^2 u_3 t_6\right) t_6 \\ & + \left(16u_3^4 - 74u_2 u_3^2 t_4 + 40u_2^2 t_4^2 + 110u_2^2 u_3 t_5 - 200u_2^3 t_6\right) t_7, \end{aligned} \tag{16}$$

of total degree 19. Let $P \mapsto \widetilde{P}$ denote the operation of expanding the t_i in terms of u_2, u_3, and $\sigma_1, \ldots, \sigma_n$ as specified by Corollary 5. This operation preserves total degree. Both $\widetilde{F}$ and $\widetilde{G}$ have degree 6 in u_3. Their resultant with respect to u_3 therefore has total degree $6 \times 19 = 114$, and it must have the polynomial α_7 of total degree $2\Delta_7 = 76$ as a factor.

The resultant $\mathrm{Res}(\widetilde{F}, \widetilde{G})$ seems to be too large to compute and factor explicitly, but we can describe its unwanted factors as follows with a little computer assistance. First observe that every term in F and G is divisible by either u_2 or u_3, and hence the same is true of $\widetilde{F}$ and $\widetilde{G}$. It follows that $\mathrm{Res}(\widetilde{F}, \widetilde{G})$ is divisible by u_2. In fact $u_2^7 \mid \mathrm{Res}(\widetilde{F}, \widetilde{G})$, as we will see in Lemma 7 below. Next, consider the polynomials

$$F_1 = 4u_3^3 - 15u_2u_3t_4 + 25u_2^2t_5, \tag{17}$$

$$G_1 = 7u_3^2t_4 - 20u_2t_4^2 - 5u_2u_3t_5 + 100u_2^2t_6, \tag{18}$$

which are closely related to the coefficients of t_7 in F and G. Specifically, $F_1 = (2u_2)^{-1}\partial F/\partial t_7$ and $G_1 = u_2^{-1}(2u_3F_1 - \partial G/\partial t_7)$. We will show that $\mathrm{Res}(\widetilde{F}_1, \widetilde{G}_1)$ divides $\mathrm{Res}(\widetilde{F}, \widetilde{G})$.

First we claim that if $F_1 = G_1 = 0$, then $F = G = 0$. The ideal $\langle F_1, G_1\rangle$ does not contain F and G, but by some easy calculations, it does contain u_2F, u_3F, u_2G, and u_3G. If $F_1 = G_1 = 0$, then all four of these polynomials vanish; so if either $u_2 \neq 0$ or $u_3 \neq 0$, we must have $F = G = 0$, while if $u_2 = u_3 = 0$, we already know that $F = G = 0$. This establishes the claim. It follows that $\widetilde{F}_1 = \widetilde{G}_1 = 0$ implies $\widetilde{F} = \widetilde{G} = 0$. Consequently, wherever $\mathrm{Res}(\widetilde{F}_1, \widetilde{G}_1)$ vanishes, so does $\mathrm{Res}(\widetilde{F}, \widetilde{G})$. Algebraically, this means that every irreducible factor of $\mathrm{Res}(\widetilde{F}_1, \widetilde{G}_1)$ divides $\mathrm{Res}(\widetilde{F}, \widetilde{G})$. The resultant of $\widetilde{F}_1$ and $\widetilde{G}_1$ with respect to u_3 is simple enough to compute explicitly. It has total degree 30, and it factors as u_2^3 times an irreducible polynomial in $\mathbb{Q}[u_2, \sigma_1, \ldots, \sigma_7]$ of total degree 24.

Thus, not only does $\mathrm{Res}(\widetilde{F}_1, \widetilde{G}_1)$ divide $\mathrm{Res}(\widetilde{F}, \widetilde{G})$, but $u_2^4\,\mathrm{Res}(\widetilde{F}_1, \widetilde{G}_1)$ does also. The quotient by the latter polynomial has total degree $114 - 8 - 30 = 76 = 2\Delta_7$, so it must be a scalar multiple of the desired polynomial α_7, β_8, or β_8^*; there are no more unwanted factors. The scalar can be computed by setting $\sigma_2, \ldots, \sigma_7$ to zero (see Theorem 8 for the result). It remains only to prove the following lemma.

Lemma 7. *With the definitions above,* $u_2^7 \mid \mathrm{Res}(\widetilde{F}, \widetilde{G}, u_3)$.

Sketch of proof. By direct calculation on a computer, u_2^7 divides $\mathrm{Res}(F, G, u_3)$ but u_2^8 does not. The only component of $V(F, G)$ lying on the hyperplane $u_2 = 0$ is the linear variety $V(u_2, u_3)$, so $V(F)$ and $V(G)$ must intersect with multiplicity 7 along $V(u_2, u_3) \subset \mathbb{A}^6$. Now, assuming $n = 7$ for definiteness, pull back via the projection $f : \mathbb{A}^9 \to \mathbb{A}^6$ that maps $(\sigma_1, \ldots, \sigma_7, u_2, u_3) \mapsto (t_4, \ldots, t_7, u_2, u_3)$. Because f is smooth, the intersection multiplicity of $V(\widetilde{F})$ and $V(\widetilde{G})$ along $f^{-1}V(u_2, u_3) = V(u_2, u_3) \subset \mathbb{A}^9$ is also 7. For fixed generic values of $\sigma_1, \ldots, \sigma_7$, we therefore have $u_2^7 \mid \mathrm{Res}(\widetilde{F}, \widetilde{G}, u_3)$. We conclude that this divisibility holds globally as well. $\square$

To summarize, the generalized Heron polynomials α_7 and $\alpha_8 = \beta_8\beta_8^*$ can be computed as follows, remembering that σ_k represents the kth elementary symmetric function in the squares $a_1^2, \dots, a_n^2$ of the side lengths, and u_2 represents -4 times the area squared.

Theorem 8 (Heptagon/Octagon Formula). *Given a crossing parity $\varepsilon \in \{-1, 0, +1\}$, define polynomials $F, G, F_1, G_1 \in \mathbb{Q}[u_2, u_3, \sigma_1, \dots, \sigma_7, \sqrt{\sigma_8}]$ by Eqs.* (15)–(18) *and the inductive definition* (13) *with $m = 3$, regarding u_2 and u_3 as indeterminates. Then*

$$\frac{2^{101}5^5 \operatorname{Res}(F, G, u_3)}{u_2^4 \operatorname{Res}(F_1, G_1, u_3)} = \begin{cases} \alpha_7, & \text{if } \varepsilon = 0, \\ \beta_8, & \text{if } \varepsilon = +1, \\ \beta_8^*, & \text{if } \varepsilon = -1. \end{cases} \tag{19}$$

Example. Consider a cyclic octagon with crossing parity $\varepsilon = 1$ and all side lengths equal to 1. We have $\sigma_k = \binom{8}{k}$, and by (13) we compute

$$\begin{aligned} t_4 &= \frac{1}{4}u_2^2 - 4u_3 - 10u_2 - 4, \\ t_5 &= \frac{1}{2}u_2u_3 + 2u_2^2 + 6u_3 + 20u_2 + 16, \\ t_6 &= \frac{1}{4}u_3^2 + 2u_2u_3 + 4u_2^2 - 4u_3 - 17u_2 - 24, \\ t_7 &= -u_3 - 4u_2 + 16. \end{aligned}$$

Now F and G can be evaluated as polynomials in u_2 and u_3. (They have 34 and 38 terms respectively, so we refrain from writing them out.) Using a computer algebra system, one can calculate and factor the resultant of F and G with respect to u_3. The result is

$$\begin{aligned} &-2^{-33}(u_2^2 + 96u_2 + 256)(u_2 + 3)^8(u_2 + 4)^{28} \\ &\quad \times u_2^7\big[19321u_2^{12} + 401584u_2^{11} + \cdots + 2^{23}u_2 + 2^{24}\big]. \end{aligned}$$

The resultant of F_1 and G_1 is $-2^{-8}5^5u_2^3$ times the same degree-12 polynomial in u_2 that appears in brackets above. Therefore Eq. (19) says

$$\beta_8 = \beta_8(-4u_2, 1, 1, \dots, 1) = 2^{76}\big(u_2^2 + 96u_2 + 256\big)(u_2 + 3)^8(u_2 + 4)^{28}.$$

The factors in this formula have the following meanings.

- The root $u_2 = -16(3 + \sqrt{8})$ of $u_2^2 + 96u_2 + 256$ corresponds to a regular octagon of side length 1, whose area is $K = \pm 2(1 + \sqrt{2})$. The other root $u_2 = -16(3 - \sqrt{8})$ corresponds to a regular eight-pointed star with vertices at $r, re^{3\pi i/4}, re^{3\pi i/2}, \dots,$ where $r^2 = 1 - \sqrt{2}/2$.
- The factor $(u_2 + 3)$ represents an equilateral triangle of side length 1 that is traversed twice by the cyclic octagon, except that one edge is traversed three times "forward"

and once "backward". The signed area is $K = \pm\sqrt{3}/2$. There are eight ways to choose the backward edge, so there are eight factors of this type.

- The factor $(u_2 + 4)$ corresponds to a square of area 1 formed by a cyclic octagon with six forward edges and two backward edges. There are $\binom{8}{2} = 28$ ways to choose the two backward edges, so this factor occurs with multiplicity 28.

In each case one can check that the crossing parity ε is 1.

For the rest of this section, we turn our attention to semicyclic $(n + 1)$-gons with $n = 3$, 4, 5, and 6, and switch to the definitions of t_i and u_j given in Corollary 6. To state the area formulas most cleanly we introduce a notion of parity for semicyclic polygons. Let n be even, and observe that the quantities $e_1 = \tau_1 + \tau_{n-1} = \sum q_i - \sum q_i^{-1}$ and $e_{n/2}/2 = \sum(-1)^i \tau_i = \prod(1 - q_i)$ are both pure imaginary. (Compute their complex conjugates using $\overline{q_i} = q_i^{-1}$.) Hence their product is real. Let $\varepsilon \in \{-1, 0, +1\}$ be its sign. Then we have

$$\begin{aligned}
\varepsilon|K|\sqrt{\sigma_n} &= \varepsilon \cdot \frac{1}{4} r^2 |\tau_1 + \tau_{n-1}| \cdot |v_1 - v_2||v_2 - v_3| \cdots |v_n - v_1| \\
&= \varepsilon \cdot \frac{1}{4} r^2 |\tau_1 + \tau_{n-1}| \cdot r^n |1 - q_1| \cdots |1 - q_n| \\
&= r^{n+2} \cdot \frac{1}{4} e_1 \cdot \frac{1}{2} e_{n/2}.
\end{aligned}$$

Define $w = 2\varepsilon|K|\sqrt{\sigma_n} = \varepsilon\sqrt{u_2 t_n}$ for n even, and let $w = 0$ for n odd. Our formulas for α_4' and α_6' factor when written in terms of w rather than σ_n. We do not know whether this type of factorization occurs in general.

Theorem 9. *Let $n \in \{3, 4, 5, 6\}$ and $m = \lfloor (n-1)/2 \rfloor$. Define polynomials $t_1, \ldots, t_{2m+1} \in \mathbb{Q}[u_2, r^2, \sigma_1, \ldots, \sigma_n]$ inductively by Eq.* (14)*. The generalized Heron polynomial α_n' for semicyclic $(n + 1)$-gons is given by*

$$\alpha_3' = 16\,\mathrm{Discr}\big(x^3 - \sigma_1 x^2 + (\sigma_2 + u_2)x - \sigma_3\big), \tag{20}$$

$$\beta_4' = 16\,\mathrm{Discr}\big(x^3 - \sigma_1 x^2 + (\sigma_2 + u_2)x - \big(\sigma_3 - 2\sqrt{u_2 t_4}\big)\big), \tag{21}$$

$$\alpha_5' = \frac{1}{4}\mathrm{Res}\big(t_3^2 - 4u_2 t_4, t_5, r^2\big), \tag{22}$$

$$\beta_6' = \frac{\mathrm{Res}(t_3^2 - 4u_2(t_4 - 2\sqrt{u_2 t_6}), u_2 t_5 - t_3\sqrt{u_2 t_6}, r^2)}{4u_2^6}, \tag{23}$$

together with $\alpha_4' = (\beta_4')(\beta_4')^*$ and $\alpha_6' = (\beta_6')(\beta_6')^*$, where the asterisk denotes negating every occurrence of $\sqrt{u_2 t_n}$.

Proof. For $n \in \{3, 4\}$, it is simplest to use the main identity directly. Defining $e_2 = 0$ if $n = 3$, we have

$$\frac{1}{4}E\left(r^2x\right)^2 = \frac{1}{4}r^4e_1^2x^2 + \frac{1}{2}r^6e_1e_2x^3 + \frac{1}{4}r^8e_2^2x^4$$
$$= u_2x^2 + 2wx^3 + \cdots$$

so, by Theorem 4, the cubic $1 - \sigma_1x + (\sigma_2 + u_2)x^2 - (\sigma_3 - 2w)x^3$ factors as $-(r^2x - 1/4)D(r^2x)^2$. In particular, its discriminant vanishes. Replacing x by $-x^{-1}$, we obtain Eq. (20) for $n = 3$, and for $n = 4$ we have factored α'_4 as the product of two discriminants β'_4 and $(\beta'_4)^*$ corresponding to $\varepsilon = +1$ and $\varepsilon = -1$ respectively.

For larger n we need Corollary 6. For $n = 5$, it says that $u_2 + t_3z + t_4z^2 + t_5z^3$ is the square of the linear polynomial $E(r^2z)/(2z)$, which yields the two equations $t_3^2 - 4u_2t_4 = 0$ and $t_5 = 0$. Their degrees in r^2 are 6 and 5 respectively, so their resultant with respect to r^2 has the correct total degree $2\Delta'_5 = 30$. (Remember that r^2 has degree 1.) It remains only to scale the resultant to be monic in $-4u_2$, and we get Eq. (22).

For $n = 6$, Corollary 6 gives us the factorization

$$u_2 + t_3z + t_4z^2 + t_5z^3 + t_6z^4 = \frac{1}{4}r^4\left(e_1 + e_2r^2z + e_3r^4z^2\right)^2.$$

Using $w = r^8e_1e_3/4$, we derive the equations

$$u_2t_5 - t_3w = 0,$$
$$u_2 + t_3z + (t_4 - 2w)z^2 = \frac{1}{4}r^4\left(e_1 + e_2r^2z\right)^2,$$

the second of which implies that the discriminant $t_3^2 - 4u_2(t_4 - 2w)$ of the left-hand side vanishes. Thus we can form the resultant of $t_3^2 - 4u_2(t_4 - 2w)$ and $u_2t_5 - t_3w$ to eliminate r^2 and obtain a multiple of the desired area formula. The resultant is small enough to compute and factor symbolically. We obtain $\alpha'_6 = (\beta'_6)(\beta'_6)^*$ where β'_6 is given by Eq. (23), and $(\beta'_6)^*$ is β'_6 with the opposite sign on $\sqrt{u_2t_6}$. □

5. Degree calculations

In this section we show by elementary means that the homogeneous polynomials α_n and α'_n have total degrees $2\Delta_n$ and $2\Delta'_n$ respectively, where $\Delta_n = \frac{n}{2}\binom{n-1}{(n-1)/2} - 2^{n-2}$ and $\Delta'_n = \frac{n}{2}\binom{n-1}{(n-1)/2}$.

First we explain why the degrees cannot be smaller. In [7], Robbins shows that $\deg(\alpha_n) \geqslant 2\Delta_n$ by constructing Δ_n cyclic n-gons with generically different squared areas from a given set of edge lengths. He takes the edge lengths to be nearly equal if n is odd, and adds a much shorter edge if n is even. For semicyclic polygons, we can take the edge lengths to be nearly equal if n is even; the argument of [7] then yields the desired number Δ'_n of semicyclic n-gons.

Suppose now that n is odd. It is not necessary (and in fact not possible) to construct Δ'_n inequivalent semicyclic n-gons with given positive real edge lengths a_j. It suffices instead to construct Δ'_n configurations $(r, q_1, \ldots, q_n)$ of complex numbers satisfying

$$a_j^2 = r^2\left(2 - q_j - q_j^{-1}\right), \quad j = 1, \ldots, n,$$
$$q_1 \cdots q_n = -1, \tag{24}$$

since it is from these equations, together with the relation $16K^2 = -r^4(\sum q_j - \sum q_j^{-1})^2$, that one derives the existence and uniqueness of the irreducible polynomial α'_n. In our configurations r is always real and positive, but sometimes $r < \min\{a_j/2\}$, in which case the q_j are negative real numbers instead of complex numbers of norm 1. The plan is to regard each q_j as a function of r by choosing a branch of Eq. (24), and then find values of r such that $q_1 \cdots q_n = -1$.

Let $n = 2m + 1$, let the first $2m$ edge lengths be large and nearly equal, and let $a_n = 2$. To find configurations with $r > \max\{a_j/2\}$, choose arbitrarily whether $0 < \arg q_n < \pi$ (the short edge goes "forward") or $-\pi < \arg q_n < 0$ ("backward"), and likewise choose a set of $k < m$ of the long edges to go backward. Then there exist $m - k$ semicyclic polygons with the given edge lengths and edge directions whose angle sums $\sum \arg q_j$ are $\pi, 3\pi, \ldots, (2m - 2k - 1)\pi$. (Apply the Intermediate Value Theorem to $\sum \arg q_j$ as r varies from $\max\{a_j/2\}$ to ∞.) The total number of such configurations is

$$\sum_{k=0}^{m-1} 2\binom{2m}{k}(m-k) = m\binom{2m}{m}.$$

To find configurations with $r < \min\{a_j/2\} = 1$, choose the branch $q_j < -1$ for exactly m of the long edges, and choose the branch $q_j > -1$ for the other m long edges. Let $\varepsilon_j = +1$ or $\varepsilon_j = -1$ respectively. As $r \to 0$, the product $q_1 \cdots q_{2m}$ approaches the constant $\prod_{j=1}^{2m} a_j^{2\varepsilon_j}$, and hence $q_1 \cdots q_n$ approaches 0 if $q_n > -1$ or $-\infty$ if $q_n < -1$. By choosing the branch for q_n according to whether $q_1 \cdots q_{2m}$, evaluated at $r = 1$, is greater or less than 1, we guarantee that $q_1 \cdots q_n = -1$ for some intermediate value of r. Thus we obtain another $\frac{1}{2}\binom{2m}{m}$ configurations. (The factor of $1/2$ is present because inverting every q_j preserves the radius and the squared area; it corresponds to reversing the orientation.) The total number of configurations is therefore at least

$$\left(m + \frac{1}{2}\right)\binom{2m}{m} = \frac{n}{2}\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} = \Delta'_n.$$

To establish matching upper bounds on the degrees of α_n and α'_n, we proceed indirectly. First we revive an argument of Möbius from the 19th century [5], which produces a polynomial of degree Δ_n that relates r^2 for a cyclic polygon to the squared side lengths. (Another version of this argument appears in [3].) Hence there are generically at most Δ_n circumradii for a given set of edge lengths. For generic side lengths (in particular, no two equal) and a radius r that admits a solution $(q_1, \ldots, q_n)$ to the system of equations (24), the solution is unique up to inverting all the q_j. (Any other solution would differ by inverting a proper subset of the q_j, so those q_j would need to have product ± 1.) Thus, because r and the q_j determine the area, there are generically at most $2\Delta_n$ possible signed areas, so $\deg(\alpha_n) \leqslant 2\Delta_n$. The same argument applied to semicyclic polygons will yield $\deg(\alpha'_n) \leqslant 2\Delta'_n$.

Given a cyclic n-gon with circumradius r and side lengths $a_j = 2y_j$ for $1 \leqslant j \leqslant n$, let $\theta_j = \sin^{-1}(y_j/r)$ be half the angle subtended by the jth side. Let $\varepsilon_2, \dots, \varepsilon_n \in \{-1, +1\}$ be chosen according to whether the jth side goes "backward" or "forward" relative to the first side. Then the sum $\theta_1 + \varepsilon_2\theta_2 + \cdots + \varepsilon_n\theta_n$ is a multiple of π. Therefore

$$\prod_{\varepsilon_j = \pm 1} r^n \sin(\theta_1 + \varepsilon_2\theta_2 + \cdots + \varepsilon_n\theta_n) = 0. \tag{25}$$

The factors of r make this a polynomial relation over $\mathbb{Q}$ between r^2 and the squared side lengths. To see why, recall that $y_j = r\sin\theta_j$ and let $x_j = r\cos\theta_j = (r^2 - y_j^2)^{1/2}$. Expand (25) using $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$ to get

$$\prod_{\varepsilon_2,\dots,\varepsilon_n} \frac{1}{2i}\left[\prod_{j=1}^{n}(x_j + i\varepsilon_j y_j) - \prod_{j=1}^{n}(x_j - i\varepsilon_j y_j)\right] = 0 \tag{26}$$

where $\varepsilon_1 = 1$. The left-hand side of (26) has a great deal of symmetry. Obviously, flipping the sign of y_j is equivalent to negating ε_j. Flipping the sign of any x_j is equivalent to flipping ε_j and negating each product over j. If $j = 1$, we can restore the condition $\varepsilon_1 = 1$ by flipping every ε_j and negating every bracket. All these operations just permute and possibly negate all the 2^{n-1} bracketed factors, so they leave the overall expression unchanged. Therefore, no odd powers of x_j occur in the expansion of (26), and hence each x_j^2 can be replaced by $r^2 - y_j^2$. Likewise each y_j occurs only to even powers. Thus we obtain a polynomial equation $M(r^2, y_1^2, \dots, y_n^2) = 0$.

The remaining part of Möbius' argument uses series expansion to find the degrees of the leading and trailing terms of M. Fix the ε_j, and rewrite the bracketed factor of (26) as

$$\prod_{j=1}^{n}\Bigl(\sqrt{r^2 - y_j^2} + i\varepsilon_j y_j\Bigr) - \prod_{j=1}^{n}\Bigl(\sqrt{r^2 - y_j^2} - i\varepsilon_j y_j\Bigr).$$

To find the term of highest degree in r, expand around $r = \infty$; the highest terms cancel, so the degree is $n - 1$. To find the term of lowest degree, expand around $r = 0$ to get

$$\prod_{j=1}^{n} iy_j\left(1 + \varepsilon_j - \frac{r^2}{2y_j^2} - \cdots\right) - \prod_{j=1}^{n} iy_j\left(1 - \varepsilon_j - \frac{r^2}{2y_j^2} - \cdots\right).$$

Its initial term has degree $\min(k, n-k)$ in r^2, where k is the number of ε_j equal to -1. Therefore M is a power of r^2 times a polynomial in r^2 of degree

$$2^{n-1}\frac{n-1}{2} - \sum_{k=0}^{n-1}\binom{n-1}{k}\min(k, n-k),$$

which simplifies to Δ_n. We can factor out the unwanted power of r^2 because it was not needed to make Eq. (25) hold.

For semicyclic polygons, the signed sum of the θ_j is an odd multiple of $\pi/2$. Equation (25) therefore becomes

$$\prod_{\varepsilon_j=\pm 1} r^n \cos(\theta_1 + \varepsilon_2\theta_2 + \cdots + \varepsilon_n\theta_n) = 0,$$

which expands to a polynomial relation $M'(r^2, y_1^2, \ldots, y_n^2) = 0$. Using series expansion again, one finds that M' is monic of degree $n2^{n-2}$ in r^2, and its lowest nonzero term has the same degree as that of M. Hence M' is a power of r^2 times a polynomial whose degree in r^2 is $\Delta_n + 2^{n-2} = \Delta'_n$.

6. Specializations

Corollary 5 characterizes the generalized Heron polynomial α_n as the relation among $u_2 = -4K^2$ and $\sigma_1, \ldots, \sigma_n$ which says that the polynomial $P_n(z)$ has $m-1$ double roots for some values of $u_3, \ldots, u_m$. Likewise Corollary 6 characterizes α'_n in terms of properties of $P'_n(z)$. These characterizations allow us to understand and factor certain specializations of α_n and α'_n. With a little extra work one can describe some of the factors explicitly. In this section we offer two such results concerning cyclic n-gons with n odd.

Let $n = 2m + 1 \geqslant 5$, and consider the constant term of α_n regarded as a polynomial in $16K^2$; that is, let $u_2 = 0$. Then $P_n(z)$ has $m-1$ double roots if and only if either $(P_n|_{u_2=0})/z$ has $m-1$ double roots, or $u_3 = 0$ and $(P_n|_{u_2=u_3=0})/z^2$ has $m-2$ double roots. Geometrically, the projective variety

$$X = \big\{[u_2 : \cdots : u_m : t_{m+1} : \cdots : t_{2m+1}] \\ \big| \; P_n(z) \text{ factors as } (b_0 + b_1 z)\big(c_0 + c_1 z + \cdots + c_{m-1}z^{m-1}\big)^2\big\}$$

intersects the hyperplane $\{u_2 = 0\}$ in two irreducible components, one corresponding to $b_0 = 0$ and one corresponding to $c_0 = 0$. The second component has intersection multiplicity two because X is tangent to $\{u_2 = 0\}$ along it. Chasing through the geometric interpretation of α_n (after Corollary 5), we regard $\alpha_n|_{u_2=0}$ as a polynomial in $\sigma_1, \ldots, \sigma_n$ and find that it is an irreducible polynomial times the square of another irreducible. The factors are not necessarily irreducible as polynomials in the side lengths a_i, however.

Proposition 10. *If n is odd, the constant term of α_n factors as*

$$\alpha_n|_{16K^2=0} = \gamma_n^2 \prod (a_1 \pm a_2 \pm \cdots \pm a_n)$$

where the product is over all 2^{n-1} sign patterns.

Proof. Heron's formula takes care of the case $n = 3$, so we may assume $n \geqslant 5$ and apply the analysis above. By Corollary 5, cyclic n-gons satisfy

$$P_n(z) = \left(\frac{1}{4} - r^2 z\right)\left(D\left(r^2 z\right)/z\right)^2,$$

so the factor γ_n^2 corresponds to $d_1 = 0$, and the other factor corresponds to $[1/4 : -r^2] = [0 : 1]$ and represents projective solutions at $r^2 = \infty$. The presence of the linear factors $a_1 \pm a_2 \pm \cdots \pm a_n$ in the constant term was proved in [8], and they correspond to solutions with $r^2 = \infty$: As a signed sum of edge lengths approaches zero, the polygon can degenerate to a chain of collinear line segments, which has zero area and infinite circumradius. (One can easily construct a curve of solutions to Eqs. (3) tending to any such point at infinity.) For $n \geqslant 3$, the product of these 2^{n-1} linear factors is symmetric in the a_i^2, so by irreducibility, no other factors can appear. □

The same kind of analysis applies to α_n when the side length a_n goes to zero, and so $t_n = \sigma_n = 0$. It's geometrically clear that the result should be divisible by α_{n-1}^2 (as the nþ side shrinks to zero, it can go either "forward" or "backward"), and the algebra confirms it. We evaluate t_{n-1} at $\sigma_n = 0$ using the definitions from Corollary 5. Then the intersection of X with the hyperplane $t_n = 0$ includes a component of multiplicity two where

$$t_{n-1} = -\sigma_{2m} + \frac{1}{4} t_m^2 = 0$$

and $P_n(z)$, considered as degree $2m - 3$, has $m - 2$ double roots. Substituting the solutions $t_m = \pm 2\sqrt{\sigma_{2m}}$ back into the definitions of t_{m+1} through t_{2m-1}, we recover the definitions of the t_j for $n = 2m$ and $\varepsilon = \pm 1$ and observe that u_m becomes t_m. Thus $P_n(z)$ specializes to $P_{n-1}(z)$, and so $\alpha_n|_{a_n=0}$ is divisible by $(\beta_{n-1})^2(\beta_{n-1}^*)^2 = \alpha_{n-1}^2$.

The other component of $X \cap \{t_n = 0\}$ corresponds to solutions in which the leading coefficient r^2 of the linear factor $(1/4 - r^2 z)$ vanishes. We can describe these solutions explicitly.

Proposition 11. *If $n \geqslant 5$ and n is odd, then*

$$\alpha_n|_{a_n=0} = \alpha_{n-1}^2 \prod \left(16K^2 + \left(a_1^2 \pm a_2^2 \pm \cdots \pm a_{n-1}^2\right)^2\right),$$

the product taken over all sign patterns with $(n-1)/2$ minus signs.

Proof. It suffices to show that all the factors in the product are present; the result then follows by comparing degrees (both sides being monic in $16K^2$).

Fix generic positive real values for $a_1, \ldots, a_{n-1}$ and signs $\varepsilon_1, \ldots, \varepsilon_{n-1} \in \{-1, +1\}$ with $\varepsilon_1 = +1$ and $\sum \varepsilon_j = 0$. Recall from (3) that cyclic polygons satisfy

$$q_i^2 + \left(a_i^2/r^2 - 2\right)q_i + 1 = 0 \quad (1 \leqslant i \leqslant n)$$

and $q_1 \cdots q_n = 1$, and any solution to these $n+1$ equations also satisfies α_n with the squared area given by Eq. (4). For sufficiently small positive values of a_n, the method of Section 5 shows that there exist solutions $(r^2, q_1, \dots, q_n)$ with $r^2 < 0$ and $q_j \approx (a_j^2/r^2)^{\varepsilon_j}$ for $1 \leqslant j < n$; furthermore a_n^2/r^2 tends to a constant. Equation (4) now implies

$$\lim_{a_n \to 0} 16K^2 = -\left(\sum_{\varepsilon_j = +1} a_j^2 - \sum_{\varepsilon_j = -1} a_j^2 \right)^2,$$

so the set of solutions with $a_n^2 = r^2 = 0$ includes all points on the hypersurface $16K^2 + (\sum_{j=1}^{n-1} \varepsilon_j a_j^2)^2 = 0$. □

7. Conclusions

The generalized Heron polynomials α_n for cyclic n-gons and α'_n for semicyclic $(n+1)$-gons are defined implicitly by $n+2$ equations in $n+1$ unknowns: n equations, namely (3), relating the side lengths to the vertex quotients $q_1, \dots, q_n$ and the radius r; Eq. (4) which expresses the squared area K^2, or equivalently $u_2 = -4K^2$, in the same way; and the equation $q_1 q_2 \cdots q_n = \delta$. Our analysis eliminates the variables q_j (and in the cyclic case, r also) at the cost of introducing $\lfloor (n-5)/2 \rfloor$ unwanted quantities $u_3, \dots, u_m$. The reduction in the number of auxiliary variables allows us, for small n, to eliminate them by ad hoc means and obtain formulas for α_n and α'_n.

The quantities $u_2, \dots, u_m$ appear on equal footing in this analysis, so we could equally well eliminate all but u_k for some $k > 2$ and obtain a polynomial relation, presumably of total degree $k\Delta_n$ or $k\Delta'_n$, between u_k and the squares of the side lengths. Unfortunately, we do not yet have a geometric interpretation for u_3 or the higher u_k.

For large n the goal of eliminating $u_3, \dots, u_m$ seems rather distant, but Corollaries 5 and 6 still illuminate aspects of the polynomials α_n and α'_n. In particular, Corollary 5 establishes a close relationship between α_{2m+1} and α_{2m+2}, generalizing those between Heron's and Brahmagupta's formulas and between Robbins' pentagon and hexagon formulas.

It may be of some interest to know how our main results were obtained. Robbins solved many combinatorial and algebraic problems in his lifetime by what he called the "Euler method": calculate examples, using a computer if convenient; discover a general pattern; and prove it, with hints from further calculations if necessary. His work on generalized Heron polynomials took this approach but was somewhat frustrated by lack of data from which to generalize. As explained in [7], Robbins first found α_5 and the closely related polynomial α_6 by interpolating from several dozen numerical examples, and he rewrote them concisely in terms of variables t_i (slightly different from ours) by interacting with a computer algebra system. Neither step is particularly feasible for α_7, whose expansion in terms of the symmetric functions σ_k has almost a million coefficients. It was possible, however, to evaluate certain specializations of α_7 and α_8 by interpolation, and to conjecture Propositions 10 and 11. For instance, we discovered that the constant term of α_7 (the specialization $u_2 = 0$) is divisible by the square of the discriminant of

$$t_4 + t_5 z + t_6 z^2 + t_7 z^3,$$

where the t_i are defined as in Corollary 5 but with $u_2 = u_3 = 0$. Unfortunately, the hidden presence of u_3 made it difficult to guess the rest of α_7.

We introduced semicyclic polygons and their area polynomials α'_n in an effort to obtain more data to study. In particular, we noticed that the mysterious cubic discriminant so prominent in α_5 appeared already in the simpler polynomial α'_3, and we hoped that whatever new phenomenon arose in α_7 would also appear in α'_5, which we could compute by interpolation. In fact α_7 turns out to be rather different from α'_5, but it was the struggle to simplify α'_5 that led us to manipulate the relations between the σ_i and τ_j by hand and thence to discover the main identity (Theorem 4). In the end, the most crucial calculations turned out to be those we did on the blackboard.

Postscript

David Robbins (1942–2003) had an exceptional ability to see and communicate the simple essence of complicated mathematical issues, and to discover elegant new results about seemingly well-understood problems. He taught and inspired a long sequence of younger mathematicians including the two surviving authors. His interest in cyclic polygons began at age 13 when he derived a version of Heron's formula. In the early 1990s he discovered the area formulas for cyclic pentagons and hexagons. When diagnosed with a terminal illness in the spring of 2003, he chose to work on this topic once again. Sadly, he did not live to see the discovery of the main identity or the heptagon formula. We dedicate this paper to the memory of our friend and colleague, whose loss is keenly felt.

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