Lie-Admissible Algebras and Kac–Moody Algebras

Kyeonghoon Jeong, Seok-Jin Kang, and Hyeonmi Lee

Department of Mathematics, Seoul National University, Seoul, 151-742, Korea

Communicated by Georgia Benkart

Received December 9, 1996

In this paper, we determine all third power-associative Lie-admissible algebras whose commutator algebras are Kac–Moody algebras.

INTRODUCTION

In 1948, A. A. Albert introduced a new family of (non-associative) algebras whose commutator algebras are Lie algebras [A]. These algebras are called Lie-admissible algebras, and they arise naturally in various areas of mathematics and mathematical physics such as differential geometry of affine connections on Lie groups. In the theory of Lie-admissible algebras, the additional conditions such as flexibility or power-associativity play an important role, and in [A], Albert proposed the problem of classifying all power-associative flexible Lie-admissible algebras whose commutator algebras are semisimple Lie algebras. Since then, one of the main themes of the research activities on Lie-admissible algebras has been the problem of classifying all Lie-admissible structures on a given Lie algebra structure.

For finite dimensional algebras over an algebraically closed field of characteristic 0, Albert's problem was solved by P. J. Laufer and M. L. Tomber [LT], and such algebras turn out to be just Lie algebras. In [M1, M3], H. C. Myung extended their result to finite dimensional algebras over algebraically closed fields with positive characteristic when the commutator algebras are classical Lie algebras or generalized Witt algebras. In [BO, MO], the assumption of power-associativity was removed for these algebras.

*This research was supported in part the Basic Science Research Institute Program, Ministry of Education, BSR1-96-1414, and by the University Development Research Fund at Seoul National University, Korea.
bras and all flexible third power-associative Lie-admissible algebras were classified when the commutator algebras are simple. Furthermore, in [B3], without assuming flexibility, G. Benkart classified all third power-associative Lie-admissible algebras whose commutator algebras are semisimple. Recently, H. C. Myung classified all third power-associative Lie-admissible algebras associated with the Virasoro algebra [M5].

In this paper, we consider a generalization of Albert’s problem and determine all third power-associative Lie-admissible algebras whose commutator algebras are Kac–Moody algebras. In particular, we prove Myung’s conjecture for the third power-associative Lie-admissible structures on affine Kac–Moody algebras [M5].

This work was greatly inspired by [B3, M5], and our approach is based on the techniques (and their generalizations) developed in [B3]. We express our sincere gratitude to Professors G. Benkart and H. C. Myung for their interest in this work and sending us the reprints and preprints of their work.

1. LIE-ADMISSIBLE ALGEBRAS

We begin with some of the basic facts about Lie-admissible algebras. Let $(L, *)$ be a (non-associative) algebra over the complex field $\mathbb{C}$ with multiplication $*$, and let $(L^-, [], )$ be its commutator algebra, where the underlying space is the same as $L$ and the bracket operation $[ , ] : L \times L \rightarrow L$ is defined by

\[ [x, y] = x * y - y * x \quad \text{for all } x, y \in L. \quad (1.1) \]

The algebra $(L, *)$ is called Lie-admissible if its commutator algebra $(L^-, [], )$ is a Lie algebra. Equivalently, the algebra $(L, *)$ is Lie-admissible if and only if the bracket operation on $L$ satisfies the Jacobi identity

\[ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{for all } x, y, z \in L. \quad (1.2) \]

We say that the algebra $(L, *)$ is flexible if

\[ (x * y) * x = x * (y * x) \quad \text{for all } x, y \in L, \quad (1.3) \]

and that it is power-associative if every element of $(L, *)$ generates an associative subalgebra. Moreover, the algebra $(L, *)$ is said to be third power-associative if

\[ x * (x * x) = (x * x) * x \quad \text{for all } x \in L, \quad (1.4) \]
and fourth power-associative if
\[ x \ast (x \ast (x \ast x)) = (x \ast x) \ast (x \ast x) \quad \text{for all } x \in L. \quad (1.5) \]

Note that if the algebra \((L, \ast)\) is flexible, then it is third power-associative. Moreover, the algebra \((L, \ast)\) is power-associative if and only if it is both third power-associative and fourth power-associative.

In this paper, we would like to determine all third power-associative Lie-admissible algebras whose commutator algebras are Kac–Moody algebras. In general, for a Lie-admissible algebra \((L, \ast)\), the multiplication \(\ast\) can be written as
\[ x \ast y = \frac{1}{2}[x, y] + x \circ y, \quad (1.6) \]
where \(\circ\) is a commutative bilinear multiplication on \(L\) given by
\[ x \circ y = \frac{1}{2}(x \ast y + y \ast x). \quad (1.7) \]
Conversely, for any commutative bilinear multiplication \(\circ\) on \(L\), we can define a Lie-admissible multiplication \(\ast\) on \(L\) by (1.6). Hence to find a Lie-admissible multiplication on \(L\), it suffices to find a commutative bilinear multiplication \(\circ\) on \(L\) which would yield a desired Lie-admissible multiplication \(\ast\) on \(L\) given by (1.6).

If a Lie-admissible algebra \((L, \ast)\) satisfies third power-associativity (1.4), then since \(x \circ x = x \ast x\) for all \(x \in L\), we have
\[ [x, x \circ x] = 0 \quad \text{for all } x \in L, \quad (1.8) \]
which can be linearized to the identity
\[ 2[x, x \circ y] + [y, x \circ x] = 0 \quad \text{for all } x, y \in L. \quad (1.9) \]
Again, (1.9) can be linearized to
\[ [x, y \circ z] + [y, z \circ x] + [z, x \circ y] = 0 \quad \text{for all } x, y, z \in L. \quad (1.10) \]
We will use these identities frequently in determining the third power-associative Lie-admissible multiplications on Kac–Moody algebras.

2. KAC–MOODY ALGEBRAS

In this section, we recall the fundamental properties of Kac–Moody algebras (cf. [K]). Let \(\mathcal{I}\) be a finite index set. An integral matrix \(\mathcal{A} = (a_{ij})_{i,j \in \mathcal{I}}\) is called a generalized Cartan matrix if (i) \(a_{ii} = 2\) for all \(i \in \mathcal{I}\), (ii) \(a_{ij} \leq 0\) for \(i \neq j\), (iii) \(a_{ij} = 0\) if and only if \(a_{ji} = 0\). A realization of \(\mathcal{A}\) is a triple \((\mathfrak{h}, \Pi, \Pi^\vee)\), where \(\mathfrak{h}\) is a complex vector space of dimension \(2|\mathcal{I}| -
rank \( A \), and \( \Pi = \{ \alpha_i \mid i \in I \} \) and \( \Pi^\vee = \{ h_i \mid i \in I \} \) are linearly independent subsets of \( \mathfrak{h}^* \) and \( \mathfrak{h} \), respectively, satisfying \( \alpha_i(h_j) = \delta_{ij} \) for \( i, j \in I \).

In this paper, we assume that the generalized Cartan matrix \( A \) is symmetric, i.e., there exists an invertible diagonal matrix \( D \) such that \( DA \) is symmetric.

The Kac–Moody algebra \( g = g(A) \) associated with a generalized Cartan matrix \( A = (a_{ij})_{i,j \in I} \) is the Lie algebra generated by the elements \( e_i, f_i, (i \in I) \) and \( h \in \mathfrak{h} \) with the defining relations

\[
\begin{align*}
[h, h'] &= 0 \quad (h, h' \in \mathfrak{h}), \\
[h, e_i] &= \alpha_i(h)e_i, \quad [h, f_i] = -\alpha_i(h)f_i \quad (i \in I), \\
[e_i, f_j] &= \delta_{ij}h_i \quad (i, j \in I), \\
(\text{ad } e_i)^{1-a_{ij}}(e_j) &= (\text{ad } f_i)^{-a_{ij}}(f_j) = 0 \quad (i \neq j).
\end{align*}
\]

The elements of \( \Pi \) (resp. \( \Pi^\vee \)) are called the simple roots (resp. simple coroots) of \( g \). Let \( Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i, \) \( Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i, \) and \( Q^- = -Q^+ \). We define a partial ordering \( \geq \) on \( \mathfrak{h}^* \) by \( \lambda \geq \mu \) if and only if \( \lambda - \mu \in Q^+ \).

The Kac–Moody algebra \( g = g(A) \) has the root space decomposition

\[ g = \bigoplus_{\alpha \in Q^+} g_\alpha, \quad \text{where } g_\alpha = \{ x \in g \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h} \}. \]

An element \( \alpha \in Q \) is called a root if \( \alpha \neq 0 \) and \( g_\alpha \neq 0 \). A root \( \alpha > 0 \) (resp. \( \alpha < 0 \)) is called positive (resp. negative). All the roots are either positive or negative. We denote by \( \Phi, \Phi^+, \) and \( \Phi^- \), the set of all roots, positive roots, and negative roots, respectively. It is known that the center \( Z(g) \) of \( g \) is given by

\[ Z(g) = \bigcap_{i \in I} \ker \alpha_i = \{ h \in \mathfrak{h} \mid \alpha_i(h) = 0 \text{ for all } i \in I \} \]

\[ = \bigcap_{\alpha \in \Phi} \ker \alpha = \{ h \in \mathfrak{h} \mid \alpha(h) = 0 \text{ for all } \alpha \in \Phi \}. \]

A generalized Cartan matrix \( A = (a_{ij})_{i,j \in I} \) is called indecomposable if for any pair of indices \( i, j \in I \), there exist indices \( i_1, i_2, \ldots, i_s \) such that \( a_{i_1i_2}a_{i_2i_3} \cdots a_{i_si} \neq 0 \). An indecomposable generalized Cartan matrix \( A \) is said to be of finite type if all its principal minors are positive, of affine type if all its proper principal minors are positive and \( \det A = 0 \), and of indefinite type if \( A \) is of neither finite nor affine type. \( A \) is of hyperbolic type if it is of indefinite type and all its proper principal submatrices are of finite or affine type.
3. THIRD POWER-ASSOCIATIVE LIE-ADMISSIBLE
STRUCTURES ON KAC–MOODY ALGEBRAS

Let \( A = (a_{ij})_{i,j \in I} \) be a generalized Cartan matrix and \( g = g(A) \) be the Kac–Moody algebra associated with \( A \). In this section, we will determine all the third power-associative Lie-admissible structures on the Kac–Moody algebra \( g = g(A) \). When \( A \) is of finite type, this problem was settled by G. Benkart [B3]. Hence we will assume that the generalized Cartan matrix is not of finite type.

We start with two technical lemmas.

**Lemma 3.1.** The algebra of linear functionals on \( h \) is an integral domain. In particular, if \( a \in \Phi \) is a root of \( g \) and \( \tau : h \to \mathbb{C} \) is a linear functional on \( h \) satisfying

\[
\tau(h) \alpha(h') + \tau(h') \alpha(h) = 0 \quad \text{for all } h, h' \in h,
\]

then \( \tau = 0 \).

**Proof.** Suppose \( \tau, \lambda : h \to \mathbb{C} \) are linear functionals such that

\[
\lambda(h) \tau(h) = 0 \quad \text{for all } h \in h. \tag{3.1}
\]

Then (3.1) can be linearized to

\[
\lambda(h) \tau(h') + \lambda(h') \tau(h) = 0 \quad \text{for all } h, h' \in h. \tag{3.2}
\]

Assume \( \lambda \neq 0 \). If \( h \in \ker \lambda \), choose \( h' \notin \ker \lambda \), and (3.2) yields \( \tau(h) = 0 \). If \( h \notin \ker \lambda \), (3.1) yields \( \tau(h) = 0 \). Hence \( \tau = 0 \) on \( h \). \( \square \)

**Lemma 3.2.** Let \( A = (a_{ij})_{i,j \in I} \) be a generalized Cartan matrix which is not of finite type. If \( a_{ij} \neq 0 \) for some indices \( i, j \in I \), then there exists a root \( \alpha = \sum_{k \in I} c_k \alpha_k \in \Phi \) of \( g \) such that \( c_i > 0 \), \( c_j > 0 \), and \( c_i \neq c_j \).

**Proof.** If \( A \) is of affine type \( X_{i,j}^{(r)} \), then \( \delta = \sum_{k=0}^{I} a_k \alpha_k \) is a positive root of \( g \), where \( a_k \) are the labels given in [K, Chap. 4]. If \( a_i \neq a_j \), we are done. If \( a_i = a_j \), then by [K, Proposition 6.3], \( \beta = r\delta + \alpha_i \) is a root of \( g \), which is in the desired form.

Suppose \( A \) is of indefinite type. Then by [K, Theorem 5.6], there exists a root \( \alpha = \sum_{k \in I} a_k \alpha_k \) such that \( a_k > 0 \) and \( \alpha(h_k) < 0 \) for all \( k \in I \). If \( a_i \neq a_j \), we are done. If \( a_i = a_j \), since \( \alpha(h_j) < 0 \), \( \alpha + \alpha_i \) is a root of \( g \), which is in the desired form. \( \square \)

**Remark.** Lemma 3.2 holds for all generalized Cartan matrices that are not of type \( A_n \) (cf. [H, K]).
As we have seen in Section 1, the third power-associative Lie-admissible structures on the Kac–Moody algebra \( g = g(A) \) are determined by the commutative bilinear multiplications \( \circ \) on \( g \) satisfying the identities (1.8), (1.9), and (1.10). We will determine such commutative bilinear multiplications \( \circ \) on \( g \) in the following lemmas.

**Lemma 3.3.** (a) The Cartan subalgebra \( \mathfrak{h} \) is closed under \( \circ \).

(b) For each \( \alpha \in \Phi \), there exist a linear functional \( \tau_\alpha: \mathfrak{h} \rightarrow \mathbb{C} \) and a bilinear map \( u_\alpha: \mathfrak{h} \times g_\alpha \rightarrow \mathfrak{h} \) such that

\[
h \circ x = \tau_\alpha(h)x + u_\alpha(h, x) \quad \text{for all } h \in \mathfrak{h}, x \in g_\alpha.
\]

(3.3)

(c) The linear functional \( \tau_\alpha \) satisfies

\[
\tau_\alpha(h)\alpha(h') + \tau_\alpha(h')\alpha(h) = \alpha(h \circ h') \quad \text{for all } h, h' \in \mathfrak{h}.
\]

(3.4)

**Proof.** (a) By bilinearity and commutativity of \( \circ \), it suffices to show that \( h \circ h \in \mathfrak{h} \) for all \( h \in \mathfrak{h} \). Write \( h \circ h = h' + \sum_{\alpha \in \Phi} y_\alpha \), where \( h' \in \mathfrak{h} \) and \( y_\alpha \in g_\alpha \). Since \([h, h \circ h] = 0\), we have \( \sum_{\alpha \in \Phi} \alpha(h)y_\alpha = 0 \), which implies

\[
h \circ h = h' + \sum_{\alpha \in \Phi} y_\alpha.
\]

If \( h_1 \in \mathfrak{h} \), write \( h \circ h_1 = h'' + \sum_{\beta \in \Phi} \beta z_\beta \), where \( h'' \in \mathfrak{h} \), \( z_\beta \in g_\beta \). By (1.9), we have

\[
0 = 2[h, h \circ h_1] + [h_1, h \circ h] = 2 \sum_{\beta \in \Phi} \beta(h)z_\beta + \sum_{\alpha \in \Phi} \alpha(h_1)y_\alpha
\]

\[
= 2 \sum_{\alpha \in \Phi} \alpha(h)z_\alpha + \sum_{\alpha \in \Phi} \alpha(h_1)y_\alpha.
\]

Hence, if \( \alpha(h) = 0 \), then \( \alpha(h_1)y_\alpha = 0 \) for all \( h_1 \in \mathfrak{h} \), which implies \( y_\alpha = 0 \). Therefore, \( h \circ h = h' \in \mathfrak{h} \).

(b), (c). Let \( \alpha \in \Phi \) be a root and \( h \in \mathfrak{h} \), \( x \in g_\alpha \). Write \( h \circ x = h' + \sum_{\beta \in \Phi} \beta y_\beta \), where \( h' \in \mathfrak{h} \), \( y_\beta \in g_\beta \). By (1.9), we have

\[
0 = 2[h, h \circ x] + [x, h \circ h] = 2 \sum_{\beta \in \Phi} \beta(h)y_\beta - \alpha(h \circ h)x
\]

which implies

\[
h \circ x = h' + y_\alpha + \sum_{\beta \neq \alpha} y_\beta.
\]
where $h' \in \mathfrak{h}$, $y_\alpha \in \mathfrak{g}_\alpha$, $y_\beta \in \mathfrak{g}_\beta$. Similarly, if $h_1 \notin \mathfrak{h}$, then

$$h_1 \circ x = h'' + z_\alpha + \sum_{\gamma \neq \alpha, \gamma(h_1) = 0} z_\gamma,$$

where $h'' \in \mathfrak{h}$, $z_\alpha \in \mathfrak{g}_\alpha$, and $z_\gamma \in \mathfrak{g}_\gamma$.

By (1.10), we have

$$[h, h_1 \circ x] + [h_1, x \circ h] + [x, h \circ h_1] = 0,$$

which yields

$$\sum_{\beta \neq \alpha, \beta(h_1) = 0} \beta(h_1) y_\beta + \sum_{\gamma \neq \alpha, \gamma(h_1) = 0} \gamma(h) z_\gamma = 0.$$

If $\beta \neq \alpha$, choose $h_1 \in \mathfrak{h}$ such that $\beta(h_1) \neq 0$. Then $\beta(h_1) y_\beta$ appears in the first sum and does not appear in the second. Hence $y_\beta = 0$ and we have $h \circ x = u_\alpha (h, x) + y_\alpha (h, x)$ where $y_\alpha (h, x) \in \mathfrak{g}_\alpha$ and $u_\alpha (h, x) \in \mathfrak{h}$. So from

$$[h, h' \circ x] + [h', x \circ h] + [x, h \circ h'] = 0,$$

one obtains

$$\alpha(h) y_\alpha (h', x) + \alpha(h') y_\alpha (h, x) = \alpha(h \circ h') x. \quad (3.5)$$

Fixing $h'$ so that $\alpha(h') = 1$ we see that if $\alpha(h) = 0$, then

$$y_\alpha (h, x) = \alpha(h \circ h') x.$$

Also, we see $2y_\alpha (h', x) = \alpha(h' \circ h') x$. Let us define $\tau_\alpha (h) = \alpha(h \circ h')$ for all $h \in \ker \alpha$ and $\tau_\alpha (h') = \frac{i}{2} \alpha(h' \circ h')$. Since $\mathfrak{h} = \ker \alpha \oplus \mathfrak{ch}'$, this defines $\tau_\alpha$. Then $h \circ x = \tau_\alpha (h) x + u_\alpha (h, x)$ since $y_\alpha (h, x) = \tau_\alpha (h) x$ for all $h \in \mathfrak{h}$, $x \in \mathfrak{g}_\alpha$, and (3.5) shows $\alpha(h) \tau_\alpha (h'') + \alpha(h') \tau_\alpha (h) = \alpha(h \circ h'')$ for all $h, h'' \in \mathfrak{h}$.

In fact, the linear functionals $\tau_\alpha$ are the same for all $\alpha \in \Phi$ as we can see in the next lemma.

**Lemma 3.4.** (a) The linear functionals $\tau_\alpha : \mathfrak{h} \rightarrow \mathbb{C}$ are the same for all $\alpha \in \Phi$. We will denote this linear functional by $\tau$. 

There exists a symmetric bilinear map $\sigma: \mathfrak{h} \times \mathfrak{h} \to Z(\mathfrak{g}) \subseteq \mathfrak{h}$ such that
\[ h \cdot h' = \tau(h')h + \tau(h)h' + \sigma(h, h') \quad \text{for all } h, h' \in \mathfrak{h}. \tag{3.6} \]

**Proof.** (a) Let $I = \{1, 2, \ldots, n\}$ be the index set for the simple roots of $\mathfrak{g}$. Since the set of simple roots $(\alpha_1, \ldots, \alpha_n)$ is linearly independent, it can be extended to a basis $(\beta_1, \ldots, \beta_n, \beta_{n+1}, \ldots, \beta_m)$ of $\mathfrak{h}^*$, so that $\beta_i = \alpha_i$ for $i = 1, \ldots, n$. Let $(t_1, \ldots, t_N)$ be its dual basis in $\mathfrak{h}$. Set
\[ \tau_i = \begin{cases} \tau_{\alpha_i} & \text{for } i = 1, \ldots, n, \\ 0 & \text{for } i = n + 1, \ldots, N, \end{cases} \]
and $\rho_{ij} = \tau(t_j)$ for $i, j = 1, \ldots, N$.

If $\gamma = \sum_{k=1}^n b_k \alpha_k \in \Phi$, then by (3.4), we have
\[ \tau_i(h)\gamma(h') + \tau_j(h')\gamma(h) = \gamma(h \cdot h') = \sum_{k=1}^N b_k (\tau_k(h)\alpha_k(h') + \tau_k(h')\alpha_k(h)). \]

(Here, we assume $b_{n+1} = \cdots = b_N = 0$ for convenience.) Take $h = t_r$ and $h' = t_s$ ($r, s = 1, \ldots, N$), and we get
\[ b_r \tau_r(t_r) + b_s \tau_s(t_s) = b_r \rho_{rs} + b_s \rho_{sr} \quad \text{for } r, s = 1, \ldots, N. \tag{3.7} \]

If $a_{ij} \neq 0$ for some $i, j \in I$, then $\gamma = \beta_i + \beta_j = \alpha_i + \alpha_j$ is a root, and setting $r = s = i$ in (3.7) and $\gamma = \alpha_i + \alpha_j$ yields $\tau_i(t_i) = \rho_{ii}$. Similarly, $\tau_i(t_j) = \rho_{ij}$. Also, by setting $r = i, s = j$ in (3.7), we obtain
\[ \rho_{ii} + \rho_{jj} = \rho_{ij} + \rho_{ji}. \tag{3.8} \]

Furthermore, for any index $k = 1, 2, \ldots, N$ with $k \neq i, k \neq j$, by setting $r = k, s = i$ and $r = k, s = j$ in (3.7), we obtain $\rho_{ik} = \rho_{kj}$.

Since the generalized Cartan matrix $A$ is not of finite type, by Lemma 3.2, there exists a root $\beta = \sum_{k=1}^N c_k \alpha_k$ such that $c_i > 0, c_j > 0$, and $c_i \neq c_j$. Then by the same argument as above, we have
\[ c_i \rho_{jj} + c_j \rho_{ii} = c_i \rho_{jj} + c_j \rho_{ij}. \tag{3.9} \]

Combining (3.8) and (3.9) yields $\rho_{ij} = \rho_{ji}$ and $\rho_{ii} = \rho_{jj}$. Consequently, we have $\rho_{ik} = \rho_{kj}$ for all $k = 1, \ldots, N$, which implies $\tau_i = \tau_j$. Since the generalized Cartan matrix $A$ is indecomposable, we have $\tau_{\alpha_1} = \cdots = \tau_{\alpha_n} = \tau$. 

(b) There exists a symmetric bilinear map $\sigma: \mathfrak{h} \times \mathfrak{h} \to Z(\mathfrak{g}) \subseteq \mathfrak{h}$ such that

The proof of this statement is similar to the proof for part (a).
Note that for any root $\alpha = \sum_{k=1}^{n} b_k \alpha_k$, we have
\[
\tau_\alpha(h) \alpha(h') + \tau_\alpha(h') \alpha(h) \\
= \alpha(h \circ h') \\
= \sum_{k=1}^{n} b_k (\tau_k(h) \alpha_k(h') + \tau_k(h') \alpha_k(h)) \\
= \sum_{k=1}^{n} b_k (\tau(h) \alpha_k(h') + \tau(h') \alpha_k(h)) \\
= \tau(h) \alpha(h') + \tau(h') \alpha(h) \\
\text{for all } h, h' \in \mathfrak{h}.
\]
Hence, by Lemma 3.1, $\tau_\alpha = \tau$ for all $\alpha \in \Phi$.

(b) Define a bilinear map $\sigma : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ by
\[
\sigma(h, h') = h \circ h' - \tau(h)h' - \tau(h')h \\
(h, h' \in \mathfrak{h}).
\]
By (3.4), we have $\sigma(h, h') \in \ker \alpha$ for all $\alpha \in \Phi$. Hence $\sigma(h, h') \in \bigcap_{\alpha \in \Phi} \ker \alpha = Z(\mathfrak{g})$ for all $h, h' \in \mathfrak{h}$. \hfill \qed

**Lemma 3.5.** For each root $\alpha \in \Phi$, there exist a linear functional $\lambda_\alpha : \mathfrak{g}_\alpha \to \mathbb{C}$ and a symmetric bilinear map $\sigma_\alpha : \mathfrak{g}_\alpha \times \mathfrak{g}_\alpha \to \mathfrak{h}$ such that
\[
x \circ y = \lambda_\alpha(y)x + \lambda_\alpha(x)y + \sigma_\alpha(x, y) \\
\text{for all } x, y \in \mathfrak{g}_\alpha. \quad (3.10)
\]

**Proof.** If $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$, by (1.10), Lemma 3.3, and Lemma 3.4, we have
\[
[h, x \circ y] = [h \circ x, y] + [h \circ y, x] \\
= [\tau(h)x + u_\alpha(h, x), y] + [\tau(h)y + u_\beta(h, y), x] \\
= \alpha(u_\beta(h, y))x + \beta(u_\alpha(h, x))y. \\
\]
It follows that $x \circ y \in \mathfrak{h} + \mathfrak{g}_\alpha + \mathfrak{g}_\beta$ for all $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, and in particular, $x \circ y \in \mathfrak{h} + \mathfrak{g}_\alpha$ for all $x, y \in \mathfrak{g}_\alpha$. Consider first the case that $x, y \in \mathfrak{g}_\alpha$. Then
\[
x \circ y = z_\alpha + \sigma_\alpha(x, y)
\]
for some symmetric bilinear map $\sigma_\alpha : \mathfrak{g}_\alpha \times \mathfrak{g}_\alpha \to \mathfrak{h}$ and $z_\alpha \in \mathfrak{g}_\alpha$.

Then $[h, x \circ y] = \alpha(h)z_\alpha$ for all $h \in \mathfrak{h}$. Equating that with (3.11) gives
\[
\alpha(h)z_\alpha = \alpha(u_\alpha(h, y))x + \alpha(u_\alpha(h, x))y.
\]
Fixing \( h' \) so that \( \alpha(h') = 1 \), and setting \( \lambda_a(x) = \alpha(u_a(h', x)) \) we have
\[
x \circ y = \lambda_a(y)x + \lambda_a(x)y + \sigma_a(x, y),
\]
and, from putting that into \([h, x \circ y]\), we also get
\[
\alpha(h)\lambda_a(x) = \alpha(u_a(h, x)) \quad \text{for all } x \in \mathfrak{g}_a, \ h \in \mathfrak{h}. \quad (3.12)
\]
This implies that the definition of \( \lambda_a \) does not depend on the choice of \( h' \) such that \( \alpha(h') = 1 \).

**Lemma 3.6.** (a) For each \( \alpha \in \Phi \), there exists a bilinear map \( \chi_{\alpha}: \mathfrak{h} \times \mathfrak{g}_a \rightarrow Z(\mathfrak{g}) \subset \mathfrak{h} \) such that
\[
h \circ x = \tau(h)x + \lambda_{\alpha}(x)h + \chi_{\alpha}(h, x) \quad \text{for all } h \in \mathfrak{h}, \ x \in \mathfrak{g}_a. \quad (3.13)
\]
(b) For any \( \alpha, \beta \in \Phi \), there exists a bilinear map \( \sigma_{\alpha\beta}: \mathfrak{g}_a \times \mathfrak{g}_\beta \rightarrow \mathfrak{h} \) such that
\[
x \circ y = \lambda_{\beta}(y)x + \alpha_{\beta}(x, y) + \sigma_{\alpha\beta}(x, y) \quad \text{for all } x \in \mathfrak{g}_a, \ y \in \mathfrak{g}_\beta.
\]
(3.14)

**Proof.** Let \( \beta \neq \alpha \). Here we know \( x \circ y = z_{\alpha} + z_{\beta} + \alpha_{\beta}(x, y) \) for some \( z_{\alpha} \in \mathfrak{g}_a \), \( z_{\beta} \in \mathfrak{g}_\beta \), and \( \alpha_{\beta}(x, y) \in \mathfrak{h} \). Equating \([h, x \circ y] = \alpha(h)z_{\alpha} + \beta(h)z_{\beta}\) with (3.11) we obtain \( \alpha(h)z_{\alpha} = \alpha(u_{\beta}(h, y))x \) and \( \beta(h)z_{\beta} = \beta(u_{\beta}(h, x))y \). Taking \( h' \) so that \( \alpha(h') = 1 \), we can define \( \lambda_{\beta}^\alpha(y) = \alpha(u_{\beta}(h', y)) \) for all \( y \in \mathfrak{g}_\beta \). Similarly, taking \( h'' \in \mathfrak{h} \) so that \( \beta(h'') = 1 \), we can define \( \lambda_{\alpha}^\beta(x) = \beta(u_{\alpha}(h'', x)) \). Then \( x \circ y = \lambda_{\beta}^\alpha(y)x + \lambda_{\alpha}^\beta(x)y + \sigma_{\alpha\beta}(x, y) \). From
\[
[h, x \circ y] + [x, y \circ h] + [y, h \circ x] = 0,
\]
we obtain
\[
\lambda_{\beta}^\alpha(y)\alpha(h)x + \lambda_{\beta}^\alpha(x)\beta(h)y - \alpha(u_{\beta}(h, y))x - \beta(u_{\alpha}(h, x))y = 0.
\]
In particular, this gives
\[
\lambda_{\beta}^\alpha(x)\beta(h) = \beta(u_{\alpha}(h, x)) \quad (3.15)
\]
for all \( x \in \mathfrak{g}_a, \ h \in \mathfrak{h} \). Letting \( \lambda_{\alpha}^\beta = \lambda_{\alpha} \), it follows from (3.12) that (3.15) holds for all \( \beta \in \Phi \).

Let \( \beta \) and \( \gamma \) be two roots of \( \mathfrak{g} \) such that \( \beta + \gamma \in \Phi \). We have the following two cases.
Case 1. \( \ker \beta = \ker \gamma \).
In this case, by (3.15), we have \( \lambda^\beta_a(x)h - u_a(h, x) \in \ker \beta = \ker \gamma \).
Hence we have
\[
\lambda^\beta_a(x)\gamma(h) = \gamma(u_a(h, x)) = \lambda^\gamma_a(x)\gamma(h).
\]
This leads to \( \lambda^\beta_a = \lambda^\gamma_a \).

Case 2. \( \ker \beta \neq \ker \gamma \).
Since \( \ker \beta \) and \( \ker \gamma \) are both of codimension 1, we can find \( h' \in \ker \beta \setminus \ker \gamma \) and \( h'' \in \ker \gamma \setminus \ker \beta \). Since (3.15) implies
\[
\lambda^\beta_a(x)\beta(h) + \lambda^\gamma_a(x)\gamma(h) = \lambda^\beta_a(\beta + \gamma)(h),
\]
by putting \( h = h' \) and \( h = h'' \), we obtain
\[
\lambda^\beta_a(x) = \lambda^\beta_a + \gamma(x) = \lambda^\gamma_a(x).
\]
Hence we have \( \lambda^\beta_a = \lambda^\gamma_a \) for all \( \beta \in \Phi \) by the indecomposability of the generalized Cartan matrix \( A \). Now (3.15) also implies \( \chi\alpha(h, x) = u\alpha(h, x) - \lambda\alpha(x)h \in Z(g) \).

**Lemma 3.7.** (a) For each \( \alpha \in \Phi \) and \( x, y \in g_\alpha \), we have \( \sigma_\alpha(x, y) \in Z(g) \), where \( \sigma_\alpha : g_\alpha \times g_\alpha \to h \) is the symmetric bilinear map given in Lemma 3.5.

(b) For each \( \alpha, \beta \in \Phi \) and \( x \in g_\alpha \), \( y \in g_\beta \), we have \( \sigma_{\alpha\beta}(x, y) \in Z(g) \), where \( \sigma_{\alpha\beta} : g_\alpha \times g_\beta \to h \) is the bilinear map given in Lemma 3.6.

**Proof.** (a) Fix \( \alpha \in \Phi \). By (1.8) and Lemma 3.5, we get \( \sigma_\alpha(x, y) \in \ker \alpha \) for all \( x, y \in g_\alpha \). Since the bilinear map \( \sigma_\alpha \) is symmetric, \( \sigma_\alpha(x, y) \in \ker \alpha \) for all \( x, y \in g_\alpha \).

For \( x, y \in g_\alpha \), choose \( z \in g_\alpha \), with \( \gamma \neq \alpha \). Since \( [z, x \circ y] = [z \circ y, x] + [z \circ z, y] \), we have \( \gamma(\sigma_\alpha(x, y))z = \alpha(\sigma_\alpha(z, y))x + \alpha(\sigma_\alpha(z, x))y \), which implies \( \sigma_\alpha(x, y) \in \ker \gamma \) for \( \gamma \neq \alpha, x, y \in g_\alpha \). Hence \( \sigma_\alpha(x, y) \in Z(g) \).

(b) For \( \alpha, \beta \in \Phi \) with \( \alpha \neq \beta \), let \( x \in g_\alpha \), \( y \in g_\beta \), and choose \( z \in g_\gamma \), with \( \gamma \neq \alpha, \gamma \neq \beta \). Then, by (1.10) and Lemma 3.6, we have
\[
\alpha(\sigma_{\beta\gamma}(y, z))x + \beta(\sigma_{\gamma\alpha}(z, x))y + \gamma(\sigma_{\alpha\beta}(x, y))z = 0, \quad (3.16)
\]
which implies \( \sigma_{\alpha\beta}(x, y) \in \ker \gamma \) for all \( \gamma \neq \alpha, \gamma \neq \beta \).

Furthermore, by (1.9) and Lemma 3.6, we obtain
\[
2\alpha(\sigma_{\alpha\beta}(x, y))x = 0,
\]
and hence \( \sigma_{\alpha\beta}(x, y) \in \ker \alpha \). Similarly, \( \sigma_{\beta\gamma}(x, y) \in \ker \beta \). Therefore, \( \sigma_{\alpha\beta}(x, y) \in \cap_{\gamma \in \Phi} \ker \gamma = Z(g) \).
Now summarizing the above lemmas, we state the main result of this paper.

**Theorem 3.8.** If \( * \) is a third power-associative Lie-admissible multiplication on the Kac–Moody algebra \( \mathfrak{g} = \mathfrak{g}(A) \), then there exist a linear functional \( \tau: \mathfrak{g} \to \mathbb{C} \) and a symmetric bilinear map \( \sigma: \mathfrak{g} \times \mathfrak{g} \to Z(\mathfrak{g}) \subset \mathfrak{h} \) such that
\[
x * y = \frac{1}{3}[x, y] + \tau(x)y + \tau(y)x + \sigma(x, y) \quad \text{for all } x, y \in \mathfrak{g}.
\]

(3.17)

Conversely, if \( * \) is a multiplication on \( \mathfrak{g} = \mathfrak{g}(A) \) defined by (3.17), then \( * \) is a third power-associative Lie-admissible multiplication on \( \mathfrak{g} \).  □

**Example 3.9.** (a) If \( \mathfrak{g} = \mathfrak{g}(A) \) is a hyperbolic Kac–Moody algebra, then, since \( \det A \neq 0 \), the center \( Z(\mathfrak{g}) \) is trivial, and hence \( \sigma = 0 \) in (3.17). Therefore, the third power-associative Lie-admissible multiplications \( * \) on \( \mathfrak{g} \) are given by
\[
x * y = \frac{1}{3}[x, y] + \tau(x)y + \tau(y)x \quad (x, y \in \mathfrak{g})
\]
for some linear functional \( \tau: \mathfrak{g} \to \mathbb{C} \).

(b) If \( \mathfrak{g} = \mathfrak{g}(A) \) is a Kac–Moody algebra with 1-dimensional center \( Z(\mathfrak{g}) = \mathbb{C}c \), then the third power-associative Lie-admissible multiplications \( * \) on \( \mathfrak{g} \) are given by
\[
x * y = \frac{1}{3}[x, y] + \tau(x)y + \tau(y)x + (x \mid y)c \quad (x, y \in \mathfrak{g}),
\]
(3.19)

where \( \tau: \mathfrak{g} \to \mathbb{C} \) is a linear functional on \( \mathfrak{g} \) and \( ( \mid ): \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \) is a symmetric bilinear form on \( \mathfrak{g} \). Hence Theorem 3.8 confirms the conjecture proposed by H. C. Myung [M 5].

4. **Flexible Lie-admissible Structures on Kac–Moody Algebras**

Recall that a Lie-admissible algebra \((L, *)\) is **flexible** if
\[
x * (y * x) = (x * y) * x \quad \text{for all } x, y \in L.
\]

(4.1)

It is easy to see that (4.1) is equivalent to
\[
[x, y] \circ x = [x, y \circ x] \quad \text{for all } x, y \in L,
\]

(4.2)

which can be linearized to
\[
[x, y \circ z] = [x, y] \circ z + y \circ [x, z] \quad \text{for all } x, y, z \in L.
\]

(4.3)
That is, the algebra $L$, is flexible if and only if the adjoint action is a derivation of the algebra $(L, *)$. Equivalently, the linear map $*: L \otimes L \to L$ given by $x \otimes y \mapsto x \ast y$ is a homomorphism of $L$-modules under the adjoint action. In this section, as an application of Theorem 3.8, we will determine all flexible Lie-admissible multiplications on Kac–Moody algebras.

**Proposition 4.1.** If $*$ is a flexible Lie-admissible multiplication on a Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$, then there exist a linear functional $\tau: \mathfrak{g} \to \mathbb{C}$ such that $\tau = 0$ on $[\mathfrak{g}, \mathfrak{g}]$ and a symmetric $\mathfrak{g}$-invariant bilinear map $\sigma: \mathfrak{g} \times \mathfrak{g} \to \mathbb{Z}(\mathfrak{g}) \subset \mathfrak{h}$ such that

$$x \ast y = \frac{1}{2} [x, y] + \tau(x)y + \tau(y)x + \sigma(x, y) \quad \text{for all } x, y \in \mathfrak{g}. \quad (4.4)$$

Conversely, if $*$ is a multiplication on $\mathfrak{g} = \mathfrak{g}(A)$ defined by (4.4), then $*$ is a flexible Lie-admissible multiplication on $\mathfrak{g}$.

**Proof.** By Theorem 3.8 and (4.2), we have

$$\tau([x, y])x + \sigma([x, y], x) = 0 \quad \text{for all } x, y \in \mathfrak{g}. \quad (4.5)$$

Hence if $x \in \mathfrak{g}_\alpha (\alpha \in \Phi)$, $\tau([x, y]) = 0$ for all $y \in \mathfrak{g}$. By skew-symmetry, $\tau([x, y]) = 0$ for all $x \in \mathfrak{g}$, $y \in \mathfrak{g}_\alpha$. Since $[\mathfrak{h}, \mathfrak{h}] = 0$, we conclude $\tau = 0$ on $[\mathfrak{g}, \mathfrak{g}]$. Thus (4.5) yields $\sigma([x, y], x) = 0$ for all $x, y \in \mathfrak{g}$. Replacing $x$ by $x + z$, we obtain

$$\sigma([x, y], z) = \sigma(x, [y, z]) \quad \text{for all } x, y, z \in \mathfrak{g},$$

which completes the proof. \hfill \Box

**Example 4.2.** (a) If $\mathfrak{g} = \mathfrak{g}(A)$ is a hyperbolic Kac–Moody algebra, then, since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, we have $\tau = 0$ in (4.4). Hence the flexible Lie-admissible multiplication $*$ on $\mathfrak{g}$ is given by

$$x \ast y = \frac{1}{2} [x, y] \quad \text{for all } x, y \in \mathfrak{g}, \quad (4.6)$$

which implies $(\mathfrak{g}, *)$ is a Lie algebra.

(b) If $\mathfrak{g} = \mathfrak{g}(A)$ is an affine Kac–Moody algebra with a flexible Lie-admissible multiplication $*$, then, since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathbb{C}d$, the linear functional $\tau$ in (4.4) is determined by the value $\tau(d)$, which may not be trivial. Moreover, the symmetric $\mathfrak{g}$-invariant bilinear map $\sigma$ in (4.4) is not necessarily trivial, which is different from the case of the Virasoro algebra with a flexible Lie-admissible structure (cf. [M5]).
REFERENCES


