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Bull. Sci. math. 132 (2008) 205–217

BULLETIN DES SCIENCES MATHÉMATIQUES

www.elsevier.com/locate/bulsci

A quartic system and a quintic system with fine focus of order 18

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Received 16 November 2006; accepted 18 December 2006

Available online 12 March 2007

Abstract

By using an effective complex algorithm to calculate the Lyapunov constants of polynomial systems E_n: $\dot{z} = iz + R_n(z, \bar{z})$, where R_n is a homogeneous polynomial of degree n, in this note we construct two concrete examples, E_4 and E_5, such that in both cases, the corresponding orders of fine focus can be as high as 18. The systems are given, respectively, by the following ordinary differential equations:

$$E_4:\quad \dot{z} = iz + 2iz^4 + iz\bar{z}^3 + \sqrt{\frac{52278}{20723}}e^{i\theta}\bar{z}^4,$$

where $\theta \notin \{k\pi \pm \frac{\pi}{6}, k\pi + \frac{\pi}{2}, k \in \mathbf{Z}\}$, and

$$E_5:\quad \dot{z} = iz + 3z^5 + \sqrt{\frac{20(c+3)}{9c^2-15}}z^4\bar{z} + z\bar{z}^4 + \sqrt{\frac{20(c+3)c^2}{9c^2-15}}\bar{z}^5,$$

where c is the root between $(-3, -\sqrt{5/3})$ of the equation

$$4155c^6 - 10716c^5 - 63285c^4 - 18070c^3 + 168075c^2 + 205450c + 60375 = 0.$$

Keywords: Center–focus; Fine focus; Quartic systems; Quintic systems; Lyapunov constants

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[1] Supported in part by the NSCF project number 10571002 of China.

0007-4497/$ – see front matter
doi:10.1016/j.bulsci.2006.12.006

1. Introduction

Consider the following planar polynomial system in which the origin is assumed to be a center of the linearized system:

$$\begin{cases} \dot{x} = -y + P(x, y), \\ \dot{y} = x + Q(x, y), \end{cases} \tag{1}$$

where P, Q are polynomials with real coefficients. It is well known that the above system always has either a center or a fine focus at the origin, and to distinguish between a center and a focus of (1), conventionally known as center–focus problem, is one of the most classical problems in the qualitative theory of ordinary differential equations. On the other hand, in the case of a focus, the problem to determine its highest possible order is also one of the interesting challenges in this field.

The center–focus problem, dating back to as early as the 19th century (see, for example, [9,11,14,21,22]), asks for the necessary and sufficient conditions on the nonlinear terms of the system (1), namely, on the coefficients of P and Q. Since the very beginning of the problem, it has caught much interest and attention. Throughout the whole 20th century, various kinds of methods and approaches have been attempted, different techniques and algorithms have been developed, and an extensive literature has been consequently produced. For the related material, we refer the reader to some valuable surveys and monographs, say, [7,8,13,16–18,23,28,29] and a wide range of reference therein.

Recently this problem has been again stimulated considerably not only by mathematicians in pure theory but also by experts in computation and applications, especially in the computer algebra systems. For a recent account on these techniques we refer to, for example, [4–6,10,17–20,29]. As a result, certain previously intractable systems can be treated now and, some piecewise results or observation can be collected and compared, which in turn, make a further systematical study accessible.

Strategically speaking, to solve the center–focus problem, one has to consider the following three steps.

- To establish some theoretical criteria by which one can determine if the equilibrium point of a given system is a center or a focus.
- To realize the criteria of the first step, which typically involves massive computation.
- To analyze the data obtained in the above step and to obtain the center–focus conditions in a readable way.

In what follows, we shall present a little more detailed exposition about these steps. At the same time, we shall also briefly recall the related background, theory and results, and introduce some necessary definitions.

From the time of Poincaré [22], several theoretical methods have already been given. Some widely applied techniques include, say, normal form method (focal values) (see, for example, [15]), the successive derivatives of the return map method (see, for example, [10]), the Lyapunov constant method (see, for example, [24,27]), etc. Certainly we can also mention some classical criteria such as the symmetry condition and the divergence free condition. In the former case it means that the system is invariant under either of the changes of coordinates $(x, y, t) \mapsto (x, -y, -t)$ or $(x, y, t) \mapsto (-x, y, -t)$, whereas in the latter case, it means that the system is Hamiltonian. In both cases it is clear that the equilibrium point is a center.

To apply normal form method in the center–focus problem is one of the well-known approaches. More precisely, taking system (1), under a near identity change of coordinates, one can always transform it into the following real standard normal form

$$\begin{cases} \dot{u} = -v - vR(r^2) - uG(r^2), \\ \dot{v} = u + uR(r^2) - vG(r^2), \end{cases} \tag{2}$$

where $r^2 = u^2 + v^2$, and $G(\xi) = g_1\xi + g_2\xi^2 + \cdots$. If *all* the coefficients g_k vanish, for $k = 1, 2, \ldots$, then it is easy to see that $\dot{r} = 0$. Consequently, the origin is a center. On the other hand, however, if there is an integer N satisfying $g_k = 0$ for all $k < N$ but $g_N \neq 0$, then from the relation $\dot{r} = g_N r^{2N+1} + \cdots$ we know that the system is a fine focus. Notice that although the normal form (2) is not unique, the first nonzero number N is an invariant of the system, i.e., the number N is uniquely determined by the system.

Definition 1. The constant g_k, $k \geqslant 1$, is said to be the kth focal value of system (1) at the origin. The number N is called the *order* of a fine focus of system (1) if the first nonzero focal value is g_N.

From time to time, in bibliography, one also frequently encounters another way to define focal values, i.e., by the Poincaré return map. More exactly, if we let $d(h_0) = P(h_0) - h_0$, where P is the Poincaré return map in a neighborhood of the origin, and if we denote by $v_k = d^{(k)}(0)/k!$, then the first nonzero focal value v_{2l+1} corresponds to an odd number $k = 2l + 1$. See [1,10] for more details.

Still another way to study the center–focus problem of system (1) is to calculate its Lyapunov constant which is introduced in the following way. According to [21], for the polynomial system (1), there exists a formal power series

$$F(x, y) = x^2 + y^2 + F_3(x, y) + \cdots + F_k(x, y) + \cdots, \tag{3}$$

where $F_k(x, y)$ is a homogeneous degree k polynomial of its variables, such that along the orbits of (1)

$$\left.\frac{dF}{dt}\right|_{(1)} = V_1 r^4 + V_2 r^6 + \cdots + V_n r^{2n+2} + \cdots, \tag{4}$$

where $\frac{dF}{dt} = \frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial y}\dot{y}$.

Definition 2. The coefficient V_k of the term r^{2k+2} in (4) is called the kth Lyapunov constant of system (1) at the origin.

For polynomial system (1), all the Lyapunov constants are also polynomials in the coefficients of the system, with rational coefficients. For each Lyapunov constant V_k, there is an infinite number of possibilities instead of being uniquely determined. On the other hand, however, all such V_k's are in the same coset modulo the ideal generated by $V_1, \ldots, V_{k-1}$ in the ring of polynomials with rational coefficients in the coefficients of the system (1).

In terms of the Lyapunov constants V_k, following the work of Poincaré [30], it is known that system (1) has a center at the origin if and only if *all* V_k are zero. Notice that this is equivalently to say that all the focal values are zero. In fact, one can show that the number N such that V_N is the first nonzero Lyapunov constant coincides with the order of fine focus given in Definition 1.

Moreover, V_N differs from g_N only by a positive number (see, for example, [12]). Therefore V_N and g_N equivalently characterize the order as well as the stability of fine focus of system (1) and, this gives the reason why in literature the two terms are used interchangeably.

In this paper we shall be interested in computing the Lyapunov constants instead of the focal values. This is mainly because, in practice, to normalize system (1) and consequently to find the focal values involve more tedious calculation than to evaluate the Lyapunov constant.

Although theoretically to prove a singular point to be a center we have to examine if all the Lyapunov constants vanish, it suffices to show that the first few of them are zero. This observation entirely relies on the Hilbert Basis Theorem, which says that the ideal of all V_k's in the ring of polynomials with rational coefficients in the coefficients of the system (1) is finitely generated. In other words, if, up to certain number N, the first N Lyapunov constants V_k turn out to be zero, then the equilibrium point of the system is already to be concluded to be a center.

The Hilbert Basis Theorem in fact equivalently says that given a polynomial system, the order of fine focus cannot reach as high as one wants. For instance, one can never expect a quadratic system of form (1) to have a fine focus of order 4. This is due to the result of Bautin [3]. It is proved in [3] that for quadratic systems the above ideal is determined by the values of V_j, $j \leqslant 3$. After Bautin, Sibirskii in [25] showed that cubic systems without quadratic terms cannot have a fine focus of order greater than 5.

Since Hilbert's existential result says nothing about how to decide and how to seek the order of fine focus of a given system, therefore to determine the number N aforementioned is completely a different story. Indeed, the progress of further study along the direction from lower degree systems to higher degree ones is slow and frustrating. After a study on quadratic-like cubic systems, only piecewise results dealing with some particular systems, say, Kukles system, are given. We refer to, say, [5,6,27]. Worthy to mention is a general result given by Bai and Liu [2]. In [2] it is proved that for even order of system (1), the order of fine focus can be as high as $n^2 - n$. As far as the authors know, this is the strongest results for general even n.

At this point, we can put forward the setting of our objectives. In this paper, we shall primarily consider systems of form (1) where the nonlinear part contains homogeneous degree 4 or degree 5 terms. For convenience, in this paper, we shall call them, respectively, quadratic-like quartic systems and quadratic-like quintic systems.

In recent years, these two kinds of systems have caught much interests [5,6,27]. However, a complete set of integrability conditions is far from being established. Actually, even the maximal possible order of fine focus is still open. According to [2], we only know that the order of fine focus can be greater than 12. In [27], an example of quartic system with order 15 is given. In this paper, we shall construct a particular example of quadratic-like quartic system of form (1) with the order of fine focus as high as 18 as well as an example of quadratic-like quintic system with the same order of fine focus.

Theorem 1. *There are quadratic-like quartic systems which have a fine focus of order* 18 *at the equilibrium point.*

There are quadratic-like quintic systems which have a fine focus of order 18 *at the equilibrium point.*

On the other hand, we have the following conjecture: The maximally possible order of fine focus of quadratic-like quartic systems is 21. The maximally possible order of fine focus of full quartic systems (i.e. with quadratic and cubic terms) is 21, too. The maximally possible order of fine focus of quadratic-like quartic systems is 18. The maximally possible order of fine focus of full quintic systems is 33.

Once we fix a theoretic criteria to determine the center–focus problem, it remains to take the second and the third steps. In our case, this means to compute the Lyapunov constants. To this end, we have to look for an effective algorithm and to realize it with the help of computer, for any attempt to take this step by hand requires considerable courage, patience and ingenuity.

Like the bibliography related to the first step, there is also a very rich reference in algorithms like we cited above. In this paper we shall essentially follow the algorithm developed in [27] where the authors, by putting the system into a complex form, give a method to calculate the Lyapunov constants for general planar polynomial systems. When restricting the algorithm in [27] to our quartic and quintic systems, we obtain an expression with less recursions so that the algorithm is more effectively applicable. A detailed technical explanation will be given in Section 2.

To analyze the data obtained in the second step and express the center–focus conditions in a readable way is far less trivial than it sounds. For example, we can compute the Lyapunov constants of, say, quartic systems up to quite high order. However, this means that we only obtain a series of necessary conditions for a given system to have a center at the singular point. These necessary conditions are, however, just some long and messy codes which, without further essential simplification, are typically useless. Moreover, the most difficult part is that we have no information till which order further necessary conditions in the series can be generated by the previous ones.

2. Preliminaries, the complex algorithm

Consider the planar polynomial system (1). By introducing complex variable $z = x + iy$, we can rewrite the system in the form

$$\dot{z} = iz + R(z, \bar{z}), \quad z \in \mathcal{C}, \tag{5}$$

where

$$R(z, \bar{z}) = P\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iQ\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

Let $F(x, y)$ be the formal series given in (3). Then it is easy to check that the corresponding complex power series $G(z, \bar{z}) = F(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i})$ satisfies $\overline{G(z, \bar{z})} = G(z, \bar{z})$ and

$$\left.\frac{dG}{dt}\right|_{(5)} = L_1|z|^4 + L_2|z|^6 + \cdots + L_m|z|^{2(m+1)} + \cdots. \tag{6}$$

On the other hand, if the formal power series $G(z, \bar{z}) = |z|^2 + \mathrm{O}(|z|^3)$ satisfies $\overline{G(z, \bar{z})} = G(z, \bar{z})$ and (6) where all L_k are real, then $F(x, y) = G(x + iy, x - iy)$ and L_k must satisfy (3). Therefore the numbers L_k in (6) in fact are Lyapunov constants of (1).

Lemma 1. *(See [26,27].) If $R(z, \bar{z})$ in* (5) *is a homogeneous polynomial of degree m, then $L_k = 0$ if $\frac{2k}{m-1}$ is not an integer.*

For quadratic-like quartic systems, we have the following immediate corollary.

Corollary 1. *If $R(z, \bar{z})$ in* (5) *is a homogeneous polynomial of degree* 4*, then $L_{3k+1} = 0$ and $L_{3k+2} = 0$ for all $k = 0, 1, \ldots$.*

In a similar way, when we study quadratic-like quintic systems, the following corollary is applicable.

Corollary 2. *If $R(z,\bar{z})$ in* (5) *is a homogeneous polynomial of degree* 5, *then $L_{2k+1}=0$ for all $k=0,1,\ldots$.*

The following notation is primarily from [27].

Let $\mathbf{r}_k$ be the $(k+1)$-dimensional vector with $\mathbf{r}_k(j+1)$ being the coefficient of term $z^j\bar{z}^{k-j}$, $0\leqslant j\leqslant k$, in the function $R(z,\bar{z})$. Similarly, let $\mathbf{g}_k$ be the $(k+1)$-dimensional vector with $\mathbf{g}_k(j+1)$ being the coefficient of term $z^j\bar{z}^{k-j}$, $0\leqslant j\leqslant k$, in the series $G(z,\bar{z})$.

We define a $(k+n)\times(n+1)$ matrix $R_{k,n}=(R_{k,n}(i,j))$ $(k\geqslant 2,\ n\geqslant 2)$, as follows:

$$R_{k,n}(i,j)=(j-1)\mathbf{r}_k(i-j+2)+(n-j+1)\overline{\mathbf{r}_k(k-i+j+1)}, \tag{7}$$

where $1\leqslant i\leqslant k+n$, $1\leqslant j\leqslant n+1$, and where $\mathbf{r}_k(j)$ is defined to be zero if $j<1$ or $j>k+1$. Moreover, we introduce the following $(k+1)\times(k+1)$ matrix $D_k=(D_k(i,j))$:

$$D_k(i,j)=\begin{cases}-\frac{1}{k-2i+2} & \text{if } i=j \text{ and } k-2i+2\neq 0,\\ 0 & \text{otherwise.}\end{cases} \tag{8}$$

Lemma 2. *(See [27].) The kth Lyapunov value of* (5) *at the origin can be calculated by*

$$L_k=\sum_{n=2}^{2k+1}\sum_{j=1}^{n+1}R_{2k+3-n,n}(k+2,j)\mathbf{g}_n(j), \tag{9}$$

where

$$\mathbf{g}_2=(0,1,0)^T,$$

and

$$\mathbf{g}_n=iD_n\sum_{m=2}^{n-1}R_{n+1-m,m}\mathbf{g}_m,\quad n\geqslant 3.$$

Since in this paper we shall be primarily interested in the case that $R(z,\bar{z})$ in (5) is a homogeneous polynomial of degree s. Therefore in what follows, we shall first of all apply Lemma 1 and simplify the algorithm given in Lemma 9. It turns out that we can eliminate one summation recursion in the above formula and obtain a more effective algorithm.

Lemma 3. *If $R(z,\bar{z})$ in* (5) *is a homogeneous polynomial of degree s, then the Lyapunov values at the origin can be computed by the following recursive formula*:

$$L_k=\sum_{j=1}^{2k+4-s}R_{s,2k+3-s}(k+2,j)\mathbf{g}_{2k+3-s}(j), \tag{10}$$

where

$$\mathbf{g}_2=(0,1,0)^T,$$

$$\mathbf{g}_n=iD_n\sum_{m=2}^{n-1}R_{n+1-m,m}\mathbf{g}_m,\quad n\geqslant 3.$$

Proof. If $R(z,\bar z)$ in (5) is a homogeneous polynomial of degree s, then $\mathbf{r}_k(j)=0$ and $R_{k,n}(i,j)=0$ if $k\neq s$. Therefore $\mathbf{g}_2=(0,1,0)^T$, $\mathbf{g}_n=0$, for $3\leqslant n\leqslant s+1$, and

$$\mathbf{g}_n = iD_nR_{s,n+1-s}\mathbf{g}_{n+1-s}, \quad n\geqslant s+1. \tag{11}$$

Thus relation (10) follows. □

Corollary 3. *If $s=4$ and, respectively, $s=5$, then the Lyapunov values of the system at the origin can be obtained, by the following formulas*:

$$L_k=\sum_{j=1}^{2k}R_{4,2k-1}(k+2,j)\mathbf{g}_{2k-1}(j), \quad s=4, \tag{12}$$

and, respectively,

$$L_k=\sum_{j=1}^{2k-1}R_{5,2k-2}(k+2,j)g_{2k-2}(j), \quad s=5. \tag{13}$$

3. An example of quartic system having a fine focus of degree 18

In this section, we shall mainly work on system (5) where $R(z,\bar z)$ is homogeneous polynomial of degree 4. Rewrite the system into the following form:

$$\begin{aligned}\dot z &= iz+(\alpha_1+i\alpha_2)z^4+(\alpha_3+i\alpha_4)z^3\bar z+(\alpha_5+i\alpha_6)z^2\bar z^2\\ &\quad+(\alpha_7+i\alpha_8)z\bar z^3+(\alpha_9+i\alpha_{10})\bar z^4,\end{aligned} \tag{14}$$

where all coefficients α_i are real. By computing the Lyapunov constants of the system, we shall look for systems within such a form which can have as high as possible order of fine focus.

First of all, according to Corollary 1, we know that L_k must be zero if k is not a natural number dividable by 3. On the other hand, by the algorithm (12) in Corollary 13, we can compute L_3 straightforwardly:

$$L_3=-2(\alpha_5\alpha_4+\alpha_3\alpha_6+\alpha_1\alpha_8+\alpha_2\alpha_7). \tag{15}$$

Certainly, we have various kinds of possibilities to choose these parameters to let (15) hold. Yet for the moment we want to concentrate on some particular examples under study in this note. We present the following well-chosen set of parameters such that the order of fine focus can reach as high as 18. We tested quite a few other possibilities to produce a higher order of fine focus. On the other hand, we conjecture that the highest possible order of fine focus for quartic system is 21, one step away from 18.

Now in (15) we set $\alpha_1=\alpha_7=\alpha_5=\alpha_6=0$. Then it follows that $L_3=0$. Under these conditions, we can proceed the calculation one step further to L_6. That is,

$$\begin{aligned}L_6 &= -\frac{2}{5}(\alpha_2+2\alpha_8)\big(2\alpha_3\alpha_{10}\alpha_8+4\alpha_3\alpha_{10}\alpha_2+3\alpha_3^2\alpha_9\\ &\quad-4\alpha_9\alpha_4\alpha_2-3\alpha_9\alpha_4^2-2\alpha_9\alpha_4\alpha_8-6\alpha_3\alpha_4\alpha_{10}\big).\end{aligned}$$

In order to let the relation $L_6=0$ stand, we can set $\alpha_3=\alpha_4=0$. In fact, with such an assumption, automatically, we shall have $L_9=0$. Furthermore, we can find

$$L_{12} = -\frac{4}{525}\alpha_9(\alpha_2 + 2\alpha_8)(\alpha_2 - 2\alpha_8)(4\alpha_2 - \alpha_8) \\ \times \left(\alpha_9^2 - 3\alpha_{10}^2\right)(59\alpha_8 + 30\alpha_2)(4\alpha_8 + \alpha_2).$$

Clearly, the relation $L_{12} = 0$ holds under the sufficiency condition $\alpha_2 = 2\alpha_8$. Thus we can easily move one step further to obtain L_{15} as follows:

$$L_{15} = -\frac{7}{320}\alpha_9\alpha_8^5\left(\alpha_9^2 - 3\alpha_{10}^2\right)\left(52278\alpha_8^2 - 20723\alpha_9^2 - 20723\alpha_{10}^2\right).$$

At this stage, if we put $\alpha_9 = 0$, then the origin of the system is a center because in this case the system is reversible. Below we assume that $\alpha_9 \neq 0$. Therefore, under a scale of the parameters if necessary, we can naturally impose an extra relation $\alpha_9^2 + \alpha_{10}^2 = 1$. Equivalently, we take $\alpha_9 = \cos\theta$, $\alpha_{10} = \sin\theta$. Now in order to make L_{15} vanish, one can ask a sufficient condition $\alpha_8 = \sqrt{\frac{20723}{52278}}$.

Finally we can obtain the following Lyapunov constant L_{18}:

$$L_{18} = \frac{-18933316550348377427791 1}{38179917427365007906613760 0}$$

Therefore if $\alpha_9 \neq 0$ and if $\alpha_9^2 - 3\alpha_{10}^2 \neq 0$, then $L_{18} \neq 0$. This means that indeed there are quartic systems with homogeneous linear terms having a fine focus of degree 18.

If we move some steps further, we see that L_{21} is already proportional to L_{18}. More exactly, we have

$$L_{21} = \frac{189231441601580144758357543795895753}{2055050427076514404688875987980288 0000}\sqrt{1083356994}\alpha_9\left(\alpha_9^2 - 3\alpha_{10}^2\right),$$

which means that

$$L_{21} = \frac{-152876554203535967575188481}{823306950264518820603360 0}L_{18}.$$

In conclusion, if $\alpha_9 \neq 0$ and if $\alpha_9^2 - 3\alpha_{10}^2 \neq 0$, then the system

$$\dot{z} = iz + 2i\sqrt{\frac{20723}{52278}}z^4 + i\sqrt{\frac{20723}{52278}}z\bar{z}^3 + (\alpha_9 + i\alpha_{10})\bar{z}^4$$

has the fine focus of order 18. We can put this system into the following form under a linear scale of z and $\bar{z}$:

$$\dot{z} = iz + 2iz^4 + iz\bar{z}^3 + \sqrt{\frac{52278}{20723}}e^{i\theta}\bar{z}^4.$$

Notice that the conditions $\alpha_9 \neq 0$ and $\alpha_9^2 - 3\alpha_{10}^2 \neq 0$ can be expressed as follows:

$$\theta \neq k\pi + \frac{\pi}{2}, \quad k\pi \pm \frac{\pi}{6}, \quad k \in \mathbf{Z}.$$

If we go on to find more Lyapunov values, as we have calculated up to L_{30}, all the constants are proportional to L_{18}. In fact, when $\theta = k\pi \pm \frac{\pi}{6}$, the system is reversible and indeed it has a center at the origin.[2]

[2] We thank J. Giné for pointing out the reversibility of the system in this case.

4. An example of quintic system having a fine focus of degree 18

This section is devoted to finding an example of quintic system having a fine focus of degree 18. That is, we consider system (5), where $R(z,\bar{z})$ is a homogeneous polynomial of degree 5, with the following explicit form:

$$\begin{aligned}\dot{z} = iz &+ (\alpha_1 + i\alpha_2)z^5 + (\alpha_3 + i\alpha_4)z^4\bar{z} + (\alpha_5 + i\alpha_6)z^3\bar{z}^2 \\ &+ (\alpha_7 + i\alpha_8)z^2\bar{z}^3 + (\alpha_9 + i\alpha_{10})z\bar{z}^4 + (\alpha_{11} + i\alpha_{12})\bar{z}^5,\end{aligned} \tag{16}$$

where all α_i are real constants.

We shall use the same algorithm as adopted in the last section for quartic system. Although the example we shall present has the same degree of fine focus like in the quartic case, we feel that this order is the maximally possible one for quadratic-like quintic systems.

To calculate the Lyapunov constants of system (16), it suffices only to consider L_{2l}, for $l = 1, 2, \ldots$. This is because, by Corollary 1, one always has $L_{2l+1} = 0$.

By the formula (13), some straightforward computation gives

$$L_2 = 2\alpha_5.$$

Thus we have only one possibility to take α_5, i.e., $\alpha_5 = 0$.

Now by putting $\alpha_5 = 0$, we can proceed the calculation to L_4. Namely, we have

$$L_4 = -2(\alpha_9\alpha_2 + \alpha_1\alpha_{10} + \alpha_7\alpha_4 + \alpha_3\alpha_8).$$

To let this relation hold, we have different ways to fix the parameters. Again, like in the quartic case, after certain attempts, we take the following particular set of parameters:

$$\alpha_7 = \alpha_8 = \alpha_2 = \alpha_{10} = 0.$$

Consequently, we arrive at the following stage:

$$L_6 = \frac{2}{3}(\alpha_3\alpha_{11} - \alpha_4\alpha_{12})(-\alpha_1 + 3\alpha_9).$$

There are two factors in the expression of L_6. Here we can simply assume that $\alpha_1 = 3\alpha_9$ to move one step further:

$$\begin{aligned}L_8 = {}& 8\alpha_{12}\alpha_9\alpha_6\alpha_4 + \frac{1}{3}\alpha_{11}\alpha_4^3 + \alpha_{12}\alpha_3\alpha_4^2 + 4\alpha_{12}\alpha_3\alpha_9^2 \\ &- 4\alpha_{11}\alpha_4\alpha_9^2 - \alpha_{11}\alpha_3^2\alpha_4 - \frac{1}{3}\alpha_{12}\alpha_3^3 - 8\alpha_{11}\alpha_6\alpha_3\alpha_9.\end{aligned} \tag{17}$$

In expression (17), we can put $\alpha_{12} = \alpha_6 = \alpha_4 = 0$ to make the relation $L_8 = 0$ hold. Consequently, we obtain the following relations:

$$L_{10} = -16\alpha_3\alpha_9^2 - \frac{16}{3}\alpha_9^2\alpha_{11} + \frac{12}{5}\alpha_3\alpha_{11}^2 - 4\alpha_3^3, \tag{18}$$

$$L_{12} = 0,$$

and

$$\begin{aligned}L_{14} = {}& 134\alpha_3^5 + \frac{9856}{9}\alpha_3^3\alpha_9^2 - \frac{128}{35}\alpha_3^2\alpha_{11}^3 + \frac{6464}{3}\alpha_3\alpha_9^4 - \frac{478}{105}\alpha_3\alpha_{11}^4 \\ &- \frac{11968}{45}\alpha_9^2\alpha 3\alpha_{11}^2 - \frac{4}{15}\alpha_3^4\alpha_{11} + \frac{3776}{15}\alpha_3^2\alpha_{11}\alpha_9^2 - \frac{24292}{315}\alpha_3^3\alpha_{11}^2 \\ &+ \frac{416}{135}\alpha_9^2\alpha_{11}^3 + \frac{11456}{15}\alpha_9^4\alpha_{11}.\end{aligned} \tag{19}$$

Without imposing any further restrictions, we can compute another two Lyapunov constants, L_{16} and L_{18}. They are given by

$$L_{16}=0,$$

and

$$\begin{aligned}L_{18}={}&\frac{1087}{54}\alpha_3\alpha_{11}^6+\frac{368357}{756}\alpha_3^4\alpha_{11}^3+\frac{4439}{1260}\alpha_3^2\alpha_{11}^5-\frac{204790}{3}\alpha_3^5\alpha_9^2\\&+\frac{21643673}{14175}\alpha_9^2\alpha_3\alpha_{11}^4+\frac{8471860}{189}\alpha_{11}^2\alpha_3\alpha_9^4-\frac{12581}{2}\alpha_3^7\\&-\frac{1311328}{9}\alpha_9^6\alpha_{11}-\frac{134963}{315}\alpha_{11}\alpha_3^6-\frac{4075753}{315}\alpha_{11}\alpha_3^4\alpha_9^2\\&-\frac{66195484}{945}\alpha_{11}\alpha_3^2\alpha_9^4-\frac{820984}{3}\alpha_3^3\alpha_9^4+\frac{4488437}{1260}\alpha_3^5\alpha_{11}^2\\&+\frac{54464521}{2835}\alpha_{11}^2\alpha_3^3\alpha_9^2-\frac{430048}{189}\alpha_9^4\alpha_{11}^3-\frac{67904}{8505}\alpha_9^2\alpha_{11}^5\\&+\frac{413419}{1620}\alpha_3^3\alpha_{11}^4-405024\alpha_9^6\alpha_3-\frac{371723}{945}\alpha_3^2\alpha_{11}^3\alpha_9^2.\end{aligned}\tag{20}$$

Below it suffices to prove the existence of solutions, in terms of the parameters α's, of the system of algebraic equations

$$L_{10}=0,\qquad L_{14}=0,\qquad L_{18}\neq 0.$$

To this end, we assume that $\alpha_9=\beta_1\alpha_3$, $\alpha_{11}=\beta_2\alpha_3$, and substitute them into the above expressions (18), (19) and (20). We have

$$L_{10}=\frac{4}{15}\big(9\beta_2^2-60\beta_1^2-20\beta_2\beta_1^2-15\big)\alpha_3^5\beta_1\beta_2,$$

$$\begin{aligned}L_{14}={}&\bigg(134-\frac{4}{15}\beta_2-\frac{24292}{315}\beta_2^2-\frac{128}{35}\beta_2^3-\frac{478}{105}\beta_2^4\\&+\frac{9856}{9}\beta_1^2+\frac{3776}{15}\beta_1^2\beta_2-\frac{11968}{45}\beta_1^2\beta_2^2+\frac{416}{135}\beta_1^2\beta_2^3\\&+\frac{6464}{3}\beta_1^4+\frac{11456}{15}\beta_1^4\beta_2\bigg)\alpha_3^7\beta_1\beta_2,\end{aligned}$$

and

$$\begin{aligned}L_{18}={}&\bigg(-\frac{12581}{2}-\frac{134963}{315}\beta_2+\frac{4488437}{1260}\beta_2^2+\frac{368357}{756}\beta_2^3+\frac{413419}{1620}\beta_2^4\\&+\frac{4439}{1260}\beta_2^5+\frac{1087}{54}\beta_2^6-\frac{204790}{3}\beta_1^2-\frac{4075753}{315}\beta_1^2\beta_2\\&+\frac{54464521}{2835}\beta_1^2\beta_2^2-\frac{371723}{945}\beta_1^2\beta_2^3+\frac{21643673}{14175}\beta_1^2\beta_2^4\\&-\frac{67904}{8505}\beta_1^2\beta_2^5-\frac{820984}{3}\beta_1^4-\frac{66195484}{945}\beta_1^4\beta_2+\frac{8471860}{189}\beta_1^4\beta_2^2\\&-\frac{430048}{189}\beta_1^4\beta_2^3-405024\beta_1^6-\frac{1311328}{9}\beta_1^6\beta_2\bigg)\alpha_3^9\beta_1\beta_2.\end{aligned}$$

If we take $\beta_2 \in (-3, -\sqrt{5/3})$ and

$$\beta_1^2 = \frac{3}{20}\frac{(3\beta_2^2-5)}{(\beta_2+3)}, \tag{21}$$

then it follows that

$$9\beta_2^2 - 60\beta_1^2 - 20\beta_2\beta_1^2 - 15 = 0.$$

Consequently, $L_{10} = 0$. Moreover, by substituting the relation (21) into L_{14} and L_{18}, we have

$$L_{14} = -\frac{2\alpha_3^7\beta_1\beta_2}{2625(\beta_2+3)^2}F(\beta_2),$$
$$L_{18} = \frac{\alpha_3^7\beta_1\beta_2^2}{47250(\beta_2+3)^3}G(\beta_2),$$

where

$$\begin{aligned} F(\beta_2) = {} & 4155\beta_2^6 - 10716\beta_2^5 - 63285\beta_2^4 - 18070\beta_2^3 \\ & + 168075\beta_2^2 + 205450\beta_2 + 60375, \end{aligned}$$

and

$$\begin{aligned} G(\beta_2) = {} & 781365\beta_2^8 + 18402357\beta_2^7 - 39352164\beta_2^6 \\ & - 295090087\beta_2^5 - 206364414\beta_2^4 + 680783545\beta_2^3 \\ & + 1213555470\beta_2^2 + 666947025\beta_2 + 115223175. \end{aligned}$$

The possibility to take $\beta_2 \in (-3, -\sqrt{5/3})$ can be verified from the observation that $F(-3) = 1951488 > 0$, and $F(-3/2) < -322 < 0$. This means that indeed there exists $\beta_0 \in (-3, -\sqrt{5/3})$ such that $F(\beta_0) = 0$. Since $F(x)$ and $G(x)$ have no common factors, therefore $G(\beta_0) \neq 0$ and consequently $L_{18} \neq 0$. Moreover, because

$$\frac{3}{20}\frac{(3\beta_0^2-5)}{(\beta_0+3)} > 0,$$

it follows that if we take

$$\alpha_2 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = \alpha_{10} = \alpha_{12} = 0,$$

and

$$\alpha_3 = \mp\sqrt{\frac{20}{3}\frac{(\beta_0+3)}{(3\beta_0^2-5)}}, \qquad \alpha_1 = 3, \qquad \alpha_9 = 1, \qquad \alpha_{11} = \beta_0\alpha_3,$$

where β_0 is the root of $F(\beta) = 0$ in $(-3, -\sqrt{5/3})$, then

$$L_k = 0, \quad k = 1, \dots, 17, \qquad L_{18} \neq 0.$$

That means that the following system has the fine focus of degree 18.

$$E_5\colon \quad \dot{z} = iz + 3z^5 + \sqrt{\frac{20(\beta_0+3)}{9\beta_0^2-15}}z^4\bar{z} + z\bar{z}^4 + \sqrt{\frac{20(\beta_0+3)\beta_0^2}{9\beta_0^2-15}}\bar{z}^5. \tag{22}$$

Notice that system (22) contains no parameter.

References

[1] A.A. Andronov, E.A. Leontovich, I.I. Gordon, A.G. Maier, Theory of Bifurcations of Dynamical Systems on a Plane, John Wiley, New York, 1973.

[2] J. Bai, Y. Liu, A class of planar degree n (even number) polynomial systems with a fine focus of order $n^2 - n$, Chinese Sci. Bull. 12 (1992) 1063–1065.

[3] N.N. Bautin, On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type, Mat. Sb. 30 (1952) 181–196; English transl. in: Math. USSR-SB 100 (1954) 397–413.

[4] M. Caubergh, F. Dumortier, Hopf–Takens bifurcations and centres, J. Differential Equations 202 (2004) 1–31.

[5] J. Chavarriga, J. Giné, Integrability of a linear center perturbed by fourth degree homogeneous polynomial, Publ. Mat. 40 (1996) 21–39.

[6] J. Chavarriga, J. Giné, Integrability of a linear center perturbed by fifth degree homogeneous polynomial, Publ. Mat. 41 (1997) 335–356.

[7] C.S. Coleman, Hilbert's 16th problem: How many cycles? in: Differential Equation Models, Modules in Applied Mathematics, vol. 1, Springer-Verlag, New York, pp. 279–297.

[8] W.A. Coppel, A survey of quadratic systems, J. Differential Equations 2 (1966) 293–304.

[9] H. Dulac, Détermination et intégration d'une certaine classe d'équations différentielle ayant pour point singulier un centre, Bull. Sci. Math. Sér. (2) 32 (1908) 230–252.

[10] J.P. Françoise, Successive derivatives of a first return map, application to the study of quadratic vector fields, Ergodic Theory Dynam. Syst. 16 (1996) 87–96.

[11] M. Frommer, Über das Auftreten vem Wirbeln und Strudeln (geschlossener und spiralieger Integralkurven) in der Umbegung rationaler Unbestimmtheitsstellen, Math. Ann. 109 (1933/1934) 395–424.

[12] F. Gobber, K.D. Willamowskii, Lyapunov approach to multiple Hopf bifurcation, J. Math. Anal. Appl. 71 (1979) 333–350.

[13] Yu.S. Ilyashenko, Finiteness for Limit Cycles, Transl. Math. Monographs, vol. 94, Amer. Math. Soc., Providence, RI, 1991.

[14] W. Kapteyn, On the midpoints of integral curves of differential equations of the first degree, Nederl. Akad. Wetensch. Verslag Afd. Natuurk. 20 (1912) 1354–1365.

[15] W. Li, Theory and Applications of Normal Forms, Science Press, Beijing, 2000.

[16] J. Li, Hilbert's 16th problem and bifurcations of planar polynomial vector fields, Int. J. Bifurcation and Chaos 13 (2003) 47–106.

[17] J. Llibre, Integrability of polynomial differential systems, in: Handbook of Differential Equations (Ordinary Differential Equations, vol. 1), Elsevier, North-Holland, Amsterdam, pp. 437–532.

[18] N.G. Lloyd, Limit cycles of polynomial systems, some recent developments, in: T. Bedford, J. Swift (Eds.), New Directions in Dynamical Systems, Lecture Notes, vol. 127, London Mathematical Soc., pp. 192–234..

[19] N.G. Lloyd, J.M. Pearson, Conditions for a centre and the bifurcation of limit cycles in a class of cubic systems, in: Lecture Notes in Mathematics, vol. 1455, pp. 231–242.

[20] N.G. Lloyd, C.J. Christopher, J. Devlin, J.M. Pearson, N. Yasmin, Quadratic-like cubic systems, Differential Equations Dynam. Syst. 5 (1997) 329–345.

[21] M.A. Lyapunov, Problème Général de la Stabilité de Mouvement, Princeton Univ. Press, Princeton, NJ, 1947 (First printing, 1892, Harkov).

[22] H. Poincaré, Mémoire sur les courbes définies par les équations différentielles, J. Math. Pures Appl. 1 (1885) 167–244; Oeuvres de Henri Poincaré, vol. 1, Gauthier-Villars, Paris, pp. 95–114.

[23] D. Schlomiuk, Algebraic and geometric aspects of the theory of polynomial vector fields, in: D. Schlomiuk (Ed.), Bifurcations and Periodic Orbits of Vector Fields, NATO SAI Series C, vol. 408, Kluwer Academic, London, pp. 429–467.

[24] S. Shi, On the structure of Poincaré–Lyapunov constants for the weak focus of polynomial vector fields, J. Differential Equations 52 (1984) 52–57.

[25] K.S. Sibirskii, The number of limit cycles in the neighborhood of a singular point, Differential Equations 1 (1965) 36–47.

[26] D. Wang, A recursive formula and its applications to computations of normal forms and focal values in dynamical systems, in: S. Liao, T. Ding, Y. Ye (Eds.), Nankai Series in Pure, Applied Mathematics and Theoretical Physics, vol. 4, World Scientific Publishing Co. Pte. Ltd., Singapore, pp. 238–247.

[27] D. Wang, R. Mao, A complex algorithm for computing Lyapunov values, Random and Computational Dynamics 2 (1994) 261–277.

[28] Y. Ye, et al., The Theory of Limit Cycles, Transl. Math. Monographs, vol. 66, Amer. Math. Soc., Providence, RI, 1984.

[29] Y. Ye, Qualitative Theory of Polynomial Differential Systems, Shanghai Scientific and Technical Publishers, Shanghai, 1995.