

ON A FORMULA OF C. S. MEIJER

BY

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Some years ago C. S. MEIJER ([1], p. 127, formula (G); [3], p. 355, formula (113)) published a formula for generalized hypergeometric functions, which contains many known formulae on special functions. Meijer's formula is

$$(1) \quad \left\{ \begin{array}{l} {}_p\Phi_{q+l} \left(\begin{matrix} \gamma_1, \dots, \gamma_k, \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q, \delta_1, \dots, \delta_l; \lambda \zeta \end{matrix} \right) = \\ \sum_{r=0}^{\infty} \frac{1}{r!} {}_k\Phi_l \left(\begin{matrix} -r, \gamma_1, \dots, \gamma_k; \\ \delta_1, \dots, \delta_l; \lambda \end{matrix} \right) (\alpha_1)_r \dots (\alpha_p)_r (-\zeta)^r {}_p\Phi_q \left(\begin{matrix} \alpha_1+r, \dots, \alpha_p+r; \\ \beta_1+r, \dots, \beta_q+r; \zeta \end{matrix} \right). \end{array} \right.$$

Here we use the following notation:

$$(\alpha)_r = \begin{cases} \alpha(\alpha+1) \dots (\alpha+r-1) & \text{if } r \text{ is a positive integer,} \\ 1 & \text{if } r = 0. \end{cases}$$

If p and q are non-negative integers, and $p \leq q+1$ or for some $i (1 \leq i \leq p)$ α_i is a non-positive integer, then

$${}_p\Phi_q \left(\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \zeta \end{matrix} \right)$$

is the analytic function of ζ defined in a neighbourhood of $\zeta=0$ by

$$(2) \quad {}_p\Phi_q \left(\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \zeta \end{matrix} \right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n}{n! \Gamma(\beta_1+n) \dots \Gamma(\beta_q+n)} \zeta^n.$$

The series on the right of (2) has a finite radius of convergence only in the case that $p=q+1$ and no $\alpha_i (i=1, \dots, p)$ is equal to a non-positive integer. The analytic continuation for this case will not be described here. It can be found in [2], § 2 and in [4]. For our purpose it is sufficient to know, that 0, 1, ∞ are the only singularities (branchpoints in general). Hence, if C is any simple curve connecting 1 and ∞ and $0 \notin C$, there exists a unique analytic function on the complement of C , which has the power series representation (2) in a neighbourhood of $\zeta=0$. The curve C will not be mentioned explicitly in the sequel, but is assumed to be suitably chosen. (Meijer uses the rays $(1, 1+i\infty)$ and $(1, 1-i\infty)$).

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In this paper (1) is proved in a new and simple way. The conditions for the validity of (1) given by MEIJER ([1], p. 127; [3], p. 355) will be deduced anew. Finally a relation for generalized Heine series is given, which is analogous to (1).

Formula (1) is valid in each of the following eight cases Ia, ..., Ie, IIa, IIb, III:

- I. None of the numbers $\gamma_1, \dots, \gamma_k, \alpha_1, \dots, \alpha_p$ is equal to 0, -1, -2, ..., and
- $p < q + 1$ and $p + k < q + l + 1$, for all values of λ and ζ .
 - $p < q + 1, p + k = q + l + 1$, for $|\lambda\zeta| < 1$.
 - $p = q + 1, k < l$, for $\text{Re } \zeta < \frac{1}{2}$ and all values of λ .
 - $p = q + 1, k = l = 0$, for $\zeta \neq 1$ and $|(\lambda - 1)\zeta| < |\zeta - 1|$.
 - $p = q + 1, k = l > 0$, for $\text{Re } \zeta < \frac{1}{2}$ and $|(\lambda - 1)\zeta| < |\zeta - 1|$.
- II. $k \geq 1$ and at least one of the numbers $\gamma_1, \dots, \gamma_k$, but none of the numbers $\alpha_1, \dots, \alpha_p$ is equal to 0, -1, -2, ..., and
- $p < q + 1$, for all values of λ and ζ .
 - $p = q + 1$, for $\text{Re } \zeta < \frac{1}{2}$ and all values of λ .
- III. $p \geq 1$ and at least one of the numbers $\alpha_1, \dots, \alpha_p$ is equal to 0, -1, -2, ..., for all values of λ and ζ .

Proof. By (2) the right-hand member of (1) can be formally written as

$$\begin{aligned}
 & \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{n=0}^r \frac{(-r)_n (\gamma_1)_n \dots (\gamma_k)_n}{n! \Gamma(\delta_1 + n) \dots \Gamma(\delta_l + n)} \lambda^n \sum_{m=0}^{\infty} \frac{(\alpha_1)_{r+m} \dots (\alpha_p)_{r+m} (-1)^r}{m! \Gamma(\beta_1 + r + m) \dots \Gamma(\beta_q + r + m)} \zeta^{m+r} = \\
 & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{r=n}^{\infty} \frac{(\gamma_1)_n \dots (\gamma_k)_n (\alpha_1)_{r+m} \dots (\alpha_p)_{r+m}}{n! \Gamma(\delta_1 + n) \dots \Gamma(\delta_l + n) \Gamma(\beta_1 + r + m) \dots \Gamma(\beta_q + r + m)} \frac{(-1)^{r+n} \lambda^n \zeta^{m+r}}{m! (r-n)!} = \\
 (3) \quad & \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} \frac{(\gamma_1)_n \dots (\gamma_k)_n (\alpha_1)_j \dots (\alpha_p)_j \lambda^n \zeta^j}{n! \Gamma(\delta_1 + n) \dots \Gamma(\delta_l + n) \Gamma(\beta_1 + j) \dots \Gamma(\beta_q + j)} \sum_{r=n}^j \frac{(-1)^{r+n}}{(r-n)! (j-r)!}.
 \end{aligned}$$

Now using

$$\sum_{r=n}^j \frac{(-1)^{r+n}}{(r-n)! (j-r)!} = \frac{(1-1)^{j-n}}{(j-n)!} = \begin{cases} 0 & \text{if } j > n \\ 1 & \text{if } j = n, \end{cases}$$

we see that (3) equals the left-hand member of (1).

In each of the cases Ia, b, IIa and III the absolute convergence of (3) can be shown by estimates of the type

$$(4) \quad \left| \frac{(\alpha_1)_n \dots (\alpha_p)_n}{\Gamma(\beta_1 + n) \dots \Gamma(\beta_p + n)} \right| \leq C n^{\text{Re}\{(\alpha_1 + \dots + \alpha_p) - (\beta_1 + \dots + \beta_p)\}}.$$

Hence, in the following we may restrict ourselves to the case $p = q + 1$, where none of the numbers $\alpha_1, \dots, \alpha_p$ is equal to 0, -1, -2, In this case we can again prove by (4) the absolute convergence of (3), but only for small values of $|\zeta|$ and $|\lambda|$, provided that $k \leq l$ or that one of the

numbers $\gamma_1, \dots, \gamma_k$ is equal to 0, -1, -2, ... Next we shall show that (1) has a larger region of validity. We need two lemmas.

Lemma 1. If $\zeta \neq 1$, then

$$\limsup_{r \rightarrow \infty} \left| \frac{(-\zeta)^r (\alpha_1)_r \dots (\alpha_{q+1})_r}{r!} {}_{q+1}\Phi_q \left(\begin{matrix} \alpha_1 + r, \dots, \alpha_{q+1} + r; \\ \beta_1 + r, \dots, \beta_q + r; \zeta \end{matrix} \right) \right|_r^{\frac{1}{r}} = \left| \frac{\zeta}{\zeta - 1} \right|.$$

Proof. The function

$$f(w) = {}_{q+1}\Phi_q \left(\begin{matrix} \alpha_1, \dots, \alpha_{q+1}; \\ \beta_1, \dots, \beta_q; \zeta(1-w) \end{matrix} \right)$$

is analytic in w for $|w| < |1 - \zeta^{-1}|$. Using (2) we can easily derive that

$$\left[\frac{d^r}{dw^r} f(w) \right]_{w=0} = (\alpha_1)_r \dots (\alpha_{q+1})_r (-\zeta)^r {}_{q+1}\Phi_q \left(\begin{matrix} \alpha_1 + r, \dots, \alpha_{q+1} + r; \\ \beta_1 + r, \dots, \beta_q + r; \zeta \end{matrix} \right)$$

for $|\zeta| < 1$. Both members being analytic in ζ (if $\zeta \neq 1$), the equality is valid in the cut ζ -plane. Hence, the Taylor expansion of $f(w)$ in powers of w is

$$f(w) = \sum_{r=0}^{\infty} w^r \frac{(-\zeta)^r (\alpha_1)_r \dots (\alpha_{q+1})_r}{r!} {}_{q+1}\Phi_q \left(\begin{matrix} \alpha_1 + r, \dots, \alpha_{q+1} + r; \\ \beta_1 + r, \dots, \beta_q + r; \zeta \end{matrix} \right).$$

As $f(w)$ is analytic for $|w| < |1 - \zeta^{-1}|$, the radius of convergence of the Taylor series is equal to $|1 - \zeta^{-1}|$. Lemma 1 expresses this fact in a different way.

Lemma 2. If none of the numbers $\gamma_1, \dots, \gamma_k$ is equal to 0, -1, -2, ..., then

$$\limsup_{r \rightarrow \infty} \left| {}_{k+1}\Phi_l \left(\begin{matrix} -r, \gamma_1, \dots, \gamma_k; \\ \delta_1, \dots, \delta_l; \lambda \end{matrix} \right) \right|_r^{\frac{1}{r}} = \begin{cases} 1 & \text{if } k < l, \\ \max(1, |1 - \lambda|) & \text{if } k = l > 0, \\ |1 - \lambda| & \text{if } k = l = 0. \end{cases}$$

However, if one of the numbers $\gamma_1, \dots, \gamma_k$ is equal to 0, -1, -2, ..., the lim sup equals 1 in all cases.

Proof. The proof runs along the same lines as that of lemma 1. The starting point is now

$$(5) \quad \frac{1}{1-w} {}_{k+1}\Phi_l \left(\begin{matrix} 1, \gamma_1, \dots, \gamma_k; \\ \delta_1, \dots, \delta_l; \frac{\lambda w}{w-1} \end{matrix} \right) = \sum_{r=0}^{\infty} w^r {}_{k+1}\Phi_l \left(\begin{matrix} -r, \gamma_1, \dots, \gamma_k; \\ \delta_1, \dots, \delta_l; \lambda \end{matrix} \right).$$

This formula is the special case $p=1, q=0, \alpha_1=0, \zeta=w(w-1)^{-1}$ of (1). Hence, it is valid for small values of $|\lambda|$ and $|w|$. If λ is fixed, the function $g(w)$ on the left of (5) has a Taylor expansion in powers of w . It is easily seen that the coefficient of w^r in that expansion is an analytic function of λ . From these considerations it is clear that for each λ the expansion (5) holds in a certain neighbourhood of $w=0$. Now $g(w)$ has a singularity in $w=1$ if $k < l$, in 1 and $(1-\lambda)^{-1}$ if $k=l > 0$, and in $(1-\lambda)^{-1}$ if $k=l=0$.

This yields the first part of lemma 2. If one of the numbers $\gamma_1, \dots, \gamma_k$ is equal to $0, -1, -2, \dots$, then $g(w)$ has a singularity at $w=1$. This completes the proof.

From lemma 1 and lemma 2 it follows that the series on the right of (1) converges absolutely in the cases Ic, d, e and IIb. Moreover, the convergence is uniform, if λ and ζ are restricted to compact sets. Hence, this series represents a function which is analytic in λ and in ζ . As (1) holds for small values of $|\lambda|$ and $|\zeta|$, the validity of (1) is also proved in the cases Ic, d, e and IIb.

The above-mentioned generalization of (1) to Heine series (for definition and properties of Heine or basic series see [5], ch. VIII) is

$$(6) \quad \left\{ \begin{array}{l} {}_{k+p}\Psi_{l+s} \left(\begin{array}{c} \gamma_1, \dots, \gamma_k, \alpha_1, \dots, \alpha_p; \\ \delta_1, \dots, \delta_l, \beta_1, \dots, \beta_s; \lambda\zeta \end{array} \right) = \sum_{r=0}^{\infty} \frac{[\alpha_1]_r \dots [\alpha_p]_r \zeta^r}{[q^{-r}]_r [\beta_1]_r \dots [\beta_s]_r} \\ \cdot {}_{k+1}\Psi_l \left(\begin{array}{c} q^{-r}, \gamma_1, \dots, \gamma_k; \\ \delta_1, \dots, \delta_l; \lambda \end{array} \right) {}_p\Psi_s \left(\begin{array}{c} \alpha_1 q^r, \dots, \alpha_p q^r; \\ \beta_1 q^r, \dots, \beta_s q^r; \zeta \end{array} \right), \end{array} \right.$$

where $0 < q < 1$,

$$[\alpha]_r = \begin{cases} (1-\alpha)(1-\alpha q) \dots (1-\alpha q^{r-1}) & \text{if } r \geq 1, \\ 1 & \text{if } r = 0, \end{cases}$$

and

$${}_p\Psi_s \left(\begin{array}{c} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_s; \zeta \end{array} \right) = \sum_{r=0}^{\infty} \frac{[\alpha_1]_r \dots [\alpha_p]_r}{[q]_r [\beta_1]_r \dots [\beta_s]_r} \zeta^r.$$

(6) is always valid if $|\zeta| < 1$ and $|\lambda| < 1$. A proof and a more precise discussion of the validity of (6) can be given in a similar way as was done for formula (1).

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