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On the equivalence of four chaotic operators[☆]

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ABSTRACT

In this paper, we study chaos for bounded operators on Banach spaces. First, it is proved that, for a bounded operator T defined on a Banach space, Li–Yorke chaos, Li–Yorke sensitivity, spatio-temporal chaos, and distributional chaos in a sequence are equivalent, and they are all strictly stronger than sensitivity. Next, we show that T is sensitive dependence iff $\sup\{\|T^n\| : n \in \mathbb{N}\} = \infty$. Finally, the following results are obtained: (1) T is chaotic iff T^n is chaotic for each $n \in \mathbb{N}$. (2) The product operator $T_n^* = \prod_{i=1}^{n} T_i$ is chaotic iff T_k is chaotic for some $k \in \{1, 2, \ldots, n\}$.

1. Introduction and basic definitions

The complexity of a topological dynamical system become a hot issue in science since the term chaos was introduced by Li and Yorke [1] in 1975; it is known as *Li–Yorke chaos* today. In [2,3], Li–Yorke chaos was studied by Duan et al. for linear operators. In 2010, Bermúdez et al. [4] gave some equivalent conditions for Li–Yorke chaotic operators and obtained a few sufficient criteria for distributionally chaotic operators.

The central point of chaos is the impossibility of prediction of dynamics due to the divergence of close orbits. The differences in definitions of chaos begin with different understanding of this divergence. The notion of *Li–Yorke sensitivity* was mentioned for the first time by Akin and Kolyada [5] in 2003. At the same time, they also introduced the concept of *spatio-temporal chaos* and provided a question:

Question 1. Are all Li–Yorke sensitive systems Li–Yorke chaotic?

In 2007, Wang et al. [6] introduced the concept of *distributional chaos in a sequence*, and proved that it is equivalent to *Li–Yorke chaos* for continuous interval self-maps. So far, following question remains open:

Question 2. Are all Li–Yorke chaotic systems distributionally chaotic in a sequence?

In this paper, we mainly study chaos for operators on Banach spaces. First, it is proved that, for a bounded operator T, Li–Yorke chaos, Li–Yorke sensitivity, spatio-temporal chaos, and distributional chaos in a sequence are equivalent, and they are all strictly stronger than sensitivity. This partly answers Questions 1 and 2. Next, we show that T is sensitive dependence iff $\sup\{\|T^n\| : n \in \mathbb{N}\} = \infty$. Finally, we study the chaoticity of compositional and product operators and prove that the following conclusions hold.

(1) T is chaotic iff T^n is chaotic for each $n \in \mathbb{N}$.
(2) The product operator $T_n^* = \prod_{i=1}^{n} T_i$ is chaotic iff T_k is chaotic for some $k \in \{1, 2, \ldots, n\}$.

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Definition 1 ([5]). Let $f : X \to X$ be a continuous map on a metric space (X, ρ). If $x, y \in X$ and $\delta > 0$, (x, y) is called a *Li–Yorke pair of modulus* δ if

$$\lim_{n\to\infty} \inf \rho(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup \rho(f^n(x), f^n(y)) \geq \delta.$$

(1-1) (x, y) is a *Li–Yorke pair* if it is a Li–Yorke pair of modulus δ for some $\delta > 0$.
(1-2) The subset $\Gamma \subset X$ is called a *scrambled set* if, for all points $x, y \in \Gamma$ with $x \neq y$, (x, y) is a Li–Yorke pair.
(1-3) f is *Li–Yorke chaotic* if X contains an uncountable scrambled set.

The set of Li–Yorke pairs of modulus δ is denoted by $\mathrm{LY}(f, \delta)$, and the set of Li–Yorke pairs is denoted by $\mathrm{LY}(f)$.

Definition 2 ([5]). Assume that (X, ρ) is a metric space and that map $f : X \to X$ continuous.

(2-1) f is *sensitive dependence* if there exists $\epsilon > 0$ such that, for any $x \in X$ and any $\delta > 0$, there is some y which is within a distance δ of x, and, for some $n \in \mathbb{N}$, $\rho(f^n(x), f^n(y)) > \epsilon$.
(2-2) f is *spatio-temporally chaotic* if, for any $x \in X$ and any $\delta > 0$, there is some y which is within a distance δ of x such that $(x, y) \in \mathrm{LY}(f)$.
(2-3) f is *Li–Yorke sensitive* if there exists $\epsilon > 0$ that satisfies that, for any $x \in X$ and any $\delta > 0$, there is some y which is within a distance δ of x such that $(x, y) \in \mathrm{LY}(f, \epsilon)$.

Suppose that $\{p_k\}_{k\in\mathbb{N}}$ is a strictly increasing sequence of positive integers. For any pair $x, y \in X$ and any real number $t > 0$, let us put

$$F^*_{xy}(t, \{p_k\}_{k\in\mathbb{N}}, f) = \lim_{n\to\infty} \sup \frac{1}{n} \sum_{k=1}^{n} \chi_{[0,t)}(\rho(f^{p_k}(x), f^{p_k}(y))),$$

and

$$F_{xy}(t, \{p_k\}_{k\in\mathbb{N}}, f) = \lim_{n\to\infty} \inf \frac{1}{n} \sum_{k=1}^{n} \chi_{[0,t)}(\rho(f^{p_k}(x), f^{p_k}(y)))$$

where $\chi_{[0,t)}(x)$ is the characteristic function of the set $[0, t)$.

Definition 3 ([6]). Suppose that $\{p_k\}_{k\in\mathbb{N}}$ is a strictly increasing sequence of positive integers, and that f a continuous self-map on a metric space X.

(3-1) The subset D_0 of X is a *distributionally chaotic set in* $\{p_k\}_{k\in\mathbb{N}}$ if, for each pair $x, y \in D_0$ with $x \neq y$, the following two conditions are satisfied: $F^*_{xy}(t, \{p_k\}_{k\in\mathbb{N}}, f) = 1$ for all $t > 0$, and $F_{xy}(\delta, \{p_k\}_{k\in\mathbb{N}}, f) = 0$ for some $\delta > 0$.
(3-2) f is *distributionally chaotic in a sequence* if it has a distributionally chaotic set which is uncountable in a sequence.

From now on, X denotes a Banach space over $\mathbb{C}$ (or $\mathbb{R}$), and $T : X \to X$ denotes a bounded operator. In this case, the associated metric is $\rho(x, y) = \|x - y\|$ for any pair $x, y \in X$, where $\|\cdot\|$ is the norm of X. Θ denotes the zero-vector of X. Let $B(x, \epsilon)$ denote ϵ-neighborhoods of a point $x \in X$, i.e., $B(x, \epsilon) = \{y \in X : \rho(x, y) < \epsilon\}$. Write $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$.

Definition 4 ([7,8]). A vector $x \in X$ is said to be *irregular* for T if $\lim_{n\to\infty} \inf \|T^n(x)\| = 0$ and $\lim_{n\to\infty} \sup \|T^n(x)\| = \infty$.

Lemma 1 ([4]). *Let $T : X \to X$ be a bounded operator. The following assertions are equivalent.*

(1) *T is Li–Yorke chaotic.*
(2) *T admits a Li–Yorke pair.*
(3) *T admits an irregular vector.*

2. Equivalent conditions of Li–Yorke chaos and sensitivity

First, we discuss Li–Yorke's chaoticity of operators with the following result.

Theorem 1. *Assume that T is a bounded operator defined on a Banach space X over $\mathbb{C}$ (or $\mathbb{R}$). Then the following are equivalent.*

(1) *T is Li–Yorke chaotic.*
(2) *T is Li–Yorke sensitive.*
(3) *T is spatio-temporally chaotic.*
(4) *T is distributionally chaotic in a sequence.*

Proof. (1) $\Rightarrow$ (2) By Lemma 1, let us suppose that z is an irregular vector of T. For any $x \in X$ and any $r > 0$, put $z_r = x + \frac{z}{\|z\|} \cdot \frac{r}{2} \in B(x, r)$. Then

$$\lim_{n\to\infty} \sup \|T^n(x) - T^n(z_r)\| = \lim_{n\to\infty} \sup \left\| T^n\left(\frac{z}{\|z\|}\cdot\frac{r}{2}\right)\right\| = \left|\frac{r}{2\cdot\|z\|}\right| \cdot \left(\lim_{n\to\infty} \sup \|T^n(z)\|\right) = \infty$$

and

$$\lim_{n\to\infty} \inf \|T^n(x) - T^n(z_r)\| = \lim_{n\to\infty} \inf \left\| T^n\left(\frac{z}{\|z\|}\cdot\frac{r}{2}\right)\right\| = \left|\frac{r}{2\cdot\|z\|}\right| \cdot \left(\lim_{n\to\infty} \inf \left\|T^n(z)\right\|\right) = 0.$$

This implies that T is Li–Yorke sensitive.

(2) $\Rightarrow$ (3) This holds trivially.

(3) $\Rightarrow$ (4) $\mathrm{LY}(T) \neq \emptyset$ holds as T is spatio-temporally chaotic. We know from Lemma 1 that there exists $z \in X$ such that it is irregular for T. Then there exist strictly increasing sequences of positive integers $\{n_k\}_{k\in\mathbb{N}}$ and $\{m_k\}_{n\in\mathbb{N}}$ such that $\lim_{k\to\infty} \|T^{n_k}(z)\| = \lim_{n\to\infty} \sup \|T^n(z)\| = \infty$ and $\lim_{k\to\infty} \|T^{m_k}(z)\| = \lim_{n\to\infty} \inf \|T^n(z)\| = 0$. Let $b_1 = l_1 = 2$, $b_i = 2^{b_1+\cdots+b_{i-1}}$ and $l_i = \sum_{h=1}^{i} b_h$ for any $i > 1$; then we get an increasing sequence of positive integers $\{b_i\}_{i\in\mathbb{N}}$. And put $\{n'_k\}_{k\in\mathbb{N}}$ and $\{m'_k\}_{k\in\mathbb{N}}$, respectively, as subsequences of $\{n_k\}_{k\in\mathbb{N}}$ and $\{m_k\}_{k\in\mathbb{N}}$ such that $m'_i < n'_i$ when $i \le b_1$, $m'_i < n'_i$ when $l_{2k} < i \le l_{2k+1}$, and $n'_i < m'_i$ when $l_{2k-1} < i \le l_{2k}$ for any $k \in \mathbb{N}$. Let

$$p_i = \begin{cases} n'_i, & \text{if } i \le b_1 \text{ or } l_{2k} < i \le l_{2k+1},\ k \in \mathbb{N}, \\ m'_i, & \text{if } l_{2k-1} < i \le l_{2k},\ k \in \mathbb{N}. \end{cases}$$

Then $\{p_i\}_{i\in\mathbb{N}}$ is an increasing sequence of positive integers.

Now, we assert that $\Gamma = \mathrm{span}\{z\}$ is a distributionally chaotic set of T in $\{p_i\}_{i\in\mathbb{N}}$.

In fact, for any pair $x, y \in \Gamma$ with $x \neq y$, it is clear that there exists $\lambda \in \mathbb{C}$ such that $x - y = \lambda z$.

Since $\lim_{k\to\infty} \left\|T^{m'_k}(z)\right\| = 0$, it follows that, for any $t > 0$, there exists $N \in \mathbb{N}$ such that $\|T^{m'_i}(\lambda z)\| < t$ for each $i \ge N$. Thus

$$\begin{aligned} F^*_{xy}(t, \{p_i\}_{i\in\mathbb{N}}, T) &= \lim_{n\to\infty} \sup \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,t)} \|T^{p_i}(x) - T^{p_i}(y)\| \\ &\ge \lim_{k\to\infty} \sup \frac{1}{l_{2k}} \sum_{i=1}^{l_{2k}} \chi_{[0,t)} \|T^{p_i}(\lambda z)\| \\ &\ge \lim_{k\to\infty} \sup \frac{b_{2k}}{l_{2k}} \\ &= \lim_{k\to\infty} \sup \frac{2^{b_1+\cdots+b_{2k-1}}}{b_1+\cdots+b_{2k-1}+2^{b_1+\cdots+b_{2k-1}}} \\ &= 1. \end{aligned}$$

Let $\delta = 1$. Since $\lim_{i\to\infty} \|T^{n'_i}(\lambda z)\| = \infty$, there exists $M \in \mathbb{N}$ such that $\|T^{n'_i}(\lambda z)\| > \delta$ for each $i \ge M$. Thus

$$\begin{aligned} F_{xy}(\delta, \{p_i\}_{i\in\mathbb{N}}, T) &= \lim_{n\to\infty} \inf \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,\delta)} \|T^{p_i}(x) - T^{p_i}(y)\| \\ &\le \lim_{k\to\infty} \inf \frac{1}{l_{2k+1}} \sum_{i=1}^{l_{2k+1}} \chi_{[0,\delta)} \|T^{p_i}(\lambda z)\| \\ &\le \lim_{k\to\infty} \inf \frac{b_1+\cdots+b_{2k}}{l_{2k+1}} \\ &= \lim_{k\to\infty} \inf \frac{b_1+\cdots+b_{2k}}{b_1+\cdots+b_{2k}+2^{b_1+\cdots+b_{2k}}} \\ &= 0. \end{aligned}$$

Hence T is distributionally chaotic in $\{p_i\}_{i\in\mathbb{N}}$ as Γ is uncountable.

(4) $\Rightarrow$ (1) This is obvious, since distributional chaos in a sequence is stronger than Li–Yorke chaos for continuous self-maps. □

Theorem 2. *Assume that T is a bounded operator defined on a Banach space X. If T is Li–Yorke chaotic, then it is sensitive dependence. Conversely, it is not true.*

Proof. Applying Lemma 1, it follows that there exists $z \in X$ such that it is irregular for T. For any $x \in X$ and any $\delta > 0$, let $x_\delta = x + \frac{z}{\|z\|} \cdot \frac{\delta}{2} \in B(x, \delta)$. We have

$$\lim_{n\to\infty} \sup \|T^n(x) - T^n(x_\delta)\| = \lim_{n\to\infty} \sup \left\| T^n \left(\frac{z}{\|z\|} \cdot \frac{\delta}{2} \right) \right\| = \infty.$$

So

$$\|T^m(x) - T^m(x_\delta)\| > 1$$

for some $m \in \mathbb{N}$. This implies that T is sensitive dependence.

Conversely, let $T : \mathbb{R} \to \mathbb{R}$ be an operator given by $T(x) = 2x$ for each $x \in \mathbb{R}$. It is not difficult to check that T is a bound operator which is sensitive dependence on $\mathbb{R}$. For any pair $x, y \in \mathbb{R}$ with $x \neq y$,

$$\lim_{n\to\infty} \|T^n(x) - T^n(y)\| = \lim_{n\to\infty} |2^n(x - y)| = \infty.$$

So T is not a Li–Yorke chaotic operator. □

Theorem 3. *Assume T is a bounded operator defined on a Banach space X. Then T is sensitive dependence iff $\sup\{\|T^n\| : n \in \mathbb{N}\} = \infty$.*

Proof (⇒)**.** There exists $\epsilon > 0$ that satisfies that, for any $r > 0$, there exist $z_r \in B(\Theta, r)$ (where Θ is the zero-vector of X) and $n_r \in \mathbb{N}$ such that $\|T^{n_r}(z_r)\| > \epsilon$ holds as T is sensitive dependence. So $\|T^{n_r}\| \geq \frac{\|T^{n_r}(z_r)\|}{\|z_r\|} \geq \frac{\epsilon}{r}$ for each $r > 0$. Thus

$$\sup\{\|T^n\| : n \in \mathbb{N}\} \geq \sup\{\|T^{n_r}\| : r > 0\} \geq \sup\left\{\frac{\epsilon}{r} : r > 0\right\} = \infty.$$

(⇐) Since $\sup\{\|T^n\| : n \in \mathbb{N}\} = \infty$, for any $k \in \mathbb{N}$, there exist $n_k \in \mathbb{N}$ and $x_k \in X$ with $\|x_k\| = 1$ such that $\|T^{n_k}(x_k)\| \geq k$. For any $x \in X$ and any $\delta > 0$, put $n_\delta = \left[\frac{2}{\delta}\right] + 1$ and $x_\delta = x + \frac{\delta}{2} \cdot x_{n_\delta}$. Clearly, $\|x_\delta - x\| < \delta$. And, $\|T^{n_{n_\delta}}(x) - T^{n_{n_\delta}}(x_\delta)\| = \left\|T^{n_{n_\delta}}\left(\frac{\delta}{2} \cdot x_{n_\delta}\right)\right\| \geq \frac{\delta}{2} \cdot n_\delta \geq 1$. So T is sensitive dependence. □

For a bounded operator T, it is not difficult to check that, for any $m \in \mathbb{N}$,

$$\sup\{\|(T^m)^n\| : n \in \mathbb{N}\} \leq \sup\{\|T^n\| : n \in \mathbb{N}\}$$

and

$$\sup\{\|T^n\| : n \in \mathbb{N}\} \leq \sup\{\|(T^m)^n\| \cdot (\|T\| + 1)^m : n \in \mathbb{Z}^+\}.$$

So, we know from Theorem 3 that next corollary is obvious.

Corollary 1. *For a bounded operator T, the following statements are equivalent.*

(1) *T is sensitive dependence.*
(2) *T^n is sensitive dependence for any $n \in \mathbb{N}$.*
(3) *T^m is sensitive dependence for some $m \in \mathbb{N}$.*

3. Chaos in compositional and product operators

First, we deduce from Lemma 1, Theorem 1, and [8, Proposition 2.4] that Li–Yorke chaos is preserved under composition. i.e., for a bounded operator T defined on a Banach space X, T is Li–Yorke chaotic iff T^n is Li–Yorke chaotic for any $n \in \mathbb{N}$.

Assume that $\{(X_i, \|\cdot\|_i)\}_{i=1}^n$ are n Banach spaces and that Θ_i the zero-vector of X_i for each $i \in \{1, 2, \ldots, n\}$. We can define their Cartesian product $X^{(n)} = \prod_{i=1}^n X_i$ together with the norm $\|(x_1, \ldots, x_n)\|_n^* = \|x_1\|_1 + \cdots + \|x_n\|_n$. It is easy to see that $(X^{(n)}, \|\cdot\|_n^*)$ is a Banach space.

For each $i \in \{1, 2, \ldots, n\}$, let T_i be a bounded operator on X_i. We can also define their product operator $T_n^* = \prod_{i=1}^n T_i : X^{(n)} \to X^n$ by $T_n^*(x) = (T_1(x_1), \ldots, T_n(x_n))$ for any $x = (x_1, \ldots, x_n) \in X^{(n)}$. Then T_n^* is a bounded operator on $X^{(n)}$ and $\|T_n^*\| = \max\{\|T_1\|, \ldots, \|T_n\|\}$. Each T_i is called a factor operator of T_n^*.

Now, we shall discuss how chaotic conditions on operators carry over to their products.

Theorem 4. *Let T_i be a bounded operator defined on a Banach space X_i for each $i \in \{1, 2, \ldots, n\}$. Then T_n^* is Li–Yorke chaotic iff T_k is Li–Yorke chaotic for some $k \in \{1, 2, \ldots, n\}$.*

Proof (⇒)**.** Since T_n^* is Li–Yorke chaotic, applying Lemma 1, it follows that there exists $z = (z_1, \ldots, z_n) \in X^{(n)}$ such that z is irregular for T_n^*. For any $i \in \{1, 2, \ldots, n\}$ and any $m \in \mathbb{N}$, since $\|(T_i)^m(z_i)\|_i \leq \|(T_n^*)^m(z)\|_n^*$ and $\lim_{m\to\infty} \inf \|(T_n^*)^m(z)\|_n^* = 0$, $\lim_{m\to\infty} \inf \|(T_i)^m(z_i)\|_i = 0$ for any $i \in \{1, 2, \ldots, n\}$.

Now we assert that $\lim_{m\to\infty} \sup \|(T_k)^m(z_k)\|_k = \infty$ for some $k \in \{1, 2, \ldots, n\}$.

In fact, if $\lim_{m\to\infty} \sup \|(T_i)^m(z_i)\|_i < \infty$ for any $i \in \{1, 2, \dots, n\}$, then there exists $M > 0$ such that $\|(T_i)^m(z_i)\|_i \le M$ for any $i \in \{1, 2, \dots, n\}$ and any $m \in \mathbb{N}$. Thus, for any $m \in \mathbb{N}$,

$$\|(T_n^*)^m(z)\| = \|(T_1)^m(z_1)\|_1 + \cdots + \|(T_n)^m(z_n)\|_n \le n \cdot M.$$

This contradicts $\lim_{m\to\infty} \|(T_n^*)^m(z)\|_n^* = \infty$.

Hence, there exists $k \in \{1, 2, \dots, n\}$ such that z_k is irregular for T_k. This, together with Lemma 1, leads to T_k being Li–Yorke chaotic.

($\Leftarrow$) Let the vector ν_k be irregular for T_k, and put $\nu = (\nu_1^*, \dots, \nu_n^*)$, where $\nu_k^* = \nu_k$ and $\nu_i^* = \Theta_i$ when $i \ne k$. Then

$$\lim_{m\to\infty} \sup \|(T_n^*)^m(\nu)\|_n^* = \lim_{m\to\infty} \sup \|(T_k)^m(\nu_k)\|_k = \infty$$

and

$$\lim_{m\to\infty} \inf \|(T_n^*)^m(\nu)\|_n^* = \lim_{m\to\infty} \inf \|(T_k)^m(\nu_k)\|_k = 0.$$

That is, υ is irregular for T_n^*. By Lemma 1, we have that T_n^* is Li–Yorke chaotic.

We know from Theorem 1 that this also holds for Li–Yorke sensitivity, spatio-temporal chaos, and distributional chaos in a sequence. □

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