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# Extremal problems involving vertices and edges on odd cycles

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*Abstract*

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We investigate the minimum, taken over all graphs $G$ with $n$ vertices and at least $\lfloor n^2/4\rfloor+1$ edges, of the number of vertices and edges of $G$ which are on cycles of length $2k+1$.

## 1. Introduction

A classical problem of extremal graph theory is that of finding, for a given graph $H$, the extremal function $\mathrm{ex}(n; H)$ giving the maximum number of edges in a graph of order $n$ not containing $H$. One of the oldest and simplest results of this type is that of Turán: $\mathrm{ex}(n; K_3)=\lfloor n^2/4\rfloor$ and the only graph with $n$ vertices and $\lfloor n^2/4\rfloor$ edges which does not contain $K_3$ is the complete bipartite graph with $\lfloor n/2\rfloor$ vertices in one class and $\lceil n/2\rceil$ in the other. This graph is known as the Turán graph and denoted $T_2(n)$. Various questions are then suggested by the fact that a graph with $n\geqslant 3$ vertices and $\lfloor n^2/4\rfloor+1$ or more edges must contain a triangle, in fact an odd cycle $C_{2k+1}$ for every $k\leqslant\lfloor(n+1)/4\rfloor$, whereas $T_2(n)$ contains no odd cycle. One would like to know the minimum value, over all graphs with $n$ vertices and $\lfloor n^2/4\rfloor+1$ or more edges, of various graph theoretic functions which vanish in case there are no triangles (more generally, cycles of length $2k+1$). For example, in an early result in extremal graph theory, Radamacher proved that every graph with $n$ vertices and $\lfloor n^2/4\rfloor+1$ edges contains at least $\lfloor n/2 \rfloor$ triangles. This was generalized by one of the authors, who proved that for $n > ck$ every graph with $n$ vertices and $\lfloor n^2/4 \rfloor + k$ edges contains $k\lfloor n/2 \rfloor$ or more triangles [5].

One might expect that at least for the case of triangles, all basic questions of this sort have been answered already. However, the following simple question is open. Determine or accurately estimate the minimum, over all graphs with $n$ vertices and $\lfloor n^2/4 \rfloor + 1$ edges of $\max\{\deg(x_1) + \deg(x_2) + \deg(x_3)\}$ where $x_1, x_2, x_3$ are the vertices of a triangle. Currently, the best known results [7] only show that this minimum is between $21n/16$ and $2(\sqrt{3} - 1)n + 5$. Related questions were investigated by Caccetta, Erdős and Vijanan in [2]. In this paper, we investigate the minimum number of vertices and edges on triangles and, more generally, cycles of length $2k + 1$.

Every effort has been made to use standard terminology and notation. Generally speaking, we follow the notation of [3].

## 2. Vertices and edges on odd cycles

The main result summarizes what is known concerning vertices and edges on odd cycles of a graph with $n$ vertices and $\lfloor n^2/4 \rfloor + 1$ or more edges.

**Theorem 1.** *Let $G$ be a graph with $n$ vertices and at least $\lfloor n^2/4 \rfloor + 1$ edges.*

(a) *At least $\lfloor n/2 \rfloor + 2$ of the vertices of $G$ are on triangles. This result is sharp.*

(b) *At least $2\lfloor n/2 \rfloor + 1$ of the edges of $G$ are on triangles. This result is sharp.*

(c) *If $k \geq 2$ and $n \geq \max\{3k(3k + 1), 216(3k - 2)\}$, then at least $2(n - k)/3$ vertices of $G$ are on cycles of length $2k + 1$. This result is asymptotically best possible.*

(d) *If $k \geq 2$ is fixed then at least $11n^2/144 - O(n)$ edges of $G$ are on cycles of length $2k + 1$ as $n \to \infty$.*

**Proof of (a).** Since the desired property is preserved by the addition of edges, it suffices to prove that every graph with $n$ vertices and precisely $\lfloor n^2/4 \rfloor + 1$ edges contains at least $\lfloor n/2 \rfloor + 2$ vertices which are on triangles. The proof is by induction. The result is vacuous for $n \leq 2$ and clear for $n = 3$. Now suppose that the theorem is true in all preceding cases and let $G$ be a graph with $n \geq 4$ vertices and $\lfloor n^2/4 \rfloor + 1$ edges. We distinguish odd and even cases.

*Case* (i): *$n$ is odd.*

If $G$ has a vertex of degree $\leq (n - 1)/2$, then the deletion of such a vertex yields a graph with $n - 1$ vertices and at least

$$\frac{n^2 - 1}{4} + 1 - \frac{n - 1}{2} = \frac{(n - 1)^2}{4} + 1$$

edges. By induction, the vertex-deleted graph, and therefore $G$ itself, contains at least $(n-1)/2+2=\lfloor n/2\rfloor+2$ vertices which are on triangles. There is such a vertex of degree $\leqslant(n-1)/2$ since, otherwise, $G$ would have at least $n(n+1)/4$ edges, contradicting the fact that $n\geqslant 4$ and $G$ has exactly $(n^2-1)/4+1$ edges.

*Case* (ii): *n is even.*

In this case, the induction step follows immediately if $G$ contains a vertex which is on a triangle and has degree $n/2$ or less. The vertex-deleted subgraph then has $n-1$ vertices and at least

$$\frac{n^2}{4}+1-\frac{n}{2}=\frac{(n-1)^2-1}{4}+1$$

edges. By induction, the vertex-deleted subgraph contains at least $(n-2)/2+2$ vertices which are on triangles and it follows that $G$ contains at least $n/2+2$ vertices which are on triangles.

To complete the proof, set $k=n/2$ and assume that $G$ has $k+1$ or fewer vertices which are on triangles and that each of these vertices has degree $k+1$ or more. Let $r$ be the clique number of $G$; clearly $r\geqslant 3$. Choose $X\subseteq V(G)$ such that $\langle X\rangle\cong K_r$ and let $Y=V(G)\backslash X$ denote the set of vertices which are external to this complete subgraph. Let us count the number of $XY$-edges, in other words edges of the form $xy$ where $x\in X$ and $y\in Y$. Since each of the vertices in $X$ has degree at least $k+1$, the number of $XY$-edges is at least $r(k-r+2)$. On the other hand, since there are at most $k+1$ vertices which are on triangles and there is no $K_{r+1}$, at most $k+1-r$ vertices in $Y$ are adjacent to two or more vertices of $X$ and no vertex in $Y$ is adjacent to all of the vertices of $X$. It follows that the number of $XY$-edges is at most $(k+1-r)(r-1)+(k-1)$. However

$$r(k-r+2)>(k+1-r)(r-1)+(k-1),$$

so we have obtained a contradiction.

To see that this result is sharp, just consider the graph obtained by adding one edge to the side of the Turán graph with $\lceil n/2\rceil$ vertices. □

**Proof of (b).** The proof is by induction and follows closely the proof of part (a). Again we distinguish odd and even cases.

*Case* (i): *n is odd.*

This proof is the same as in Part (a). Alternatively, having applied induction in case there is a vertex of degree $(n-1)/2$ or less, one can notice that if each vertex of $G$ has degree at least $(n+1)/2$, then *every* edge is on a triangle.

*Case* (ii): *n is even.*

Let us first consider the case where $G$ contains a $K_4$. In this instance, we may suppose that $n\geqslant 6$ since the result is otherwise trivial. Choose $X\subseteq V(G)$ such that $\langle X\rangle\cong K_4$ and set $Y=V(G)\backslash X$. If $\langle Y\rangle$ contains at least $(n-4)^2/4+1$ edges, then the desired result follows immediately by induction since $\langle Y\rangle$ has at least $n-4$ edges which are on triangles and thus $G$ has at least $n+2$ such edges.

So suppose $\langle Y\rangle$ contains $(n-4)^2/4$ or fewer edges. Then there are $2n-9$ or more $XY$-edges. If $k$ is the number of vertices in $Y$ which are adjacent to only one vertex of $X$, then $2n-9 \leqslant k+4(n-4-k)$ and thus $k \leqslant (2n-7)/3$. It follows that the number of $XY$-edges which are on triangles is at least $(2n-9)-(2n-7)/3$ and that the total number of edges which are on triangles is at least $(2n-9)-(2n-7)/3+6$. This numbr exceeds $n+1$, so the case where $G$ contains $K_4$ is settled.

Now consider the case in which $G$ contains a $K_3$ but no $K_4$. Choose $W \subseteq V(G)$ such that $\langle W\rangle \cong K_3$ and set $Z = V(G)\backslash W$. If $W$ contains a vertex of degree $n/2$ or less, delete it and apply induction to the resulting graph. The vertex-deleted graph has at least $n-1$ edges which are on triangles and the restoration of the deleted vertex adds two such edges. Thus we may assume that each vertex of $W$ has degree at least $n/2+1$, which means that the number of $WZ$-edges is at least $3(n/2-1)$. Since $G$ contains no $K_4$, each vertex of $Z$ is adjacent to at most two vertices of $W$. If $k$ is the number of vertices of $Z$ which are adjacent to one vertex of $W$, then $3(n/2-1) \leqslant k+2(n-3-k)$ and thus $k \leqslant n/2-3$. Thus there are $3(n/2-1)-(n/2-3) = n$ or more $WZ$-edges which are on triangles so $G$ has at least $n+3$ such edges.

The example used in (a) shows that this result is sharp. □

Before giving the proof of (c), we state three results which will be used in the argument. The following three results are found in [6], [8] and [9] respectively.

**Lemma 1** (Erdős-Gallai). *Every graph with $n$ vertices and more than $(r-2)n/2$ edges contains a path of order $r$.*

**Lemma 2** (Gyárfás, Rousseau and Schelp). *If $a$, $b$, $r$ satisfy $a, b \geqslant 2r$, then every subgraph of $K(a, b)$ with more than $(a+b-2r)r$ edges contains a path of order* $2(r+1)$.

**Lemma 3** (Woodall). *Every graph with $n \geqslant 3$ vertices and at least $\lfloor n^2/4\rfloor + 1$ edges contains a cycle of length $l$ for each $l$ satisfying $3 \leqslant l \leqslant \lceil (n+3)/2\rceil$. In particular, every graph with $n \geqslant 4k-1$ vertices and at least $\lfloor n^2/4\rfloor+1$ edges contains a cycle of length* $2k+1$.

**Proof of (c).** Suppose that the statement is false and for $k \geqslant 2$ and

$$n \geqslant \max\{3k(3k+1), 216(3k-2)\},$$

let $G$ be a graph with $n$ vertices and $\lfloor n^2/4\rfloor + 1$ or more edges in which the number of vertices on cycles of length $2k+1$ is less than $2(n-k)/3$.

By deleting appropriate vertices from $G$, we can obtain a graph $H$ with $p > n/3$ vertices and $\lfloor p^2/4\rfloor + 1$ or more edges which contains a cycle of length $2k+1$ and in which every vertex which lies on such a cycle has degree at least $\lfloor p/2\rfloor + 1$. The procedure is as follows. Having deleted $r$ vertices from $G$ to obtain a graph of order $n-r$, delete a vertex which lies on a cycle of length $2k+1$ and has degree at most $\lfloor (n-r)/2 \rfloor$ in this graph if such a vertex exists. The new graph has $n-r-1$ vertices and at least $\lfloor (n-r-1)^2/4 \rfloor + 1$ edges. This procedure cannot continue to $r \geq 2n/3$ for that would imply that $G$ had $2n/3$ vertices on cycles of length $2k+1$. However, since $n/3 > 3k^2 > 4k+1$, Lemma 3 shows that each graph obtained in the procedure contains a cycle of length $2k+1$. Thus the procedure terminates with the graph $H$ as specified.

Let $S$ denote the set of all vertices of $H$ which are on cycles of length $2k+1$ and let $T = V(H)\backslash S$. Denote $|S|$ by $s$, so $|T| = p - s$. By assumption, $s < 2(p-k)/3$ since otherwise $G$ would have at least

$$2(p-k)/3 + (n-p) \geq 2(n-k)/3$$

vertices on cycles of length $2k+1$. Thus

$$2k+1 \leq s < 2(p-k)/3.$$

**Claim.** *No vertex of $T$ is adjacent to more than $k$ vertices of $S$.*

To prove this claim, suppose to the contrary that $w \in T$ is adjacent to more than $k$ vertices of $S$. If all of these vertices are on a common cycle of length $2k+1$, then two of them would have to be next nearest neighbors on this cycle and this would imply that $w$ is on a cycle of length $2k+1$. Thus we assume that $w$ is adjacent to $x_1$ which belongs to the cycle $C = (x_1, x_2, \ldots, x_{2k+1}, x_1)$ and $w$ is adjacent to $v \in S$ where $v$ is not on $C$. We consider three cases. In the first two cases, the following notion is used. Call a path on $2k$ vertices $(u_1, u_2, \ldots, u_{2k})$ *$w$-forcing* if (i) $u_1, u_{2k} \in S$, (ii) $w$ is one of the internal vertices of the path, (iii) $u_1u_{2k} \notin E(H)$ and the number of edges joining $u_1, u_{2k}$ to internal vertices on the path is at most $2k$, or $u_1u_{2k} \in E(H)$ and the number of edges joining $u_1, u_{2k}$ to internal vertices on the path is at most $2k-2$. If $(u_1, u_2, \ldots, u_{2k})$ is $w$-forcing, then the number of edges from $u_1, u_{2k}$ to vertices not on the cycle exceeds $p-2k$, so there is a vertex $z \in V(H)\backslash\{u_1, u_2, \ldots, u_{2k}\}$ which is adjacent to both $u_1$ and $u_{2k}$. This places $w$ on the $(2k+1)$-cycle $(u_1, u_2, \ldots, u_{2k}, z, u_1)$, contradicting the fact that $w \in T$. Hence $H$ contains no $w$-forcing paths.

*Case* (i): $k = 2$.

If $x_2v \notin E(H)$ then $(v, w, x_1, x_2)$ is $w$-forcing. Hence $x_2v \in E(H)$ and, in exactly the same way, $x_5v \in E(H)$. If $x_2w \in E(H)$ then $(w, x_2, v, x_5, x_1, w)$ is a five-cycle containing $w$. Hence $x_2w \notin E(H)$ and, by the same token, $x_5w \notin E(H)$. Since $w$ is not on a five-cycle, $x_3w \notin E(H)$ and $x_4w \notin E(H)$ as well. Thus $w$ is adjacent to only one vertex ($x_1$) on the cycle $C = (x_1, x_2, x_3, x_4, x_5, x_1)$. Since $w$ is adjacent to at least three vertices of $S$, it is adjacent to a second vertex $v'$ not on $C$. Now $v'$ plays the same role as does $v$, so $v'$ must be adjacent to both $x_2$ and $x_5$. Now if $x_1v \in E(H)$ then $(w, v, x_1, x_2, v', w)$ is a five-cycle containing $w$. Thus $x_1v \notin E(H)$. But then $(v, w, x_1, x_2)$ is $w$-forcing. The contradiction thus obtained completes the proof in this case.

*Case* (ii) $k = 3$.

If $x_4 v \in E(H)$ then $(w, v, x_4, x_5, x_6, x_7, x_1, w)$ is a seven-cycle containing $w$. Hence $x_4 v \notin E(H)$. Either $x_1 v \notin E(H)$ or $x_4 w \notin E(H)$; otherwise $(w, x_4, x_5, x_6, x_7, x_1, v, w)$ is a seven-cycle containing $w$. In case $x_1 v \notin E(H)$, we may assume that $x_4$ is adjacent to $x_1$ and $x_2$ and $v$ is adjacent to $x_2$ and $x_3$. Otherwise, $(v, w, x_1, x_2, x_3, x_4)$ is a $w$-forcing path. If $x_4 w \notin E(H)$ we may for the same reason assume that $x_4$ is adjacent to $x_1$ and $x_2$ and $v$ is adjacent to $x_2$ and $x_3$. Now $x_4 x_7 \notin E(H)$; otherwise $(w, v, x_2, x_3, x_4, x_7, x_1, w)$ is a seven-cycle containing $w$. Similarly, $x_2 x_7 \in E(H)$ yields the seven-cycle $(w, v, x_3, x_4, x_2, x_7, x_1, w)$ and $x_7 w \in E(H)$ yields the seven-cycle $(w, v, x_2, x_3, x_4, x_1, x_7, w)$. Hence $x_2 x_7 \notin E(H)$ and $x_7 w \notin H$. But then $(x_7, x_1, w, v, x_3, x_4)$ is a $w$-forcing path. Thus a contradiction has been obtained, concluding the proof of this case.

*Case* (iii) $k \geq 4$.

Let $u = x_5$ and note that if there were a vertex $z \notin \{x_5, \ldots, x_{2k+1}, x_1, w\}$ adjacent to both $u$ and $v$, then $(w, v, z, u, x_6, \ldots, x_{2k+1}, x_1, w)$ would be a cycle of length $2_k + 1$ containing $w$. Set

$$X_u = N(u)\setminus\{x_1, \ldots, x_{2k+1}, v, w\} \quad \text{and} \quad X_v = N(v)\setminus\{x_1, \ldots, x_{2k+1}, w\}.$$

Then $X_u$ and $X_v$ are disjoint sets. Set $a = |X_u|$, $b = |X_v|$ and $c = p - (a + b)$. Since $u$ is not adjacent to itself, $a \geq \lfloor p/2 \rfloor - 2k - 1$. Since $v$ is not adjacent to either $x_4$ or $x_{2k-1}$, we find $b \geq \lfloor p/2 \rfloor - 2k + 1$. let us call an edge of $H$ *good* if it lies on a cycle of length $2k + 1$ and *bad* otherwise. Let $B$ denote the set of bad edges. We bound $|B|$ as follows. By the Erdős-Gallai theorem (Lemma 1), in any collection of $(k - 1)a + 1$ or more edges in $\langle X_u \rangle$, some of the edges will be on paths of order $2k$ in $\langle X_u \rangle$ and hence on cycles of length $2k + 1$ in $H$. Thus the number of bad edges in $\langle X_u \rangle$ is at most $(k - 1)a$. By the same argument, the number of bad edges in $\langle X_v \rangle$ is at most $(k - 1)b$. Since $w$ is not on a cycle of length $2k + 1$, there is no path of order $2k - 6$ in the bipartite subgraph of $H$ with parts $X_u$ and $X_v$. If there were such a path, then there would be a path of order $2k - 4$ from $u$ to $v$ which together with the vertex disjoint path $(v, w, x_1, x_2, x_3, x_4, u)$ would yield a cycle of length $2k + 1$ containing $w$. Thus, by Lemma 2 (with $r = k - 4$), the total number of $X_u X_v$-edges does not exceed $(k - 4)(a + b) - 2(k - 4)^2$. Combining these bounds and using the fact that $a + b \geq 2\lfloor p/2 \rfloor - 4k$ so $c \leq 4k + 1$, we find

$$\begin{aligned} |B| &\leq (k - 1)a + (k - 1)b + (k - 4)(a + b) - 2(k - 4)^2 + \binom{c}{2} + c(a + b) \\ &= (2k - 5 + c)(p - c) - 2(k - 4)^2 + \binom{c}{2} \\ &< (2k - 5 + c)p \leq 2(3k - 2)p. \end{aligned}$$

Since $s < 2p/3$, the total number of edges (good + bad) satisfies

$$\left\lfloor \frac{p^2}{4} \right\rfloor + 1 \leq |E(H)| < \binom{2p/3}{2} + 2(3k - 2)p.$$

This inequaltiy implies

$$\frac{p}{4} < \frac{1}{3}\binom{2p}{3} + 2(3k-2),$$

and so requires $p < 72(3k-2)$. However, we have assumed $p > n/3 \geq 72(3k-2)$. Thus we have reached a contradiction, completing the proof of the claim.

Since each vertex of $T$ is adjacent to at most $k$ vertices of $S$, the number of $ST$-edges is at most $(p-s)k$. At the same time, $|T| = p - s > p/3 > 24(3k-2) > 4k-1$ and it follows from the result of Woodall (Lemma 3) that $\langle T \rangle$ has at most $\lfloor (p-s)^2/4 \rfloor$ edges. An edge count now yields

$$\left\lfloor \frac{p^2}{4} \right\rfloor + 1 \leq |E(H)| \leq \binom{s}{2} + \left\lfloor \frac{(p-s)^2}{4} \right\rfloor + (p-s)k,$$

so

$$\frac{p^2}{4} < \frac{s(s-1)}{2} + \frac{p^2 - 2ps + s^2}{4} + (p-s)k,$$

and this requires $F(s) \geq 0$ where $F$ is the quadratic function

$$F(x) = \tfrac{3}{4}x^2 - \tfrac{1}{2}(p + 2k + 1)x + pk.$$

However, $p \geq k(3k+1) > (2k+1)^2/2$ so

$$F(2k+1) = (k + \tfrac{1}{2})^2 - \frac{p}{2} < 0$$

and

$$F\left(\frac{2(p-k)}{3}\right) = \frac{k(3k+1) - p}{3} \leq 0.$$

Since $2k + 1 \leq s < 2(p-k)/3$ and $F$ is convex, $F(2k+1) < 0$ and $F(2(p-k)/3) \leq 0$ imply $F(s) < 0$. Thus we have the desired contradiction.

To see that this result is asymptotically best possible, consider the graph $G = K_s \cup T_2(n-s)$ where $s = \lceil 2n/3 \rceil + 1$. A simple calculation shows that for all $n \geq 3$,

$$\binom{s}{2} + \left\lfloor \frac{(n-s)^2}{4} \right\rfloor \geq \left\lfloor \frac{n^2}{4} \right\rfloor + 1,$$

so this graph has $n$ vertices and at least $\lfloor n^2/4 \rfloor + 1$ edges. However, only $s = \lceil 2n/3 \rceil + 1$ of the vertices of this graph are on cycles of length $2k+1$. Instead of $K_s \cup T_2(n-s)$, we may use the graph obtained by letting $K_s$ and $T_2(n+1-s)$ share one vertex. This yields a connected graph with $n$ vertices and $\binom{s}{2} + \lfloor (n+1-s)^2/4 \rfloor$ edges which has $s = \lceil 2n/3 \rceil + 1$ vertices on cycles of length $2k+1$. Thus the minimum, taken over all graphs with $n$ vertices and at least $\lfloor n^2/4 \rfloor + 1$ edges, of the number of vertices on cycles of length $2k+1$ is between $2(n-k)/3$ and $\lceil 2n/3 \rceil + 1$, at least for all sufficiently large $n$. □

The following result [4] will be used in the proof of (d).

**Lemma 4** (Edwards). *Every graph with n vertices and* $\lfloor n^2/4 \rfloor + 1$ *edges contains an edge common to at least* $n/6$ *triangles.*

**Note.** This result was conjectured by Bollobás and Erdős in [1]. To see that this is essentially best possible, consider the graph of order $n = 6k$ whose vertices are partitioned into six independent sets $X(r, c)(r = 1, 2, c = 1, 2, 3)$ with cardinalities

$$|X(2, 1)| = |X(2, 3)| = k - 1, \qquad |X(1, 3)| = |X(2, 2)| = k,$$
$$|X(1, 1)| = |X(1, 2)| = k + 1.$$

Join distinct sets $X(r, c)$ and $X(r', c')$ completely if either $r = r'$ or $c = c'$. This gives a graph with $9k^2 + 1 = n^2/4 + 1$ edges in which every edge is on at most $k + 1$ triangles. To the knowledge of the authors, the proof of Lemma 4 has never been published. In his unpublished manuscript, Edwards proves a stronger result. Let $G$ be a graph with $n$ vertices and $m$ edges and let $f_3(G)$ denote the largest integer such that $G$ contains $f_3(G)$ triangles with a common edge. Set

$$c_v^2 = \frac{1}{n} \sum_{i=1}^{n} (1 - d_i/\bar{d})^2$$

where $d_1, d_2, \ldots, d_n$ are the degrees of the vertices of $G$ and $\bar{d} = 2m/n$ is the average degree. Edwards proves that if $m \geqslant n^2/(4(1 + c_v^2))$ and $G$ is not bipartite, then

$$f_3(G) \geqslant \frac{2m}{n}(1 + c_v^2) - \frac{n}{3} = \frac{1}{2m} \sum_{i=1}^{n} d_i^2 - \frac{n}{3}.$$

As an immediate corollary, if $m > \lfloor n^2/4 \rfloor$ then

$$f_3(G) \geqslant \frac{2m}{n} - \frac{n}{3}.$$

In particular, setting $m = \rceil n^2/4 \rfloor + 1$ we have the result quoted in Lemma 4.

**Proof of (d).** From Lemma 4 we know that $G$ contains adjacent vertices $u$ and $v$ such that $|N(u) \cap N(v)| \geqslant n/6$. Choose $X \subseteq N(u) \cap N(v)$ so that $m = |X| \geqslant n/6$ and set $X' = X \cup \{u, v\}$ and $Y = V(G) \backslash X'$ and $m = |X|$. Every edge which is on a path of order $2k - 2$ in $\langle X \rangle$ is on a cycle of length $2k + 1$ in $\langle X' \rangle$ since if $(x_1, x_2, \ldots, x_{2k-2})$ is such a path and we choose a vertex $x_a \in X$ not on this path, then $(x_1, x_2, \ldots, x_{2k-2}, u, x_a, v, x_1)$ is a cycle as claimed. In view of the Erdős-Gallai result (Lemma 1), the number of edges of $\langle X' \rangle$ which are not on cycles of length $2k + 1$ is at most $(2k - 4)m/2 + 2m + 1 = km + 1$. (The $2m + 1$ edges incident with $u$ and $v$ could fail to be on cycles of length $2k + 1$.) Each edge which is on a path or order $2k - 1$ alternating between $X$ and $Y$ and beginning and ending in $X$ is on a cycle of length $2k + 1$ in $G$ since if $(x_1, y_1, \ldots, x_{k-1}, y_{k-1}, x_k)$ is such a path then $(x_1, y_1, \ldots, x_k, u, v, x_1)$ is a cycle as claimed. Using Lemma 2 it follows that if the number of $X'Y$-edges which are not on cycles of length $2k+1$ is at most $[(n-2)-2(k-1)](k-1)+2(n-m-2)$. [The $2(n-m-2)$ accounts for the possible edges joining $u$ and $v$ with $Y$ which could fail to be on cycles of length $2k+1$. Since $k$ is fixed, $n \to \infty$ and we are given the freedom to choose $X \subseteq N(u) \cap N(v)$, we may certainly assume that $\min(m+2, n-m-2) \geq 2(k-1)$, so Lemma 2 applies.] Finally, the number edges in $\langle Y' \rangle$ which are not on cycles of length $2k+1$ is at most $\lfloor (n-m-2)^2/4 \rfloor$. It follows that the number of edges of $G$ which are on cycles of length $2k+1$ is at least

$$\left\lfloor \frac{n^2}{4} \right\rfloor + 1 - (km+1) - [(n-2)-2(k-1)](k-1) - 2(n-m-2) - \left\lfloor \frac{(n-m-2)^2}{4} \right\rfloor.$$

Since $m \geq n/6$, the dominant contribution to the above expression is

$$\frac{n^2}{4} - \frac{(n-m)^2}{4} \geq \frac{11n^2}{144},$$

while all others terms are $O(n)$. We thus obtain the stated result. □

## 3. An open problem

The example used to show that the result in part (c) is asymptotically best possible also suggests that the desired minimum in (d) is asymptotic to $2n^2/9$ and not $11n^2/144$. This example has $\binom{s}{2}$ edges on cycles of length $2k+1$ where $s = \lceil 2n/3 \rceil + 1$, and we know of no better example. Although our result in (d) shows at least that the desired minimum grows quadratically with $n$, there is no suggestion that it is best possible. We thus close with the following conjecture.

**Conjecture 1.** If $k \geq 2$ and $G$ is a graph with $n$ vertices and at least $\lfloor n^2/4 \rfloor + 1$ edges, then at least $2n^2/9 - O(n)$ edges of $G$ are on cycles of length $2k+1$.

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