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Note

Decomposition formulas of zeta functions of graphs and digraphs

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Abstract

We give a decomposition formula of the zeta function of a regular covering of a graph G with respect to equivalence classes of prime, reduced cycles of G. Furthermore, we give a decomposition formula of the zeta function of a g-cyclic Γ-cover of a symmetric digraph D with respect to equivalence classes of prime cycles of D, for any finite group Γ and $g \in \Gamma$.

Keywords: Graph covering; Digraph covering; Zeta function

1. Introduction

Graphs and digraphs treated here are finite and simple. Let $G=(V(G),E(G))$ be a connected graph with vertex $V(G)$ and arc set $E(G)$, and D the symmetric digraph corresponding to G. Note that $E(G)=E(D)$. We also refer D as a graph G. For $e=(u,v)\in D(G)$, set $u=o(e)$ and $v=t(e)$. Furthermore, let $e^{-1}=(v,u)$ be the *inverse* of $e=(u,v)$.

A *path P of length n* in D (or G) is a sequence $P=(e_1,\dots,e_n)$ of n arcs such that $e_i\in E(G), t(e_i)=o(e_{i+1})(1\leqslant i\leqslant n-1)$. Set $|P|=n, o(P)=o(e_1)$ and $t(P)=t(e_n)$. Also, P is called $(o(P),t(P))$*-path*. We say that a path $P=(e_1,\dots,e_n)$ has a *backtracking* if $e_{i+1}^{-1}=e_i$ for some $i(1\leqslant i\leqslant n-1)$. A (v,w)-path is called a *v-cycle* (or *v-closed path*) if $v=w$. The *inverse cycle* of a cycle $C=(e_1,\dots,e_n)$ is the cycle $C^{-1}=(e_n^{-1},\dots,e_1^{-1})$.

We introduce an equivalence relation between cycles. Such two cycles $C_1=(e_1,\dots,e_m)$ and $C_2=(f_1,\dots,f_m)$ are called *equivalent* if $f_j=e_{j+k}$ for all j. The inverse cycle of C is not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle

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C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is *reduced* if both C and C^2 have no backtracking. Furthermore, a cycle C is *prime* if it is not a multiple of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G for a vertex v of G.

The (*Ihara*) *zeta function* of a graph G is defined to be a function of $u \in \mathbf{C}$ with u sufficiently small, by

$$\mathbf{Z}(G,u) = \mathbf{Z}_G(u) = \prod_{[C]} (1 - u^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G, and $|C|$ is the length of C (see [1,11]).

Zeta functions of graphs started from zeta functions of regular graphs by Ihara [6]. In [6], he showed that their reciprocals are explicit polynomials. Hashimoto [5] treated multivariable zeta functions of bipartite graphs. Bass [1] generalized the Ihara's result on the zeta function of a regular graph to an irregular graph, and showed that its reciprocal is a polynomial. Various proofs of Bass's Theorem were given by [3,7,11]. Mizuno and Sato [9] obtained a decomposition formula for the zeta function of a regular covering of a graph.

Cycles, prime cycles and reduced cycles in a simple digraph which is not symmetric are defined similarly to the case of a symmetric digraph. Let D be a connected digraph, and N_m the number of all cycles with length m in D (we do not require that cycles are reduced). Then, the *zeta function* of a digraph D is defined to be a function of $u \in \mathbf{C}$ with u sufficiently small, by

$$\mathbf{Z}(D,u) = \mathbf{Z}_D(u) = \exp\left(\sum_{m \geqslant 1} \frac{N_m}{m} u^m \right).$$

Let D have n vertices $v_1, \dots, v_n$. The *adjacency matrix* $\mathbf{A} = \mathbf{A}(D) = (a_{ij})$ of D is the square matrix of order n such that $a_{ij} = 1$ if there exists an arc starting at the vertex v_i and terminating at the vertex v_j, and $a_{ij} = 0$ otherwise. Mizuno and Sato [10] gave a determinant expression and an Euler product expression of the zeta function of a digraph: $\mathbf{Z}(D,u)^{-1} = \det(\mathbf{I} - \mathbf{A}(D)u) = \prod_{[C]}(1 - u^{|C|})$, where $[C]$ runs over all equivalence classes of prime cycles of D.

Kotani and Sunada [7] treated zeta functions of strongly connected digraphs. In [7], they stated a connection between zeta functions of graphs and that of strongly connected digraphs, and gave a new proof of Bass's Theorem by using the connection. Let $G = (V, E)$ be a connected non-circuit graph. Then the *oriented line graph* $L(\vec{G}) = (V_L, E_L)$ of G is defined as follows:

$$V_L = E; \; E_L = \{(e_1, e_2) \in E \times E \,|\, \bar{e}_1 \neq e_2, t(e_1) = o(e_2)\}.$$

There exist no reduced cycles in the oriented line graph. Thus, there is a one-to-one correspondence between prime cycles in $L(\vec{G})$ and prime, reduced cycles in G, and so $\mathbf{Z}_G(u) = \mathbf{Z}_{L(\vec{G})}(u)$.

2. Zeta functions of regular coverings of graphs

Let G be a graph and Γ a finite group. Let $E(G)$ be the arc set of the symmetric digraph corresponding to G. Then a mapping $\alpha : E(G) \to \Gamma$ is called an *ordinary voltage assignment* if $\alpha(v,u) = \alpha(u,v)^{-1}$ for each $(u,v) \in E(G)$. The pair (G,α) is called an *ordinary voltage graph*. The *derived graph* G^α of the ordinary voltage graph (G,α) is defined as follows: $V(G^\alpha) = V(G) \times \Gamma$ and $((u,h),(v,k)) \in E(G^\alpha)$ if and only if $(u,v) \in E(G)$ and $k = h\alpha(u,v)$. The *natural projection* $\pi : G^\alpha \to G$ is defined by $\pi(u,h) = u, (u,h) \in V(G^\alpha)$. The graph G^α is called a *derived graph covering* of G with voltages in Γ or a Γ*-covering* of G. The natural projection π commutes with the right multiplication action of the $\alpha(e), e \in E(G)$ and the left action of Γ on the fibers: $g(u,h) = (u,gh), g \in \Gamma$, which is free and transitive. Thus, the Γ-covering G^α is a $|\Gamma|$-fold regular covering of G with covering transformation group Γ. Furthermore, every regular covering of a graph G is a Γ-covering of G for some group Γ (see [4]).

Let G be a connected graph, Γ a finite group and $\alpha : E(G) \to \Gamma$ an ordinary voltage assignment. Then we define the *net voltage* $\alpha(P)$ of each path $P = (v_1, \ldots, v_l)$ of G by $\alpha(P) = \alpha(v_1, v_2) \cdots \alpha(v_{l-1}, v_l)$. We denote the order of $g \in \Gamma$ by *ord*(g).

Theorem 1. *Let G be a connected graph, Γ a finite group with n elements, and $\alpha : E(G) \to \Gamma$ an ordinary voltage assignment. Suppose that the Γ-covering G^α of G is connected. Then the reciprocal of the zeta function of G^α is*

$$\mathbf{Z}(G^\alpha, u)^{-1} = \prod_{[C]} (1 - u^{|C| \mathrm{ord}(\alpha(C))})^{n/\mathrm{ord}(\alpha(C))},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G.

Proof. Let C be any prime, reduced cycle of G^α and $\pi(C) = C_0^k$, where C_0 is a prime, reduced cycle of G and $\pi : G^\alpha \to G$ is the natural projection. Let $m = \mathrm{ord}(\alpha(C_0))$. By [4, Theorem 2.1.3], the preimage of C_0 in G^α is the union of n/m disjoint cycles with length $m|C_0|$, and so $k = m$. Therefore, it follows that

$$\mathbf{Z}(G^\alpha, u)^{-1} = \prod_{[C_0]} (1 - u^{|C_0| \mathrm{ord}(\alpha(C_0))})^{n/\mathrm{ord}(\alpha(C_0))},$$

where $[C_0]$ runs over all equivalence classes of prime, reduced cycles of G. □

3. Zeta functions of cyclic Γ-covers

Let D be a symmetric digraph and Γ a finite group. A function $\alpha : E(D) \to \Gamma$ is called *alternating* if $\alpha(y,x) = \alpha(x,y)^{-1}$ for each $(x,y) \in E(D)$. For $g \in \Gamma$, a *g-cyclic Γ-cover* $D_g(\alpha)$ of D is the digraph defined as follows (see [8]): $V(D_g(\alpha)) = V(D) \times \Gamma$, and $((v,h),(w,k)) \in E(D_g(\alpha))$ if and only if $(v,w) \in E(D)$ and $k = h\alpha(v,w)g^{-1}$. The *natural projection* $\pi : D_g(\alpha) \to D$ is a function from $V(D_g(\alpha))$ onto $V(D)$ which erases the second coordinates. A digraph D' is called a *cyclic Γ-cover* of D if D' is a g-cyclic Γ-cover of D for some $g \in \Gamma$.

The pair (D,α) of D and α can be considered as the ordinary voltage graph $(\tilde{D},\alpha)$ of the underlying graph $\tilde{D}$ of D. Thus the 1-cyclic Γ-cover $D_1(\alpha)$ corresponds to the Γ-covering $\tilde{D}^{\alpha}$, where 1 is the unit of Γ.

Let D be a connected symmetric digraph, Γ a finite group and $\alpha : E(D) \to \Gamma$ an alternating function. Furthermore, let $g \in \Gamma$. Then we define the function $\alpha_g : E(D) \to \Gamma$ as follows: $\alpha_g(v,w) = \alpha(v,w)g^{-1}, (v,w) \in E(D)$. For each path $P=(v_1,\dots,v_l)$ of D, let $\alpha_g(P) = \alpha(v_1,v_2)g^{-1}\cdots\alpha(v_{l-1},v_l)g^{-1}$. Note that, if $g^2 \neq 1$, then α_g is not alternating, and so $D_g(\alpha)$ is not a Γ-covering of the underlying graph of D.

For $h \in \Gamma$, the *permutation matrix* $\mathbf{P}_h = (p_{ij})$ of h in Γ is the square matrix of order n such that $p_{ij}=1$ if $g_i h = g_j$, and $p_{ij}=0$ otherwise, where $n=|\Gamma|$ and $\Gamma=\{g_1=1,g_2,\dots,g_n\}$. A cyclic permutation $(h_1\ h_2 \dots h_m)$ is the permutation such that $h_1 \to h_2 \to \cdots \to h_m \to h_1$.

Theorem 2. *Let D be a connected symmetric digraph, Γ a finite group with n elements, $g \in \Gamma$ and $\alpha : E(D) \to \Gamma$ an alternating function. Then the reciprocal of the zeta function of $D_g(\alpha)$ is*

$$\mathbf{Z}(D_g(\alpha),u)^{-1} = \prod_{[C]} (1 - u^{|C|\operatorname{ord}(\alpha_g(C))})^{n/\operatorname{ord}(\alpha_g(C))},$$

where $[C]$ runs over all equivalence classes of prime cycles of D.

Proof. By Theorem 3 and Corollary 3 in [10], we have

$$\mathbf{Z}(D_g(\alpha),u)^{-1} = \prod_{[C]} \prod_{\rho} \det(\mathbf{I} - \rho(\alpha_g(C))u^{|C|})^f,$$

where ρ runs over all irreducible representations of Γ and $f = \deg \rho$. The property of the right regular representation of a finite group implies that

$$\prod_{\rho} \det(\mathbf{I} - \rho(\alpha_g(C))u^{|C|})^f = \det(\mathbf{I}_n - \sigma(\alpha_g(C))u^{|C|}) = \det(\mathbf{I}_n - \mathbf{P}_{\alpha_g(C)}u^{|C|}),$$

where $\mathbf{P}_h$ is the permutation matrix of h in Γ, and σ is the right regular representation of Γ (see [2]).

Let $\gamma = \alpha_g(C)$, $H = \langle \gamma \rangle$ the subgroup of Γ generated by γ, $m = \operatorname{ord}(\gamma)$ and $k = n/m$. Furthermore, Let $\{h_1 = 1, h_2, \dots, h_k\}$ be a set of all representatives of Γ/H. Then the disjoint cycle decomposition of $\sigma(\gamma)$ is

$$\sigma(\gamma) = (1\ \gamma \cdots \gamma^{m-1})(h_2\ h_2\gamma \cdots h_2\gamma^{m-1}) \cdots (h_k\ h_k\gamma \cdots h_k\gamma^{m-1}).$$

Thus,

$$\det(\mathbf{I}_n - \mathbf{P}_\gamma u^{|C|}) = \det(\mathbf{I}_m - \mathbf{P}'_\gamma u^{|C|})^k = (1 - u^{|C|m})^k,$$

where $\mathbf{P}'_\gamma$ is the permutation matrix of γ in H. Therefore, the result follows. □

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