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A Continuity Property Related to Kuratowski's Index of Non-compactness, Its Relevance to the Drop Property, and Its Implications for Differentiability Theory

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We define and study the properties of α upper semi-continuity, a new continuity property for set-valued mappings from a topological space into subsets of a metric space, expressed in terms of Kuratowski's index of non-compactness. This α upper semi-continuity is related closely to the usual upper semi-continuity but significantly α upper semi-continuous minimal weak* cuscos from a Baire space into subsets of the dual of any Banach space are generically single-valued. Kuratowski's index of non-compactness has been used to study the drop property and α upper semi-continuity is dual to property α studied there. Uniform α upper semi-continuity of the duality mapping is dual to nearly uniform rotundity properties. Importantly, α upper semi-continuity has application in differentiability theory, providing another characterisation for Asplund spaces. An examination of an associated concept, α-denting point for sets, yields still further advances concerning the differentiability of convex functions on a large class of Banach spaces.

1. INTRODUCTION

For a bounded set E in a metric space X, the Kuratowski index of non-compactness of E is

$$\alpha(E) \equiv \inf\{r : E \text{ is covered by a finite family of sets of diameter less than } r\}.$$

Recently, this index of non-compactness has been used to give characterisations of the drop and uniform version of the drop properties for a Banach space.

Given a Banach space X over the real numbers, with dual X^*, we denote the closed unit ball of X by $B(X) \equiv \{x \in X : \|x\| \leqslant 1\}$ and the unit sphere of X by $S(X) \equiv \{x \in X : \|x\| = 1\}$. Given $f \in S(X^*)$ and $\delta > 0$, the slice of $B(X)$

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defined by f and δ is the subset $S(B(X), f, \delta) \equiv \{x \in B(X) : f(x) > 1 - \delta\}$. X is said to have property α if given $\varepsilon > 0$ and $f \in S(X^*)$ there exists a $\delta(\varepsilon, f) > 0$ such that $\alpha(S(B(X), f, \delta)) < \varepsilon$. Rolewicz has shown, [15, p. 32], that X has the drop property if and only if it has property α.

Given a Banach space X, for each $x \in S(X)$ we denote by $D(x)$ the set $\{f \in S(X^*) : f(x) = 1\}$. The set-valued mapping $x \to D(x)$ from $S(X)$ into subsets of $S(X^*)$ is called the duality mapping on $S(X)$. Recently it has been shown [5, p. 505] that X has the drop property if and only if the duality mapping $f \to D(f)$ on $S(X^*)$ is upper semi-continuous and compact valued in $S(X^{**})$. Now X is said to have property $U\alpha$ if given $\varepsilon > 0$ there exists a $0 < \delta(\varepsilon) < 1$ such that $\alpha(S(B(X), f, \delta)) < \varepsilon$ for all $f \in S(X^*)$. It is natural to ask about the corresponding duality mapping characterisation for this uniform case. The solution to this problem involves the definition of a generalised continuity condition given in terms of the Kuratowski index of non-compactness. But the significance of this generalised continuity condition is much more far-reaching than the solution of the original problem.

We begin in Section 2 with a definition of this generalised continuity condition called α upper semi-continuity for set-valued mappings from a toplogical space into subsets of a metric space. We give a characterisation in terms of upper semi-continuity and show that any minimal weak (or weak *) usco (or cusco) from a Baire space into subsets of the dual of a Banach space, which is α upper semi-continuous on a dense subset of its domain, is single-valued and norm upper semi-continuous on a dense G_δ subset of its domain.

The drop property has been investigated for a bounded closed convex set K with $0 \in \operatorname{int} K$ in a Banach space X and has been characterised in terms of the upper semi-continuity of the subdifferential mapping $f \to \partial p(f)$ from X^* into subsets of X^{**} of the gauge p of the polar K° in X^*, [6, p. 383]. In Section 3 we show that there is a satisfying duality between the α upper semi-continuity of the subdifferential mapping $x \to \partial p(x)$ from X into subsets of X^* of the gauge p of a bounded closed convex set K with $0 \in \operatorname{int} K$ and the property α of its polar K° in X^*. We continue to explore this duality and solve our initial problem by showing that a Banach space is nearly uniformly rotund if and only if the duality mapping $f \to D(f)$ on X^* is uniformly α upper semi-continuous on $S(X^*)$. We observe that although there is in general a close relation between upper semi-continuity and α upper semi-continuity for set-valued mappings from a Baire subset of a Banach space into subsets of its dual, the uniformisation of these two continuity properties produces quite distinct conditions.

The single-valuedness property established in Section 2 suggests an application in determining conditions under which continuous convex functions on a Banach space are generically Fréchet differentiable. In Section 4 we use the Kuratowski index of non-compactness to define α-denting points of a set in a Banach space and we derive another characterisation for Asplund spaces. We also extend a recent result, [8, Theorem 3.5], to show that on a Banach space X which can be equivalently renormed to have every point of $S(X)$ an α-denting point of $B(X)$, a continuous convex function on an open convex subset of X^* is generically Fréchet differentiable provided that the set of points where the function has a weak * continuous subgradient is residual in its domain.

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2. α Upper Semi-Continuity

a. *The Relation with USCO Mappings*

Consider a set-valued mapping Φ from a topological space A into subsets of a topological space X. Φ is said to be upper semi-continuous at $t \in A$ if, given an open set W containing $\Phi(t)$ there exists an open neighbourhood U of t such that $\Phi(U) \subseteq W$. For brevity we refer to Φ being upper semi-continuous at $t \in A$ as an usco mapping at t if $\Phi(t)$ is non-empty and compact. If X is a linear topological space we refer to Φ being upper semi-continuous at $t \in A$ as a cusco mapping at t if $\Phi(t)$ is non-empty, convex and compact. For a set-valued mapping Φ from A into subsets of a metric space X we introduce another upper semi-continuity property defined by the Kuratowski index of non-compactness. We say that Φ is α upper semi-continuous at $t \in A$ if, given $\varepsilon > 0$ there exists an open neighbourhood U of t such that $\alpha(\Phi(U)) < \varepsilon$.

The characteristic function on the rational numbers is a simple example of a single-valued mapping on the real line which is nowhere continuous but everywhere α upper semi-continuous. Nevertheless, for a set-valued mapping into subsets of a complete metric space there is a close relation between these two continuity properties when the mapping satisfies a particular intersection property for images of a local base. The following property of usco mappings reveals the form of the intersection property required.

2.1. Proposition. *An usco mapping Φ from a topological space A into subsets of a Hausdorff space X satisfies the property that for each $t \in A$, $\Phi(t) = \bigcap \{\overline{\Phi(U)} : U \in \mathscr{B}\}$ where $\mathscr{B}$ is a local base for t and also for closure with respect to any topology stronger than that given for X.*

Proof. Suppose there exists for some $t \in A$ an $x_0 \in \bigcap \{\overline{\Phi(U)} : U \in \mathscr{B}\} \backslash \Phi(t)$. Then since $\Phi(t)$ is compact and X is Hausdorff there exist disjoint open sets V and W in X such that $\Phi(t) \subseteq V$ and $x_0 \in W$. Now Φ is upper semi-continuous at t so there exists some $U \in \mathscr{B}$ such that $\Phi(U) \subseteq V$. But this contradicts $x_0 \in \bigcap \{\overline{\Phi(U)}: U \in \mathscr{B}\}$. ∎

For α upper semi-continuous mappings we show that the intersection property of the form given in this proposition implies a nested sequence property which is more amenable for our purposes.

We will use the following elementary properties of the Kuratowski index of non-compactness for bounded subsets of a metric space X (see [1, pp. 4–8]),

(i) if $E \subseteq F$ then $\alpha(E) \leqslant \alpha(F)$,

(ii) $\alpha(\bar{E}) = \alpha(E)$, where $\bar{E}$ denotes the closure of E,

for X complete,

(iii) $\alpha(E) = 0$ if and only if $\bar{E}$ is compact,

for X a normed linear space,

(iv) $\alpha(kE) = |k|\, \alpha(E)$ for all real k,

(v) $\alpha(\text{co } E) = \alpha(E)$, where co E denotes the convex hull of E.

It is often more convenient to work with what is sometimes called the ball index of non-compactness which we define as

$$\chi(E) \equiv \inf\{r : E \text{ is covered by a finite family of open balls of radius less than } r\}$$

where the centres of covering balls do not necessarily belong to E. Now χ has all the properties (i)–(v). Both indices are equivalent in the sense that $\chi(E) \leqslant \alpha(E) \leqslant 2\chi(E)$. Because of this equivalence we will generally speak of Kuratowski's index of non-compactness but mostly work with covers by open balls. Sometimes we will find it convenient for our argument to consider a cover by closed balls rather than open balls but clearly the ball index of non-compactness remains the same. For a metric space (X, d), given $x_0 \in X$ and $r > 0$ we will denote by $B(x_0; r)$ the open ball $\{x \in X: d(x, x_0) < r\}$ and by $B[x_0; r]$ the closed ball $\{x \in X: d(x, x_0) \leqslant r\}$. For any set E in a topological space X we will denote by $C(E)$ the complement of E in X.

2.2. Lemma. *Consider a set-valued mapping Φ from a topological space A into subsets of a complete metric space X which is α upper semi-continuous at $t \in A$. If $\Phi(t) = \bigcap \{\overline{\Phi(U)}: U \in \mathscr{B}\}$ where $\mathscr{B}$ is a local base for t, then there exists a nested sequence $\{U_n\}$ of open neighbourhoods of t such that $\Phi(t) = \bigcap_1^\infty \overline{\Phi(U_n)}$ and* $\lim_{n \to \infty} \alpha(\overline{\Phi(U_n)}) = 0$.

Proof. Since Φ is α upper semi-continuous at t we can choose a nested sequence $\{U_n\}$ of open neighbournoods of t such that $\lim_{n\to\infty} \alpha(\overline{\Phi(U_n)}) = 0$. Then $\bigcap_1^\infty \overline{\Phi(U_n)}$ is compact and we consider for $k \in \mathbb{N}$,

$$K \equiv \bigcap_1^\infty \overline{\Phi(U_n)} \Big\backslash \bigcup \left\{ B\left(x; \frac{1}{k}\right) : x \in \Phi(t) \right\}$$

which is also compact (and possibly empty). Suppose that for every $U \in \mathscr{B}$, $\overline{\Phi(U)} \cap K \neq \varnothing$. Then for all finite subfamilies $\{U_{\alpha_1}, ..., U_{\alpha_n}\}$ of $\mathscr{B}$, $\varnothing \neq \overline{\Phi(\bigcap_1^n U_{\alpha_i})} \cap K \subseteq \bigcap_1^n \overline{\Phi(U_{\alpha_i})} \cap K$. So by the finite intersection property, $\varnothing \neq \bigcap \{\overline{\Phi(U)} : U \in \mathscr{B}\} \cap K = \Phi(t) \cap K$. But this contradicts the fact that $\Phi(t) \subseteq C(K)$. So there is some $V_k \in \mathscr{B}$ such that $\overline{\Phi(V_k)} \cap K = \varnothing$. We now define another nested sequence $\{U_n'\}$ of open neighbourhoods of t by $U_n' \equiv U_n \cap \bigcap_{i=1}^n V_i$. For $k \in \mathbb{N}$ we have

$$\begin{aligned} \Phi(t) &\subseteq \bigcap_{n=1}^\infty \overline{\Phi(U_n')} \subseteq \bigcap_{n=1}^\infty \overline{\Phi(U_n \cap V_k)} \\ &\subseteq \bigcap_{n=1}^\infty \overline{\Phi(U_n)} \cap \overline{\Phi(V_k)} \subseteq \bigcup \left\{ B\left(x; \frac{1}{k}\right) : x \in \Phi(t) \right\}. \end{aligned}$$

So $\Phi(t) = \bigcap_{n=1}^\infty \overline{\Phi(U_n')}$. Clearly as $U_n' \subseteq U_n$, $\lim_{n\to\infty} \alpha(\overline{\Phi(U_n')}) = 0$. ∎

To establish the relation between α upper semi-continuity and upper semi-continuity we will use the following generalisation of Cantor's intersection theorem due to Kuratowski, [9, p. 303].

2.3. LEMMA. *Given a complete metric space and a nested sequence of non-empty closed sets* $\{F_n\}$, $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$ *with the property that* $\lim_{n\to\infty} \alpha(F_n) = 0$, *then* $\bigcap_1^\infty F_n$ *is non-empty and compact.*

2.4. THEOREM. *A set-valued mapping Φ from a topological space A into subsets of a complete metric space X is an usco at $t \in A$ if and only if Φ is α upper semi-continuous at t and $\Phi(t) = \bigcap \{\overline{\Phi(U)} : U \in \mathscr{B}\}$ where $\mathscr{B}$ is a local base for t.*

Proof. If $\Phi(t)$ is compact, given $\varepsilon > 0$ there exists $\{x_1, x_2, ..., x_n\} \subseteq X$ such that $\Phi(t) \subseteq \bigcup_1^n B(x_i; \varepsilon)$. If Φ is also upper semi-continuous at t there exists an open neighbourhood U of t such that $\Phi(U) \subseteq \bigcup_1^n B(x_i; \varepsilon)$, so $\alpha(\Phi(U)) < 2\varepsilon$; that is, Φ is α upper semi-continuous at t. From Proposition 2.1 we have that $\Phi(t) = \bigcap \{\overline{\Phi(U)} : U \in \mathscr{B}\}$ where $\mathscr{B}$ is a local base for t.

Conversely, if Φ is α upper semi-continuous at t then $\alpha(\Phi(t)) = 0$ and so $\Phi(t)$ is compact. If $\Phi(t) = \bigcap \{\overline{\Phi(U)}; U \in \mathscr{B}\}$ where $\mathscr{B}$ is local base for t then from Lemma 2.2 we have that there exists a nested sequence $\{U_n\}$

of open neighbourhoods of t such that $\Phi(t)=\bigcap_1^\infty \overline{\Phi(U_n)}$ and $\lim_{n\to\infty} \alpha(\overline{\Phi(U_n)})=0$. If there exists an open set W in X containing $\Phi(t)$ such that $\overline{\Phi(U_n)}\backslash W\neq\varnothing$ for each $n\in\mathbb{N}$, then $\{\overline{\Phi(U_n)}\backslash W\}$ is a nested sequence of closed sets, and since $\overline{\Phi(U_n)}\backslash W\subseteq\overline{\Phi(U_n)}$ we have that $\lim_{n\to\infty}\alpha(\overline{\Phi(U_n)}\backslash W)=0$. So from Lemma 2.3 we conclude that $\bigcap_1^\infty\{\overline{\Phi(U_n)}\backslash W\}\neq\varnothing$. But this contradicts the fact that $\Phi(t)=\bigcap_1^\infty\overline{\Phi(U_n)}$. So given an open set W in X containing $\Phi(t)$ there exists an $n_0\in\mathbb{N}$ such that $\Phi(U)\subseteq W$ for all $U\subseteq U_{n_0}$; that is, Φ is upper semi-continuous at t. ∎

We note that without upper semi-continuity or α upper semi-continuity the intersection property for images of a local base does not necessarily imply the nested sequence property. The identity mapping id from an uncountable set X with the cofinite topology into X with the discrete metric is nowhere continuous and nowhere α upper semi-continuous. But for every $t\in X$, $\text{id}(t)=\bigcap\{\overline{\text{id}(U)}; U\in\mathscr{B}\}$ where $\mathscr{B}$ is a local base for t in the cofinite topology and for any sequence $\{U_n\}$ in the local base $\mathscr{B}$, $\text{id}(t)\neq\bigcap_1^\infty\overline{\text{id}(U_n)}$.

b. *Single-Valued Implications*

One of the great advantages of the generalised continuity condition we have introduced is that significant classes of α upper semi-continuous set-valued mappings from a Baire space into subsets of a metric space are single-valued on a dense G_δ subset of their domain. This is a consequence of the compactness type arguments which are entailed by Kuratowski's index of non-compactness.

An usco (cusco) from a topological space A into subsets of a topological space X (linear topological space X) is said to be minimal if its graph does not contain the graph of any other usco (cusco) with the same domain. We will use the property of minimal uscos (cuscos) given in the following characterisation.

2.5. Lemma. *Consider an usco (cusco) Φ from a topological space A into subsets of a Hausdorff space (separated linear topological space) X. Then Φ is a minimal usco (cusco) if and only if for any open set V in A and closed (closed and convex) set K in X where $\Phi(V)\nsubseteq K$ there exists a non-empty open set $V'\subseteq V$ such that $\Phi(V')\cap K=\varnothing$.*

Proof. Given that Φ is a minimal usco (cusco), suppose that there exists a $t_1\in A$ such that $\Phi(t_1)\cap K=\varnothing$. Then $C(K)$ is open and contains $\Phi(t_1)$. Since Φ is upper semi-continuous at t_1 there exists an open neighbourhood V' of t_1 contained in V such that $\Phi(V')\subseteq C(K)$. Suppose that $\Phi(t)\cap K\neq\varnothing$ for each $t\in V$. Then the set-valued mapping Φ' from A into subsets of X defined by

$$\begin{aligned} \Phi'(t) &= \Phi(t) \cap K \qquad \text{for} \quad t \in V \\ &= \Phi(t) \qquad\quad\;\; \text{for} \quad t \notin V \end{aligned}$$

is an usco (cusco) whose graph is strictly contained in that of Φ. But Φ is minimal so $\Phi(V) \subseteq K$.

Conversely, given that Φ satisfies the condition, suppose that it is not a minimal usco (cusco). Then there exists an usco (cusco) Φ' whose graph is contained in that of Φ but for some $t \in A$, $\Phi'(t) \neq \Phi(t)$. Consider $x_0 \in \Phi(t) \setminus \Phi'(t)$. Since $\Phi'(t)$ is compact and X is Hausdorff there exist disjoint open sets W_1 and W_2 such that $\Phi'(t) \subseteq W_1$ and $x_0 \in W_2$. If Φ' is a cusco then $\Phi'(t)$ is convex so we can choose W_2 to be an open half-space. Now since Φ' is upper semi-continuous at t there exists an open neighbourhood V of t such that $\Phi'(V) \subseteq W_1$. But then for each $s \in V$, $\Phi(s) \cap C(W_2) \neq \emptyset$. Therefore, by the assumed condition $\Phi(V) \subseteq C(W_2)$, but this contradicts $x_0 \in \Phi(t) \cap W_2$. ∎

We present the next theorem in general form. Particular cases where we will want to apply the result are when the range space is a Banach space and the τ-topology is the weak topology, or more significantly when the range space is the dual of a Banach space and the τ-topology is the weak * topology.

2.6. THEOREM. *Consider a Baire space A, a metric space (normed linear space) X and X with a topology τ where the metric closed balls are also τ-closed. Consider a minimal τ-usco (τ-cusco) Φ from A into subsets of X. If Φ is α upper semi-continuous on a dense subset of A then there exists a dense G_δ subset of A on which Φ is single-valued and metric (norm) upper semi-continuous.*

Proof. Given $\varepsilon > 0$, consider $O_\varepsilon \equiv \bigcup \{\text{open sets } U \text{ in } A : \operatorname{diam} \Phi(U) \leqslant 2\varepsilon\}$. Now O_ε is open; we show that O_ε is also dense in A. Consider W a non-empty open set in A. Now there exists a $t \in W$ where Φ is α upper semi-continuous. So there exists an open neighbourhood U_1 of t contained in W such that $\alpha(\Phi(U_1)) < \varepsilon$. Therefore, there exists $\{x_1, ..., x_n\} \subseteq X$ such that $\Phi(U_1) \subseteq \bigcup_1^n B[x_i; \varepsilon]$. Now if $\Phi(U_1) \subseteq B[x_1; \varepsilon]$ write $V \equiv U_1$, but if not we have by Lemma 2.5 that there exists a non-empty open set $U_2 \subseteq U_1$ such that $\Phi(U_2) \cap B[x_1; \varepsilon] = \emptyset$. Now if $\Phi(U_2) \subseteq B[x_2; \varepsilon]$ write $V \equiv U_2$, but if not we have by Lemma 2.5 that there exists a non-empty open set $U_3 \subseteq U_2$ such that $\Phi(U_3) \cap B[x_2; \varepsilon] = \emptyset$. Continue in this way. We will have defined V by the nth step because if not, we define a non-empty open set $U_{n+1} \subseteq U_n \subseteq \cdots \subseteq U_1$ such that $\Phi(U_{n+1}) \cap \bigcup_1^n B[x_i; \varepsilon] = \emptyset$ and this contradicts $\Phi(U_1) \subseteq \bigcup_1^n B[x_i; \varepsilon]$. So W contains a non-empty open subset V with $\operatorname{diam} \Phi(V) \leqslant 2\varepsilon$. So O_ε is dense in A. Then since A is a Baire space,

$\bigcap_1^\infty O_{1/n}$ is a dense G_δ subset of A on which Φ is single-valued and metric (norm) upper semi-continuous. ∎

An important application of our theory so far concerns the differentiability of convex functions. A continuous convex function ϕ on an open convex subset A of a Banach space X is said to be Fréchet differentiable at $x \in A$ if

$$\lim_{t \to 0} \frac{\phi(x+ty)-\phi(x)}{t} \text{ exists and is approached uniformly for all } y \in S(X).$$

A subgradient of ϕ at $x_0 \in A$ is a continuous linear functional f on X such that $f(x-x_0) \leqslant \phi(x)-\phi(x_0)$ for all $x \in A$. The subdifferential of ϕ at x_0 is denoted by $\partial\phi(x_0)$ and is the set of subgradients of ϕ at x_0. The subdifferential mapping $x \to \partial\phi(x)$ is a minimal weak * cusco from A into subsets of X^* [13, p. 100]. Further, ϕ is Fréchet differentiable at $x \in A$ if and only if the subdifferential mapping $x \to \partial\phi(x)$ is single-valued and norm upper semi-continuous at x [13, p. 18]. So from Proposition 2.1 and Theorem 2.6 we have the following result.

2.7. COROLLARY. *A continuous convex function ϕ on an open convex subset A of a Banach space X whose subdifferential mapping $x \to \partial\phi(x)$ is α upper semi-continuous on a dense subset of A is Fréchet differentiable on a dense G_δ subset of A.*

It is interesting to note that this corollary implies the well-known result that every convex function on an open convex subset of a finite dimensional normed linear space is Fréchet differentiable on a dense G_δ subset of its domain.

3. THE DUALITY OF α UPPER SEMI-CONTINUITY AND PROPERTY α

a. *Duality for Convex Sets and Gauges*

A closed bounded convex set K in a Banach space X is said to have the drop property if for every closed set C disjoint from K there exists an $x \in C$ such that $\operatorname{co}\{x, K\} \cap C = \{x\}$. Given $f \in X^* \setminus \{0\}$ and $\delta > 0$, the slice of K defined by f and δ is the subset $S(K, f, \delta) \equiv \{x \in K: f(x) > \sup f(K) - \delta\}$. We say that K has property α for $f \in S(X^*)$ if given $\varepsilon > 0$ there exists a $\delta(\varepsilon, f) > 0$ such that $\alpha(S(K, f, \delta)) < \varepsilon$. Kutzarova has shown [10, p. 284] that K with $\operatorname{int} K \neq \varnothing$ has the drop property if and only if it has property α for each $f \in S(X^*)$.

Consider a closed bounded convex set K with $0 \in \operatorname{int} K$ in a Banach space X. The gauge p of K is defined by $p(x) = \inf\{\lambda > 0: x \in \lambda K\}$ and is a continuous sublinear functional on X. We define the polar of K as the set $K^\circ \equiv \{f \in X^* : f(x) \leqslant 1 \text{ for all } x \in K\}$. Then K° is weak * compact convex and $0 \in \operatorname{int} K^\circ$. We denote by $K^{\circ\circ}$ the polar of K° in X^{**}.

In [6, p. 384] it was shown that a closed bounded convex set K with $0 \in \operatorname{int} K$ in a Banach space X has the drop property if and only if the subdifferential mapping $f \to \partial p(f)$ for the gauge p of the polar K° is a norm cusco from $X^* \setminus \{0\}$ into subsets of X^{**}. This suggests that, in view of Theorem 2.4, we explore the duality between α upper semi-continuity for the subdifferential mapping $x \to \partial p(x)$ for the gauge p of K and property α for the polar K°. We show that there is a local duality between the two concepts. But to do so we need properties relating the Kuratowski index of non-compactness for slices of the sets K and $K^{\circ\circ}$. We use ^ to denote natural embedding elements in X^{**}.

3.1. LEMMA. *Consider a closed bounded convex set K with $0 \in \operatorname{int} K$ in a Banach space X. Then*

(i) *$\hat{K}$ is weak * dense in $K^{\circ\circ}$ and*

(ii) *given $f \in X^* \setminus (0)$ and $0 < r < \sup f(K) = \sup \hat{f}(K^{\circ\circ})$*

(a) *$S(\hat{K}, \hat{f}, r)$ is weak * dense in $S(K^{\circ\circ}, \hat{f}, r)$ and*

(b) *$\alpha(S(K, f, r)) = \alpha(S(K^{\circ\circ}, \hat{f}, r))$*

Proof. (i) Now $K^{\circ\circ} \equiv \{F \in X^{**} : F(f) \leqslant 1 \text{ for all } f \in K^\circ\}$ is weak * closed so $\bar{\hat{K}}^{w^*} \subseteq K^{\circ\circ}$. Suppose there exists an $F_0 \in K^{\circ\circ} \setminus \bar{\hat{K}}^{w^*}$. Then since $\bar{\hat{K}}^{w^*}$ is convex there exists an $f \in X^*$ which strongly separates F_0 and $\bar{\hat{K}}^{w^*}$; that is, there exists an $\alpha > 0$ and an $\varepsilon > 0$ such that

$$\hat{f}(F) \leqslant \alpha - \varepsilon < \alpha + \varepsilon \leqslant \hat{f}(F_0) \qquad \text{for all} \quad F \in \bar{\hat{K}}^{w^*}.$$

Then $f(x) \leqslant \alpha$ for all $x \in K$; that is $(f/\alpha)(x) \leqslant 1$ so $f/\alpha \in K^\circ$. But then $F_0(f/\alpha) \leqslant 1$; that is, $F_0(f) \leqslant \alpha$. But this contradicts the separation inequality.

(ii) (a) For any $F \in S(K^{\circ\circ}, \hat{f}, r)$ consider a weak * neighbourhood N of F. Then $W \equiv N \cap \{F \in X^{**} : F(f) > \sup \hat{f}(K^{\circ\circ}) - r\}$ is a weak * neighbourhood of F. But since by (i), $\hat{K}$ is weak * dense in $K^{\circ\circ}$ then W contains an element of $\hat{K}$ necessarily of $S(\hat{K}, \hat{f}, r)$.

(b) Our result follows from (a) using the fact that for any bounded set E in X, $\operatorname{diam} \overline{\operatorname{co}}^{w^*} \hat{E} = \operatorname{diam} E$.

3.2. THEOREM. *Consider a closed bounded convex set K with $0 \in \operatorname{int} K$ in a Banach space X.*

(i) *For p the gauge of K on X, the subdifferential mapping* $x \to \partial p(x)$ *is* α *upper semi-continuous at* $x \in X \setminus \{0\}$ *if and only if* K^o *has property* α *for* $\hat{x}$.

(ii) *For p the gauge of* K^o *on* X^*, *the subdifferential mapping* $f \to \partial p(f)$ *is* α *upper semi-continuous at* $f \in X^* \setminus \{0\}$ *if and only if K has property* α *for f.*

Proof. (i) Given $x \in X \setminus \{0\}$ and $0 < \delta^2 < \sup \hat{x}(K^o)$, consider $f \in S(K^o, \hat{x}, \delta^2)$. Then we have $f(x) > \sup \hat{x}(K^o) - \delta^2 = p(x) - \delta^2$ and since $f \in K^o, f(y) \leqslant p(y)$ for all $y \in X$ so $f(y - x) \leqslant p(y) - p(x) + \delta^2$ for all $y \in X$. By the Brøndsted–Rockafellar Theorem [13, p. 51], there exists an $x_0 \in X$ and $f_0 \in \partial p(x_0)$ such that $\|x - x_0\| < \delta$ and $\|f - f_0\| < \delta$. So $f_0 \in \partial p(B(x; \delta))$ and $S(K^o, \hat{x}, \delta^2) \subseteq \partial p(B(x; \delta)) + \delta B(X^*)$. If $x \to \partial p(x)$ is α upper semi-continuous at x, given $\varepsilon > 0$ there exists a $0 < \delta < \varepsilon$ and $\{f_1, ..., f_n\} \subseteq X^*$ such that $\partial p(B(x; \delta)) \subseteq \bigcup_1^n f_i + \varepsilon B(X^*)$. Then $S(K^o, \hat{x}, \delta^2) \subseteq \bigcup_1^n f_i + 2\varepsilon B(X^*)$; that is, K^o has property α for $\hat{x}$.

Conversely, if K^o has property α for $\hat{x}$, given $\varepsilon > 0$ there exists an $0 < r < \sup \hat{x}(K^o)$ and $\{f_1, ..., f_n\} \subseteq X^*$ such that $S(K^o, \hat{x}, r) \subseteq \bigcup_1^n f_i + \varepsilon B(X^*)$. Consider the weak * open half space $W \equiv \{f \in X^* : f(x) > \sup \hat{x}(K^o) - r\}$. Now $\partial p(x) \subseteq W$. As $x \to \partial p(x)$ is weak * upper semi-continuous at x there exists a $\delta > 0$ such that $\partial p(B(x; \delta)) \subseteq W$. So $\partial p(B(x; \delta)) \subseteq W \cap K^o = S(K^o, \hat{x}, r)$. Then $\partial p(B(x; \delta)) \subseteq \bigcup_1^n f_i + \varepsilon B(X^*)$; that is, $x \to \partial p(x)$ is α upper semi-continuous at x.

(ii) If $f \to \partial p(f)$ on X^* is α upper semi-continuous at f then by (i), K^{oo} has property α for $\hat{f}$. That is, given $\varepsilon > 0$ there exists an $0 < r < \sup f(K) = \sup \hat{f}(K^{oo})$ and $\{F_1, ..., F_n\} \subseteq X^{**}$ such that $S(K^{oo}, \hat{f}, r) \subseteq \bigcup_1^n F_i + \varepsilon B(X^{**})$. So $S(\hat{K}, \hat{f}, r) \subseteq (\bigcup_1^n F_i \in B(X^{**})) \cap \hat{X}$ which is a finite family of open sets in $\hat{X}$ of diameter less than 2ε. So K has property α for f.

Conversely, if K has property α for f, given $\varepsilon > 0$ there exists an $0 < r < \sup f(K) = \sup \hat{f}(K^{oo})$ such that $\alpha(S(K, f, r)) \leqslant \varepsilon$. But by Lemma 3.1(ii)(b), $\alpha(S(K^{oo}, \hat{f}, r)) \leqslant \varepsilon$. So K^{oo} has property α for $\hat{f}$. Then by (i), $f \to \partial p(f)$ is α upper semi-continuous at f. ∎

Using the characterisation of the drop property given in [10, p. 284], we derive directly from Theorem 3.2(ii) the following characterisation which compares with that of [6, p. 384].

3.3. Corollary. *A bounded closed convex set K with* $\operatorname{int} K \neq \varnothing$ *in a Banach space X has the drop property if and only if the subdifferential mapping* $f \to \partial p(f)$ *for the gauge p of the polar* K^o *is* α *upper semi-continuous on* $X^* \setminus \{0\}$.

Consider a bounded closed convex set K with $\operatorname{int} K \neq \emptyset$ in a Banach space X. It is known that if K has the drop property then X is reflexive [10, p. 284]. It follows from Corollary 3.3, but it can be proved directly, that if the subdifferential mapping $f \to \partial p(f)$ for the gauge p of the polar K° is α upper semi-continuous on $X^* \backslash \{0\}$ then X is reflexive.

For the Banach space c_0, the duality mapping $x \to D(x)$ on $S(c_0)$ is a norm cusco [4, p. 105], so from Theorem 2.4 it is α upper semi-continuous on $S(c_0)$. From Theorem 3.2(i) we deduce that the Banach space l_1 has property α for all $\hat{x} \in S(c_0^{\wedge})$. So given a closed bounded convex set K with $0 \in \operatorname{int} K$ in a Banach space X, when the conditions of Theorem 3.2(ii) hold for all $f \in X^* \backslash \{0\}$ then X is reflexive, but it is clear that when the conditions of Theorem 3.2(i) hold for all $x \in X \backslash \{0\}$ then X is not necessarily reflexive.

b. *Uniform α Upper Semi-continuity and Geometry*

Knowing that a Banach space X has property α if and only if the duality mapping $f \to D(f)$ on $S(X^*)$ is norm upper semi-continuous and compact valued [5, p. 504], it is tempting to conjecture that property $U\alpha$ on X can be characterised by the duality mapping $f \to D(f)$ on $S(X^*)$ being uniformly norm upper semi-continuous and compact valued. That this is false is a consequence of the following examination of the implications of uniform norm upper semi-continuity.

We say that an usco mapping Φ from a metric space (A, d) into subsets of a Banach space X is uniformly norm upper semi-continuous on A if given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\Phi(y) \subseteq \Phi(x) + \varepsilon B(X)$ for all $d(x, y) < \delta$.

3.4. PROPOSITION. *Consider a metric space A, a normed linear space X and X with a linear topology τ where the norm closed balls are also τ-closed. Consider a minimal τ-usco (τ-cusco) mapping Φ from A into subsets of X. If Φ is uniformly norm upper semi-continuous on A then Φ is single-valued on A.*

Proof. Suppose Φ is not single-valued at $t_0 \in A$. Then there exist $x_1, x_2 \in \Phi(t_0)$ and $x_1 \neq x_2$. Consider $0 < \varepsilon < \frac{1}{3} \|x_1 - x_2\|$. Since Φ is uniformly norm upper semi-continuous on A there exists a $\delta(\varepsilon) > 0$ such that $\Phi(B(t_0; \delta)) \subseteq \Phi(t_0) + \varepsilon B(X)$. But $\Phi(B(t_0; \delta)) \cap C(B[x_2, 2\varepsilon]) \neq \emptyset$ and since Φ is a minimal τ-usco (τ-cusco) on A it follows from Lemma 2.5 that there exists an open subset V_1 of $B(t_0; \delta)$ such that $\Phi(V_1) \subseteq C(B[x_2, 2\varepsilon])$. For any $t_1 \in V_1$ we have $x_2 \notin \Phi(t_1) + \varepsilon B(X)$. But $t_0 \in B(t_1; \delta)$ which contradicts the uniform norm upper semi-continuity of Φ on A. ∎

This proposition has interesting geometrical consequences for a Banach space X when we consider the duality mapping on $S(X)$. To demonstrate

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this we need to establish appropriate properties of the duality mapping on $S(X)$.

3.5. LEMMA. *For a Banach space X, the duality mapping $x \to D(x)$ on $S(X)$ is a minimal weak * cusco.*

Proof. We need only prove minimality. Suppose there exists another weak * cusco D' whose graph is contained in the graph of D and there exists an $x_1 \in S(X)$ such that $f_1 \in D(x_1)$ but $f_1 \notin D'(x_1)$. Then $D'(x_1)$ and f_1 can be strongly separated by a weak * closed hyperplane generated by some $x_0 \in S(X)$. So there exists a $0<\delta<1$ such that $f_1(x_0) > 1-\delta$ and $f(x_0) < 1-\delta$ for all $f \in D'(x_1)$. Since D' is weak * upper semi-continuous at x_1, there exists an open neighbourhood V of x_1 in $S(X)$ such that $D'(V) \subseteq \{f \in X^* : f(x_0) < 1-\delta\}$. Consider any $t>0$ such that $x(t) = (x_1 + tx_0)/\|x_1 + tx_0\| \in V$. Since the duality mapping $x \to D(x)$ is monotone, for any $f \in D(x(t))$, $(f - f_1)(x_0) = (1/t)(f - f_1)(x_1 + tx_0 - x_1) \geqslant 0$. So $f(x_0) \geqslant f_1(x_0) > 1-\delta$ for all $f \in D(x(t))$. But this contradicts the fact that the graph of D' is contained in the graph of D. ▮

A Banach space X is said to have uniformy Fréchet differentiable norm if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists and is approached uniformly for all $x, y \in S(X)$. It is known that the norm of X is uniformly Fréchet differentiable if and only if there exists a selection $x \to f_x \in D(x)$ which is uniformly continuous on $S(X)$ [3, p. 36].

So we have the following corollary from Proposition 3.4.

3.6. COROLLARY. *A Banach space X has uniformly Fréchet differentiable norm if and only if the duality mapping $x \to D(x)$ is uniformly norm upper semi-continuous on $S(X)$.*

A Banach space X is said to be uniformly rotund if given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\|x - y\| < \varepsilon$ for all $\|x + y\| > 2 - \delta$. A Banach space X has uniformly Fréchet differentiable norm if and only if its dual X^* is uniformly rotund. Furthermore, a uniformly rotund Banach space is reflexive, [3, pp. 36, 37].

So then the proposed dual to property $U\alpha$ for a Banach space X implies that the dual space X^* has uniformly Fréchet differentiable norm on $S(X^*)$ and that X is uniformly rotund. However, a Banach space X is said to be nearly uniformly rotund if given $\varepsilon > 0$ there exists a $0 < \delta < 1$ such that for every closed convex set E in $B(X)$ with $\alpha(E) > \varepsilon$ we have

$E \cap (1-\delta) B(X) \neq \varnothing$. Rolewicz has shown [16, p. 185] that a Banach space X has property $U\alpha$ if and only if X is nearly uniformly rotund. But a Banach space which is nearly uniformly rotund is not necessarily uniformly rotund. In fact a Banach space which is nearly uniformly rotund cannot necessarily be equivalent renormed to be uniformly rotund [7, p. 747].

The required dual to property $U\alpha$ for a Banach space X is a uniformisation of α upper semi-continuity for the duality mapping $f \to D(f)$ on $S(X^*)$. It is interesting to see that two properties such as α upper semi-continuity and upper semi-continuity with compact values, which are equivalent for duality mappings, should be so different when they are given the obvious uniform definitions. Given a Banach space X, the duality mapping $x \to D(x)$ on $S(X)$ is said to be uniformly α upper semi-continuous if given $\varepsilon > 0$ there exists a $0 < \delta(\varepsilon) < 1$ such that $\alpha(D(B(x; \delta))) < \varepsilon$ for all $x \in S(X)$.

The following theorem is the uniform analogue of Theorem 3.2 for $B(X)$ of a Banach space X. We say that the dual space X^* has property weak * $U\alpha$ if given $\varepsilon > 0$ there exists a $0 < \delta(\varepsilon) < 1$ such that $\alpha(S(B(X^*), \hat{x}, \delta)) < \varepsilon$ for all $x \in S(X)$.

3.7. THEOREM. *For a Banach space X,*

(i) *the duality mapping $x \to D(x)$ on $S(X)$ is uniformly α upper semi-continuous if and only if the dual space X^* has property weak * $U\alpha$,*

(ii) *the duality mapping $f \to D(f)$ on $S(X^*)$ is uniformly α upper semi-continuous if and only if X has property $U\alpha$.*

Proof. (i) Given $x \in S(X)$ and $0 < \delta < 1$, consider $f \in D(y)$ where $y \in B(x; \delta)$. Then $f(x) = f(y) + f(x) - f(y) = 1 + f(x-y) > 1 - \delta$, so $D(B(x; \delta)) \subseteq S(B(X^*), \hat{x}, \delta)$. If, given $\varepsilon > 0$ there exists a $0 < \delta < 1$ such that $\alpha(S(B(X^*), \hat{x}, \delta)) < \varepsilon$ for all $x \in S(X)$ then $\alpha(D(B(x; \delta))) < \varepsilon$ for all $x \in S(X)$.

Conversely, given $x \in S(X)$ and $0 < \delta < 1$, consider $f \in S(B(X^*), \hat{x}, \delta^2)$. Then as in Theorem 3.2(i) we have $S(B(X^*), \hat{x}, \delta^2) \subseteq D(B(x; \delta)) + \delta B(X^*)$. So if given $\varepsilon > 0$ there exists a $0 < \delta(\varepsilon) < \varepsilon$ and $\alpha(D(B(x; \delta))) < \varepsilon$ for all $x \in S(X)$ then $\alpha(S(B(X^*), \hat{x}, \delta^2)) < 2\varepsilon$ for all $x \in S(X)$.

(ii) If X has property α then X is reflexive [11, p. 95], so if X has property $U\alpha$ then by (i) the duality mapping $f \to D(f)$ on $S(X^*)$ is uniformly α upper semi-continuous.

Conversely, if the duality mapping $f \to D(f)$ on $S(X^*)$ is uniformly α upper semi-continuous then by (i) the second dual X^{**} has property weak * $U\alpha$; that is, given $\varepsilon > 0$ there exists a $0 < \delta(\varepsilon) < 1$ such that $\alpha(S(B(X^{**}),$

$\hat{f}, \delta)) < \varepsilon$ for all $f \in S(X^*)$. So $\alpha(S(B(X), f, \delta)) < \varepsilon$ for all $f \in S(X^*)$; that is, X has property $U\alpha$. ∎

As with Theorem 3.2 we notice that a Banach space satisfying the conditions of Theorem 3.7(ii) is reflexive. But a Banach space satisfying the conditions of Theorem 3.7(i) is not necessarily reflexive and it is interesting to see that again the Banach space c_0 illustrates this. Given $x \equiv \{\lambda_1, \lambda_2, ..., \lambda_n, ...\} \in S(c_0)$, the set $E \equiv \{n \in \mathbb{N} : |\lambda_n| \geqslant \frac{1}{2}\}$ is finite. For any $y \equiv \{\mu_1, \mu_2, ..., \mu_n, ...\} \in B(x; \frac{1}{2})$, $\text{supp}(y) \equiv \{n \in \mathbb{N} : |\mu_n| = 1\} \subseteq E$. Now

$$\begin{aligned} D(y) &= \text{co}\{\{0, ..., 0, \text{sgn}\, \mu_k, 0, ...\} : k \in \text{supp}(y)\} \\ &\subseteq \text{co}\{\{0, ..., 0, \underset{k\text{th place}}{\pm 1}, 0, ...\} : k \in E\} \end{aligned}$$

which is compact since E is finite. So $\alpha(D(B(x; \frac{1}{2}))) = 0$ and we conclude that the duality mapping $x \to D(x)$ on $S(c_0)$ is uniformly α upper semi-continuous.

We noted that a Banach space X has property $U\alpha$ if and only if X is nearly uniformly rotund. Theorem 3.7(ii) gives us the following dual characterisation of nearly uniform rotundity.

3.8. Corollary. *A Banach space X is nearly uniformly rotund if and only if the duality mapping $f \to D(f)$ on $S(X^*)$ is uniformly α upper semi-continuous.*

However, Theorem 3.7(i) prompts us to consider a weakened form of nearly uniform rotundity for a dual, which would be equivalent to property weak * $U\alpha$. We say that the dual X^* of a Banach space X is weak * nearly uniformly rotund if, given $\varepsilon > 0$ there exists $0 < \delta(\varepsilon) < 1$ such that for every weak * closed convex set E in $B(X^*)$ with $\alpha(E) > \varepsilon$ we have $E \cap (1 - \delta)\, B(X^*) \neq \emptyset$.

3.9. Corollary. *The dual X^* of a Banach space X is weak * nearly uniformly rotund if and only if the duality mapping $x \to D(x)$ on $S(X)$ is uniformly α upper semi-continuous.*

Proof. From Theorem 3.7(i) it is sufficient to show that X^* is weak * nearly uniformly rotund if and only if it is weak * $U\alpha$.

Suppose that X^* is weak * nearly uniformly rotund. Then since for each $x \in S(X)$, $\overline{S(B(X^*), \hat{x}, \delta/2)}$ is weak * closed and $\overline{S(B(X^*), \hat{x}, \delta/2)} \cap (1 - \delta)\, B(X^*) = \emptyset$ we have $\alpha(S(B(X^*), \hat{x}, \delta/2)) < \varepsilon$, so X^* has property weak * $U\alpha$.

Conversely, suppose that X^* has property weak * $U\alpha$ and consider a weak * closed convex set E in $B(X^*)$ such that $E \cap (1 - \delta)\, B(X^*) = \emptyset$. Then since E is weak * closed and convex there exists a weak * continuous linear functional on X^* separating E and $(1-\delta)\,B(X^*)$. That is, there exists an $x \in S(X)$ such that $S(B(X^*), \hat{x}, \delta) \supseteq E$. Since $\alpha(S(B(X^*), \hat{x}, \delta)) < \varepsilon$ then $\alpha(E) < \varepsilon$ and X^* is weak * nearly uniformly rotund. ∎

4. α Upper Semi-continuity and Differentiability Properties of Convex Functions

a. *The Characterisation of Asplund Spaces*

A Banach space X is called an Asplund space if every continuous convex function on an open convex domain in X is Fréchet differentiable on a dense G_δ subset of its domain. In Theorem 2.6 we showed that minimal weak * uscos (or weak * cuscos) from a Baire space into subsets of the dual of a Banach space which are α upper semi-continuous on a dense subset of their domain are single-valued and norm upper semi-continuous on a dense G_δ subset of their domain. In Corollary 2.7 we showed that this has differentiability implications for convex functions on open convex subsets of a Banach space whose subdifferential mapping is α upper semi-continuous on a dense subset of their domain. This suggests that we explore further α upper semi-continuity and related properties in detemining conditions under which a Banach space is an Asplund space or has similar differentiability properties.

Part of the study of Asplund spaces is to determine norm properties which imply that a Banach space is an Asplund space. In particular, a Banach space X with weak cusco duality mapping $x \to D(x)$ on $S(X)$ is an Asplund space [4, p. 106]. So we can make the following deduction from Theorem 2.4 and Proposition 2.1.

4.1. Theorem. *A Banach space X with duality mapping $x \to D(x)$ α upper semi-continuous on $S(X)$ is an Asplund space.*

The classical characterisation theorem for Asplund spaces was given by Namioka and Phelps [12, p. 737]. We present an extended characterisation using Theorem 2.6.

4.2. Theorem. *For a Banach space X the following are equivalent,*

(i) *every continuous convex function ϕ on an open convex subset A of X is Fréchet differentiable on a dense G_δ subset of A,*

(ii) *every non-empty bounded set in X^* has weak * slices of arbitrarily small diameter,*

(iii) *every non-empty bounded set in X^* has weak * slices whose Kuratowski index of non-compactness is arbitrarily small.*

Proof. In view of the classical characterisation and because it is obvious that (ii) $\Rightarrow$ (iii), it will be sufficient to prove (iii) $\Rightarrow$ (i). Consider a continuous convex function ϕ on an open convex subset A in X and given $\varepsilon > 0$, $O_\varepsilon \equiv \bigcup \{\text{open sets } V \text{ in } A : \alpha(\partial\phi(V)) < \varepsilon\}$. Now O_ε is open; we show that O_ε is dense in A. Consider a non-empty open set W in A. Since the subdifferential mapping $x \to \partial\phi(x)$ is locally bounded [13, p. 29], there exists a non-empty open subset U of W for which $\partial\phi(U)$ is bounded. Now by the hypothesis in (iii) there exists a $z \in X \backslash \{0\}$ and a $\delta > 0$ such that $\alpha(S(\partial\phi(U), z, \delta)) < \varepsilon$. Now $\partial\phi(U) \not\subseteq \{f \in X^* : f(z) \leqslant \sup \hat{z}(\partial\phi(U)) - \delta\}$ so from Lemma 2.5 there exists a non-empty open subset V of U and so of W such that $\partial\phi(V) \subseteq S(\partial\phi(U), z, \delta)$. Then $\alpha(\partial\phi(V)) < \varepsilon$. We conclude that the subdifferential mapping $x \to \partial\phi(x)$ is α upper semi-continuous on the dense G_δ subset $\bigcap_1^\infty O_{1/n}$ of A. It follows again from Corollary 2.7 that ϕ is Fréchet differentiable on a dense G_δ subset of A. ∎

The dual theorem which corresponds to the classical Asplund space characterisation was given by Collier in [2, p. 103]. A similar characterisation results from an extension of Theorem 4.2.

4.3. THEOREM. *For a Banach space X the following are equivalent,*

(i) *every continuous weak * lower semi-continuous convex function ϕ on an open convex subset A of X^* is Fréchet differentiable on a dense G_δ subset of A,*

(ii) *every non-empty bounded set in X has slices of arbitrarily small diameter,*

(iii) *every non-empty bounded set in X has slices whose Kuratowski index of non-compactness is arbitrarily small.*

Proof. Again in view of the classical characterisation and because it is obvious that (ii) $\Rightarrow$ (iii), it will be sufficient to prove (iii) $\Rightarrow$ (i). Consider a continuous weak * lower semi-continuous function ϕ on an open convex subset A in X^* and given $\varepsilon > 0$, $O_\varepsilon \equiv \bigcup \{\text{open sets } V \text{ in } A : \alpha(\partial\phi(V)) < 2\varepsilon\}$. Now O_ε is open; we show that O_ε is dense in A. Consider a non-empty open set W in A. Since the subdifferential mapping $f \to \partial\phi(f)$ is locally bounded there exists a non-empty open subset U of W for which $\partial\phi(U)$ is bounded. It follows from the form of the Bishop–Phelps theorem given in [14, p. 180] that $\partial\phi(U) \cap \hat{X} \neq \varnothing$. By the hypothesis in (iii) there exists an $f \in X^* \backslash \{0\}$ and a $\delta > 0$ such that $\alpha(S(\partial\phi(U) \cap \hat{X}, \hat{f}, \delta)) < \varepsilon$. Again since the subdifferential mapping $f \to \partial\phi(f)$ is a minimal weak * cusco, as in the proof of Theorem 4.2 there exists a non-empty open subset V of U such that $\partial\phi(V) \subseteq \{F \in X^{**} : F(f) > \sup \hat{f}(\partial\phi(U) \cap \hat{X}) - \delta\}$. Now there exists $\{x_1, ..., x_n\} \subseteq X$ such that $S(\partial\phi(U) \cap \hat{X}, \hat{f}, \delta) \subseteq \bigcup_1^n \hat{x}_i + \varepsilon B(\hat{X}) \subseteq \bigcup_1^n \hat{x}_i + \varepsilon B(X^{**})$. If $\partial\phi(V) \not\subseteq \text{co}(\bigcup_1^n \hat{x}_i + \varepsilon B(X^{**}))$ then by Lemma 2.5

there exists a non-empty open subset V' of V such that $\partial\phi(V') \cap \text{co}(\bigcup_1^n \hat{x}_i + \varepsilon B(X^{**})) = \varnothing$. Again from the form of the Bishop–Phelps theorem given in [14, p. 180], $\partial\phi(V') \cap \hat{X} \neq \varnothing$. But then we have contradicted the fact that $\partial\phi(V') \cap S(\partial\phi(U) \cap \hat{X}, \hat{f}, \delta) \neq \varnothing$. So we conclude that $\partial\phi(V) \subseteq \text{co}(\bigcup_1^n \hat{x}_i + \varepsilon B(X^{**}))$ and by the Kuratowski index property (v), $\alpha(\partial\phi(V)) < 2\varepsilon$. We conclude that the subdifferential mapping $x \to \partial\phi(x)$ is α upper semi-continuous on the dense G_δ subset $\bigcap_1^\infty O_{1/n}$ of A. It follows again from Corollary 2.7 that ϕ is Fréchet differentiable on a dense G_δ subset of A. ∎

b. *A More General Class of Differentiability Spaces*

Although there are separable Banach spaces which do not have the differentiability properties of Theorem 4.3, it has recently been shown [8, Theorem 3.5] that there is a large class of Banach spaces, including the separable spaces, where every continuous convex function on an open convex subset of the dual is Fréchet differentiable on a dense G_δ subset of its domain provided that the set of points where the function has a weak * continuous subgradient is residual in its domain. Such spaces are those which can be equivalently renormed to have every point of the unit sphere a denting point of the closed unit ball. We generalise this result using Kuratowski's index of non-compactness and Theorem 2.6.

Given a Banach space X and $r > 0$ we say that $x \in rS(X)$ is a denting point of $rB(X)$ if given $\varepsilon > 0$, x is contained in a slice of $rB(X)$ of diameter less than ε. Generalising we say that $x \in rS(X)$ is an α-denting point of $rB(X)$ if given $\varepsilon > 0$, x is contained in a slice of $rB(X)$ with Kuratowski index less than ε. Similarly, we say that $f \in rS(X^*)$ is a weak * denting point of $rB(X^*)$ if given $\varepsilon > 0$, f is contained in a weak * slice of $rB(X^*)$ of diameter less than ε, and $f \in rB(X^*)$ is a weak * α-denting point of $rB(X^*)$ if given $\varepsilon > 0$, f is contained in a weak * slice of $rB(X^*)$ with Kuratowski index less than ε. It is known that the denting points of $rB(X)$ map to weak * denting points of $rB(X^{**})$ under the natural embedding [8, Lemma 3.3], and the generalisation follows from Lemma 3.1(ii)(b) and the Kuratowski index property (iv), that the α-denting points of $rB(X)$ map to weak * α-denting points of $rB(X^{**})$ under the natural embedding.

We will need the following property of minimal weak * cuscos [8, Lemma 3.4(iii)].

4.4. LEMMA. *Given a minimal weak * cusco Φ from a Baire space A into subsets of the dual X^* of a Banach space X, there exists a dense G_δ subset D of A such that at each $t \in D$ the real valued mapping defined on A by*

$$\rho(t) = \inf\{\|f\| : f \in \Phi(t)\}$$

is continuous and $\Phi(t)$ lies in the face of a sphere of X^* of radius $\rho(t)$.

4.5. THEOREM. *Consider a Banach space X which can be equivalently renormed to have every point of $S(X)$ an α-denting point of $B(X)$. Then every minimal weak * cusco Φ from a Baire space A into subsets of X^{**} for which the set $G \equiv \{t \in A : \Phi(t) \cap \hat{X} \neq \varnothing\}$ is residual in A, is single-valued and norm upper semi-continuous on a dense G_δ subset of A. In particular, every continuous convex function ϕ on an open convex set A in X^* for which the set $G \equiv \{f \in A : \partial\phi(f) \cap \hat{X} \neq \varnothing\}$ is residual in A, is Fréchet differentiable on a dense G_δ subset of A.*

Proof. Consider X so renormed. Given $\varepsilon > 0$, consider $O_\varepsilon \equiv \bigcup\{$open sets V in $A : \alpha(\Phi(V)) < \varepsilon\}$. Now O_ε is open; we show that O_ε is dense in A. From Lemma 4.4 there exists a dense G_δ subset G_1 of A such that at every point $t \in G_1$ the mapping ρ where $\rho(t) = \inf\{\|f\| : f \in \Phi(t)\}$, is continuous and $\Phi(t)$ lies in the face of a sphere of X^{**} of radius $\rho(t)$. Now $G \cap G_1$ is residual in A. Consider a non-empty open subset W of A and $t_0 \in G \cap G_1 \cap W$. There exists some $\hat{x}_0 \in \Phi(t_0) \cap \hat{X}$. If $x_0 = 0$, then since ρ is continuous at t_0 there exists an open neighbourhood U of t_0 such that $\Phi(t) \cap \varepsilon B(X^{**}) \neq \varnothing$ for all $t \in U$. Then by Lemma 2.5, $\Phi(U) \subseteq \varepsilon B(X^{**})$ so $U \subseteq O_\varepsilon \cap W$. If $x_0 \neq 0$ then x_0 is an α-denting point of $\rho(t_0)\, B(X)$ so $\hat{x}_0$ is a weak * α-denting point of $\rho(t_0)\, B(X^{**})$. Then there exists a $g \in S(X^*)$ and a $\delta > 0$ such that $\hat{x}_0 \in S(\rho(t_0)\, B(X^{**}), \hat{g}, \delta)$ and $\alpha(S(\rho(t_0)\, B(X^{**}), \hat{g}, \delta) < \varepsilon/2$. We can choose $1 < \lambda < 2$ such that $\hat{x}_0 \in S(\lambda\rho(t_0)\, B(X^{**}), \hat{g}, \lambda\delta) = \lambda S(\rho(t_0)\, B(X^{**}), \hat{g}, \delta)$ and then by the Kuratowski index property (iv), $\alpha(\lambda S(\rho(t_0)\, B(X^{**}), \hat{g}, \delta)) < \varepsilon$. Since ρ is continuous at t_0 there exists an open subset V' of W containing t_0 such that $\Phi(t) \cap \lambda\rho(t_0)\, B(X^{**}) \neq \varnothing$ for all $t \in V'$. So by Lemma 2.5, $\Phi(V') \subseteq \lambda\rho(t_0)\, B(X^{**})$. Since $\Phi(V') \nsubseteq \{F \in X^{**} : F(g) \leqslant \lambda\rho(t_0) - \lambda\delta\}$ then again by Lemma 2.5, there exists a non-empty open subset V of V' and so of W such that $\Phi(V) \subseteq S(\lambda\rho(t_0)\, B(X^{**}), \hat{g}, \lambda\delta)$ and so $\alpha(\Phi(V)) < \varepsilon$. We conclude that Φ is α upper semi-continuous on the dense G_δ subset $\bigcap_1^\infty O_{1/n}$ of A and our result follows from Theorem 2.6.

We noted previously that the subdifferential mapping $f \to \partial\phi(f)$ of a continuous convex function ϕ on an open convex subset A of X^* is a minimal weak * cusco from A into subsets of X^{**}, so from Corollary 2.7, ϕ is Fréchet differentiable on a dense G_δ subset of A. ▌

The question now arises whether the class of Banach spaces we have been considering in this theorem is larger than the class in the original theorem we have generalised. It is an open question whether spaces of our class can be equivalently renormed to have this more restricted condition.

We do have a differentiability property for the dual norm of a Banach space X where every point of $S(X)$ is an α-denting point of $B(X)$. This result generalises [8, Theorem 3.2]. We need the following elementary property for slices.

4.6. LEMMA. *For a Banach space X and $x \in S(X)$ and any slice of $B(X)$ determined by $f \in S(X^*)$ and containing x, there exists an $\varepsilon > 0$ such that for all $g \in S(X^*)$ where $\|g - f\| < \varepsilon$ there exists a slice of $B(X)$ determined by g which contains x and is contained in the slice determined by f.*

Proof. There exists some $0 < \delta < 1$ such that $x \in S(B(X), f, \delta)$. Choose $\varepsilon < \frac{1}{2}(f(x) - (1-\delta))$. Then for $\|f - g\| < \varepsilon$ we have $g(x) > f(x) - \varepsilon > 1 - (\delta - \varepsilon)$. So $x \in S(B(X), g, \delta - \varepsilon)$. But also for $y \in S(B(X), g, \delta - \varepsilon)$ we have $f(y) \geqslant g(y) - \varepsilon > 1 - \delta$. So $y \in S(B(X), f, \delta)$. ∎

Given $\varepsilon > 0$, we denote by $\alpha_\varepsilon(S(X^*))$ the set of points of $S(X^*)$ which determine slices of $B(X)$ with Kuratowski index less than ε. From Lemma 4.6 we see that $\alpha_\varepsilon(S(X^*))$ is open in $S(X^*)$. Further $B(X)$ has property α for all $f \in \bigcap_1^\infty \alpha_{1/n}(S(X^*))$.

4.7. THEOREM. *A Banach space X where every point of $S(X)$ is an α-denting point of $B(X)$ has dual norm Fréchet differentiable on a dense G_δ subset of X^*.*

Proof. Consider $f \in S(X^*)$ which attains its norm on $S(X)$ say at $x \in S(X)$. Then x is an α-denting point of $B(X)$, so given $0 < \varepsilon < 1$ there exists a $g \in S(X^*)$ and $0 < \delta < 1$ such that $x \in S(B(X), g, \delta)$ and $\alpha(S(B(X), g, \delta)) < \varepsilon$. For $0 < \eta < \varepsilon$ consider $h = \eta g + (1-\eta) f$. Then $\|h - f\| < 2\eta$. Writing $K \equiv B(X) \backslash S(B(X), g, \delta)$ we have $\sup h(K) \leqslant \eta \sup g(K) + (1-\eta) \sup f(K) < \eta g(x) + (1-\eta) f(x) = h(x)$. So h separates x from K and defines a slice of $B(X)$ containing x but contained in $S(B(X), f, \delta)$. Then $h \in \alpha_\varepsilon(S(X^*))$. From the Bishop–Phelps Theorem we conclude that $\alpha_\varepsilon(S(X^*))$ is dense in $S(X^*)$ and since $\alpha_\varepsilon(S(X^*))$ is open in $S(X^*)$, $\bigcap_1^\infty \alpha_{1/n}(S(X^*))$ is a dense G_δ subset of $S(X^*)$. Then $B(X)$ has property α for all $f \in \bigcap_1^\infty \alpha_{1/n}(S(X^*))$ and from Theorem 3.2(ii), the duality mapping $f \to D(f)$ on $S(X^*)$ is α upper semi-continuous on $\bigcap_1^\infty \alpha_{1/n}(S(X^*))$. We deduce from Theorem 2.6 that the norm of X^* is Fréchet differentiable on a dense G_δ subset of $S(X^*)$. ∎

We should note that if a Banach space X has every point of $S(X)$ an α-denting point of $B(X)$ this does not even imply that the denting points of $B(X)$ are dense in $S(X)$. Every finite dimensional Banach space X_n has every point of $S(X_n)$ an α-denting point of $B(X_n)$ but if its unit sphere contains faces of dimension $(n-1)$ then the denting points of $B(X_n)$ are not dense in $S(X_n)$. So Theorem 4.7 is a genuine advance on [8, Theorem 3.2]. Nevertheless, Theorem 4.7 implies that the closed unit ball of a space of our class is the closed convex hull of its strongly exposed points [13, p. 87].

c. *A Generalisation of Strongly Exposed Point Structure*

Given a closed bounded convex set K in a Banach space X, $x \in K$ is a strongly exposed point of K if there exists an $f \in S(X^*)$ such that for every $\varepsilon > 0$ there exists a $0 < \delta < \sup f(K)$ such that $x \in S(K, f, \delta)$ and diam $S(K, f, \delta) < \varepsilon$; we say that f strongly exposes K at x. For a closed bounded convex set K in the dual X^* of a Banach space X, $f \in K$ is a weak * strongly exposed point of K if there exists an $\hat{x} \in S(\hat{X})$ which strongly exposes K at f. We now give the appropriate generalisation of this concept using the Kuratowski index of non-compactness. For a closed bounded convex set K in a Banach space X we say that a subset E of K is α-strongly exposed if there exists an $f \in S(X^*)$ such that K has property α for f and $E = \bigcap_{\delta > 0} S(K, f, \delta)$; we say that f α-strongly exposes K at E. For a closed bounded convex set K in the dual X^* of a Banach space X, we say that a subset E of K is weak * α-strongly exposed if there exists an $\hat{x} \in S(\hat{X})$ which α strongly exposes K at E.

It is important to recognise that the two concepts coincide for singleton sets.

4.8. Proposition. *Given a closed bounded convex set K in a Banach space X, a singleton subset $\{x\}$ of K is α-strongly exposed if and only if x is a strongly exposed point of K.*

Proof. If $x \in K$ is a strongly exposed point of K then clearly $\{x\}$ is α-strongly exposed.

Conversely, consider $\{x\}$ α-strongly exposed by $f \in S(X^*)$. Given $\varepsilon > 0$ consider $B(x; \varepsilon)$ and the nested sequence of closed sets $\{C(B(x; \varepsilon)) \cap \overline{S(K, f, 1/n)}\}$. If all the sets in this sequence are non-empty then by Lemma 2.3, $C(B(x; \varepsilon)) \cap \bigcap_1^\infty \overline{S(K, f, 1/n)}$ is non-empty. Then $C(B(x; \varepsilon)) \cap \bigcap_1^\infty S(K, f, 2/n)$ is non-empty, but this contradicts $\{x\} = \bigcap_{\delta > 0} S(K, f, \delta)$. So we conclude that one of the closed sets in our nested sequence is empty; that is, there exists a $\delta > 0$ such that $S(K, f, \delta) \subseteq B(x; \varepsilon)$ and this implies that x is a strongly exposed point of K. ∎

Namioka and Phelps, [12, p. 735], also characterised an Asplund space by the weak * strongly exposed point structure of the weak * compact convex subsets of its dual. The corresponding characterisation is in terms of weak * α-strongly exposed subsets of such sets.

4.9. Theorem. *For a Banach space X the following are equivalent,*

(i) *every continuous convex function ϕ on an open convex subset A of X is Fréchet differentiable on a dense G_δ subset of A,*

(ii) *every weak * compact convex subset of X^* is the weak * closed convex hull of its weak * strongly exposed points,*

(iii) *every weak * compact convex subset of* X^* *is the weak * closed convex hull of its weak * α-strongly exposed subsets.*

Proof. In view of the classical characterisation and because it is obvious that (ii) ⇒ (iii), it will be sufficient to prove (iii) ⇒ (i). But if (iii) holds then every weak * compact convex set in X^* has weak * slices whose Kuratowski index is arbitrarily small. But this implies that every non-empty bounded subset of X^* has weak * slices whose Kuratowski index is arbitrarily small. Then our conclusion follows from Theorem 4.2(iii) ⇒ (i). ∎

The dual theorem follows from Theorem 4.3(iii) ⇒ (i) by a similar argument.

4.10. THEOREM. *For a Banach space* X *the following are equivalent,*

(i) *every continuous weak * lower semi-continuous convex function* ϕ *on an open convex subset* A *of* X^* *is Fréchet differentiable on a dense* G_δ *subset of* A,

(ii) *every closed bounded convex set in* X *is the closed convex hull of its strongly exposed points,*

(iii) *every closed bounded convex set in* X *is the closed convex hull of its* α-*strongly exposed subsets.*

Note added in proof. Troyanski [17] has recently answered the question we raised concerning Theorem 4.5 by showing that a Banach space X where every point of $S(X)$ is an α-denting point of $B(X)$ can be equivalently renormed to be locally uniformly rotund.

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