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Delay Differential Equations with Hill's Type Growth Rate and Linear Harvesting

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Abstract—For the equation,

$$\dot{N}(t) = \frac{r(t)\,N(t)}{1+[N(t)]^{\gamma}} - b(t)\,N(t) - a(t)\,N(g(t)),$$

we obtain the following results: boundedness of all positive solutions, extinction, and persistence conditions. The proofs employ recent results in the theory of linear delay equations with positive and negative coefficients. © 2005 Elsevier Ltd. All rights reserved.

Keywords—Delay equations, Extinction and persistence, Mackey-Glass equation, Bounded solutions.

1. INTRODUCTION

Consider the following differential balance equation which is widely used [1–3] in population dynamics

$$\frac{\dot{N}}{N} = \beta(t,N) - \delta(t,N) - \lambda(t,N). \tag{1}$$

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In this model, all per capita rates depend solely on the size of the population and the current time t, where

(a) $\beta(t,N)$ is the per capita fecundity rate (the birth rate),
(b) $\delta(t,N)$ is the per capita mortality rate, and
(c) $\lambda(t,N)$ is the per capita consumer rate (harvesting rate per capita).

According to Hassell [4], models in population dynamics in a limited environment are based on two fundamental premises: populations have the potential to increase exponentially and there is a feedback that progressively reduces the actual rate of increase.

We assume that functions $\delta(N)$ and $\lambda(N)$ are monotone increasing functions.

Most population models incorporate function $\beta(t,N)$, where $\beta(t,N)$ is typically assumed to be a linear decreasing function of N. For example, if $\beta(t,N) = r(1 - N/K)$, where K is the carrying capacity of the environment, then in the absence of δ, λ, we have the logistic equation, $\dot{N}(t) = rN(1 - N/K)$.

Some models incorporate $\beta(t,N)$ which is a Hill's type function,

$$\beta(t,N) = \frac{r}{1+N^{\gamma}}, \tag{2}$$

$\gamma > 0$ (see [2,5–7]). A parameter $\gamma > 0$ is referred to as the "abruptness" parameter, sometimes [5].

In [6–8], the function of type (2) was applied to model white blood cells production (Mackey-Glass equation)

$$\frac{dN}{dt} = \frac{rN_{\tau}}{1+N_{\tau}^{\gamma}} - bN, \tag{3}$$

where $E(N) = rN_{\tau}/(1 + N_{\tau}^{\gamma})$ modeled the blood cell reproduction, the time lag $N_{\tau} = N(t - \tau)$ described the maturational phase before blood cells are released into circulation, the mortality rate is proportional to the circulation. Equation (3) was introduced to explain the oscillations in numbers of neutrophils observed in some cases of chronic myelogenous leukemia [6,8]. The reproduction function can differ from one in (3). For instance, $r/(K^{\gamma} + N^{\gamma})$ describes the red blood cells production rate [9], where three parameters, r, K, γ, are chosen to match the experimental data.

Generally, models with the delay in the reproduction term recognize that for real organisms, it takes time to develop from newborns to reproductively active adults. The harvesting term may also include time lag θ, where θ can be the time to develop a consumer strategy.

When we multiply both sides of (1) by N and take into account delays, we get the following time-lag model based on equation (1),

$$\frac{dN}{dt} = N_{\tau}\beta(t,N_{\tau}) - \delta(t,N)N - \lambda(t,N_{\theta})N_{\theta}, \tag{4}$$

where $N_{\tau} = N(t - \tau)$, τ is time to recover.

In further considerations, we assume β is a Hill's type function (2) and δ, λ depend on t only,

$$\frac{dN}{dt} = \frac{r(t)N_{\tau}}{1+N_{\tau}^{\gamma}} - b(t)N - a(t)N_{\theta}, \tag{5}$$

where r (a fecundity factor), a (a hunting factor), b (a mortality factor), and γ (an abruptness factor) are positive.

More general form of equation (5) is

$$\frac{dN}{dt} = \frac{r(t)N_{\eta}}{1+N_{\tau}^{\gamma}} - b(t)N - a(t)N_{\theta}, \tag{6}$$

where two different delays are involved in the reproduction term (which may be important especially at the early stages of population growth).

For example, for $a(t) \equiv 0$, $r(t) \equiv r$, equation (5) is the Mackey-Glass equation (3).

Our goal is to investigate an equation of type (5). In the present paper, we consider this equation, where a delay is included in the harvesting term aN_θ only. Hereafter, we do not assume that parameters of differential equations are continuous functions.

We study the existence of global solutions for the initial value problem, boundedness of these solutions, and also extinction and persistence conditions. The following three possibilities are described: a positive solution eventually exceeds some positive number, there are positive solutions approaching zero, there are solutions which intersect the x-axis. In the first case, the solution is persistent while in the second and the third cases the population extincts, either eventually or in finite time.

Such problems for another delay equation with harvesting were considered in [10,11] and for equations with Richard's nonlinearity in [12,13].

The model (5),(6) incorporates variable coefficients which can describe, for instance, seasonal changes, daily and periodical changes in the disease process, and changing in time harvesting policy. A specific feature of equation (5),(6) is that the linearized equations contain both positive and negative coefficients. To analyze it, we apply some recent oscillation results for such linear equations (see [14,15]).

The research of the present paper can be extended in the following directions. In addition to extinction and persistence problems, it is interesting to obtain stability and oscillation conditions and to study equations (5) and (6) with delays in the reproduction term ($\eta, \tau \neq 0$) as well.

2. PRELIMINARIES

Consider a scalar delay differential equation

$$\dot{N}(t) = \frac{r(t)\, N(t)}{1 + [N(t)]^\gamma} - b(t)\, N(t) - a(t)\, N(g(t)), \qquad t \geq 0, \tag{7}$$

with the initial function and the initial value,

$$N(t) = \varphi(t), \qquad t < 0, \qquad N(0) = N_0, \tag{8}$$

under the following conditions:

(a1) $\gamma > 0$;

(a2) $r(t) \geq 0, b(t) \geq 0, a(t) \geq 0$ are Lebesgue measurable and essentially bounded on $[0, \infty)$ functions, $\liminf_{t\to\infty} b(t) \geq b > 0$;

(a3) $g(t)$ is a Lebesgue measurable function, $g(t) \leq t$, $\limsup_{t\to\infty} g(t) = \infty$;

(a4) $\varphi : (-\infty, 0) \to R$ is a Borel measurable bounded function, $\varphi(t) \geq 0$, $N_0 > 0$.

DEFINITION 1. *A locally absolutely continuous function $N : R \to R$ is called a solution of problem (7),(8), if it satisfies equation (7), for almost all $t \in [0, \infty)$ and equalities (8), for $t \leq 0$.*

If t_0 is the first point, where the solution $N(t)$ of (7),(8) vanishes, i.e., $N(t_0) = 0$, then we consider this solution only in the interval $[0, t_0)$. It means that we consider only positive solutions of the problem (7),(8).

We will present below lemmas which will be used in the proof of the main results.

Consider the scalar linear delay differential equation,

$$\dot{x}(t) + \sum_{l=1}^{n} c_l(t)\, x(g_l(t)) = f(t), \qquad t \geq 0, \tag{9}$$

with the initial condition,

$$x(t) = \varphi(t), \qquad t < 0, \qquad x(0) = x_0, \tag{10}$$

and the corresponding differential inequality,

$$\dot{y}(t) + \sum_{l=1}^{n} c_l(t)\, y(g_l(t)) \leq f(t), \qquad t \geq 0. \tag{11}$$

DEFINITION 2. *A solution $X(t,s)$ of the problem,*

$$\dot{x}(t) + \sum_{l=1}^{n} c_l(t)\, x(g_l(t)) = 0, \qquad t \geq s, \qquad x(t) = 0, \qquad t < s, \qquad x(s) = 1,$$

is called a fundamental function of (9).

DEFINITION 3. *We say that a function is nonoscillatory if it is either eventually positive or eventually negative.*

LEMMA 1. *(See [16].) Suppose functions c_l, f are Lebesgue measurable and essentially bounded on $[0,\infty)$, $\varphi : (-\infty, 0) \to R$ is a Borel measurable bounded function, for g_l Condition (a3) holds. Then, for the solution of (9),(10), we have the following representation,*

$$x(t) = X(t,0)\, x_0 - \int_0^t X(t,s) \sum_{l=1}^{n} c_l(s)\, \varphi(g_l(s))\, ds + \int_0^t X(t,s)\, f(s)\, ds, \tag{12}$$

where $\varphi(t) = 0$, $t \geq 0$.

LEMMA 2. *(See [17].) Suppose functions c_l, f are Lebesgue measurable and essentially bounded on $[0,\infty)$, $\varphi : (-\infty, 0) \to R$ is a Borel measurable bounded function, (a3) holds for g_l. Let $c_l^+(t) = \max\{c_l(t), 0\}$.*

1. *If $c_l(t) \geq 0$, $y(t)$ is a positive solution of (11), for $t \geq t_0 \geq 0$, then $y(t) \leq x(t)$, $t \geq t_0$, where $x(t)$ is a solution of (9) and $x(t) = y(t)$, $t \leq t_0$.*
2. *If $c_l(t) \geq 0$, $\int_0^\infty \sum_{l=1}^n c_l(t)\, dt = \infty$, then for any nonoscillatory solution $x(t)$ of (9) with $f(t) \equiv 0$, we have $\lim_{t\to\infty} x(t) = 0$.*
3. *If*

$$\sup_{t>0} \sum_{l=1}^{n} \int_{\min_k g_k(t)}^{t} c_l^+(s)\, ds \leq \frac{1}{e}, \tag{13}$$

then, equation (9) with $f(t) \equiv 0$ has a nonoscillatory solution.

4. *If either (13) holds or there exists a nonnegative solution of the inequality,*

$$u(t) \geq \sum_{l=1}^{n} c_l^+(t) \exp\left\{ \int_{g_l(t)}^{t} u(s)\, ds \right\}, \quad t \geq 0; \qquad u(t) = 0, \quad t < 0, \tag{14}$$

and

$$0 \leq \varphi(t) \leq x_0, \tag{15}$$

then, the solution of initial value problem (9),(10), with $f(t) \equiv 0$, is positive.

Consider also the following linear delay equation with positive and negative coefficients,

$$\dot{x}(t) + a(t)\, x(g(t)) - c(t)\, x(t) = 0, \qquad t \geq 0, \tag{16}$$

and the corresponding differential inequality,

$$\dot{y}(t) + a(t)\, y(g(t)) - c(t)\, y(t) \leq 0, \qquad t \geq 0. \tag{17}$$

LEMMA 3. *(See [14,15].) Suppose $a(t)$, $c(t)$, $g(t)$ are Lebesgue measurable and essentially bounded on $[0,\infty)$ functions, (a3) holds,*

$$a(t) \geq c(t) \geq 0, \qquad \int_0^\infty [a(t) - c(t)]\, dt = \infty \tag{18}$$

and

$$\limsup_{t\to\infty} c(t)\,[t - g(t)] < 1. \tag{19}$$

Then,

1. *If $y(t)$ is a positive solution of (17), for $t \geq t_0 \geq 0$, then $y(t) \leq x(t)$, $t \geq t_0$, where $x(t)$ is a solution of (16) and $x(t) = y(t)$, $t \leq t_0$.*
2. *For every nonoscillatory solution $x(t)$ of (16), we have $\lim_{t\to\infty} x(t) = 0$.*

3. EXISTENCE AND BOUNDEDNESS OF SOLUTIONS

THEOREM 1. *Suppose (a1)–(a4) hold. Then, problem (7),(8) has a unique local positive solution. This solution either becomes negative or is a global positive bounded solution.*

PROOF. The existence of the unique local solution is a consequence of well-known results for nonlinear delay differential equations (see, for example, [18,19]). Since initial conditions in (8) are nonnegative, then there exists a local positive solution. So, we need only to prove the second part of the theorem. Here, we apply some ideas from [12]. Suppose for some positive local solution, we have

$$\lim_{t\to t_0 -} N(t) = +\infty. \tag{20}$$

Rewrite (7) in the left neighborhood of t_0 in the form

$$\dot{N}(t) = N(t)\left[\frac{r(t)}{1 + [N(t)]^\gamma} - b(t) - a(t)\,\frac{N(g(t))}{N(t)}\right]. \tag{21}$$

Since

$$\lim_{t\to t_0 -} \frac{r(t)}{1 + [N(t)]^\gamma} = 0$$

and $b(t) \geq b > 0$, then for some $\delta > 0$ and, for $t_0 - \delta < t < t_0$, we have $N'(t) \leq 0$, which contradicts (20).

It means that every local solution which remains positive, can be continued on $[0,\infty)$. We need only to prove that this global positive solution is bounded.

First, suppose that for this solution

$$\lim_{t\to\infty} N(t) = \infty. \tag{22}$$

Equation (21) implies that, for some $T > 0$ and, for $t \geq T$, we have $N'(t) < 0$, which contradicts (22).

Now, suppose

$$\limsup_{t\to\infty} N(t) = \infty, \tag{23}$$

but $\lim_{t\to\infty} N(t)$ does not exist. Then, there exists a sequence of intervals $I_n = [a_n, b_n]$, $b_n < a_{n+1}$, $\lim b_n = \infty$, such that

$$\sup_{t\in I_n} N(t) > K_n, \qquad \inf_{t\in I_n} |N'(t)| < \epsilon_n,$$

where $\lim_n K_n = \infty$, $\lim_n \epsilon_n = 0$.

Condition (a2) implies $r(t) \le r_0 < \infty$. Hence, for $t \in I_n$, we have

$$-\epsilon_n < \frac{r(t)\,N(t)}{1+[N(t)]^\gamma} - b(t)\,N(t) - a(t)\,N(g(t)) < \epsilon_n,$$

consequently,

$$a(t)\,N(g(t)) < \frac{r(t)\,N(t)}{1+[N(t)]^\gamma} - b(t)\,N(t) + \epsilon_n.$$

Hence, for n sufficiently large, we have

$$\frac{a(t)\,N(g(t))}{N(t)} < \frac{r(t)}{1+[N(t)]^\gamma} - b(t) + \frac{\epsilon_n}{N(t)} < \frac{r_0}{1+[K_n]^\gamma} - b_0 + \frac{\epsilon_n}{K_n} < 0.$$

This is a contradiction and the theorem is proven.

LEMMA 4. *Assume (a1),(a2) hold, where $b(t)$ is changed by $B(t)$. Then, there exists a unique positive bounded global solution of the problem*

$$\dot{x}(t) = \frac{r(t)\,x(t)}{1+[x(t)]^\gamma} - B(t)\,x(t), \qquad x(0) = x_0 > 0. \tag{24}$$

PROOF. The proof is similar to the proof of Theorem 1.

It is to be noted that, unlike Theorem 1, the solution cannot become negative due to the existence and uniqueness theorem for ordinary differential equations since the zero function is a solution of (24).

We will apply Lemma 4 for $B(t) = b(t)$ and

$$B(t) = b(t) + a(t)\exp\left\{-\int_{g(t)}^{t} [r(s) - b(s)]\,ds\right\}. \tag{25}$$

LEMMA 5. *Assume (a1)–(a4) hold and $N(t)$ is a positive solution of (7),(8). Then,*

$$N(g(t)) \ge N(t)\exp\left\{-\int_{g(t)}^{t} [r(s) - b(s)]\,ds\right\}. \tag{26}$$

PROOF. Equality (7) implies, for any $t_0 \ge 0$,

$$\dot{N}(t) \le r(t)\,N(t) - b(t)\,N(t), \qquad t \ge t_0.$$

Hence,

$$N(t) \le N(t_0)\exp\left\{\int_{t_0}^{t} [r(s) - b(s)]\,ds\right\}.$$

A substitution $t_0 = g(t)$ proves the lemma.

Denote

$$A_0 = \sup_{t\ge 0} x(t), \tag{27}$$

where $x(t)$ is the solution of (24).

If in (24) $r(t) \ge B(t)$, then the solution of this equation increases for

$$x(t) < \left(\frac{r(t)}{B(t)} - 1\right)^{1/\gamma}$$

and decreases otherwise. Hence, $x(t)$ does not exceed

$$A = \max\left\{x_0, \sup_t \left[\frac{r(t)}{B(t)} - 1\right]^{1/\gamma}\right\}$$

and so, $A_0 \le A$.

THEOREM 2. *Assume (a1)–(a4), (15) are satisfied, and either*

$$\sup_{t\geq 0}\int_{g(t)}^{t} a(s)\exp\left\{-\int_{g(s)}^{s}\left[\frac{r(\tau)}{1+A_0^{\gamma}}-b(\tau)\right]d\tau\right\}ds\leq\frac{1}{e}, \tag{28}$$

where A_0 is denoted by (27), or the inequality

$$\sup_{t\geq 0}\int_{g(t)}^{t}\left[a(s)+b(s)\right]ds\leq\frac{1}{e} \tag{29}$$

holds, with $x_0 = N_0$, where either $B(t) = b(t)$ or $B(t)$ is denoted by (25). Then, for the solution of (7),(8), we have

$$0 < N(t) \leq A_0, \qquad t\geq 0. \tag{30}$$

PROOF. Suppose the statement of the theorem is not correct. Then, there exists a solution $N(t)$, such that either $N(t)$ is a global positive solution of (7),(8) satisfying $N(\bar{t}) > A_0$, for some $\bar{t} > 0$ or $N(t)$ vanishes, for some $\bar{t} > 0$,

$$0 < N(t)\leq A_0, \qquad 0\leq t<\bar{t}; \qquad N(\bar{t}) = 0. \tag{31}$$

Suppose the first possibility holds and $B(t) = b(t)$. For any $t_0 < \bar{t}$ and for $0 \leq t < t_0$, we have

$$\dot{N}(t)\leq\frac{r(t)N(t)}{1+[N(t)]^{\gamma}}-b(t)N(t).$$

Hence, [20] $N(t) \leq x(t)$, where $x(t)$ is the solution of ordinary differential equation (24) with $x_0 = N_0$.

Since $x(t) \leq A_0$, then $N(t) \leq A_0$ and we have a contradiction.

Suppose now $B(t)$ is denoted by (25). Lemma 5 implies

$$\dot{N}(t)\leq\frac{r(t)N(t)}{1+[N(t)]^{\gamma}}-\left[b(t)+a(t)\exp\left\{-\int_{g(t)}^{t}\left[r(s)-b(s)\right]ds\right\}\right]N(t).$$

Hence, $0 < N(t) \leq A_0$ is proved similar to the previous case.

Let (28) and (31) hold. Define $y(t)$ by the following expression

$$N(t)=\exp\left\{\int_{0}^{t}\left[\frac{r(s)}{1+[N(s)]^{\gamma}}-b(s)\right]ds\right\}y(t), \qquad t\geq 0,$$
$$y(t)=\varphi(t), \qquad t<0, \qquad y(0)=N_0,$$

and substitute this expression into (7). We obtain the equation

$$\dot{y}(t)=-a(t)\exp\left\{-\int_{g(t)}^{t}\left[\frac{r(s)}{1+[N(s)]^{\gamma}}-b(s)\right]ds\right\}y(g(t)), \tag{32}$$

$$y(t)=\varphi(t), \qquad t<0, \qquad y(0)=N_0. \tag{33}$$

By assumption (31) $N(t) \leq A_0$, thus, (28) implies

$$\int_{g(t)}^{t} a(s)\exp\left\{-\int_{g(s)}^{s}\left[\frac{r(\tau)}{1+[N(\tau)]^{\gamma}}-b(\tau)\right]d\tau\right\}ds$$
$$\leq\int_{g(t)}^{t} a(s)\exp\left\{-\int_{g(s)}^{s}\left[\frac{r(\tau)}{1+[A_0]^{\gamma}}-b(\tau)\right]d\tau\right\}ds\leq\frac{1}{e}.$$

Lemma 2, Part 4, implies that for the solution of (32),(33), we have $y(t) > 0$, $t \geq 0$. Hence, $N(t) > 0$, $t \geq 0$. We have a contradiction.

Suppose now (29) and (31) hold. Solution $N(t)$ of (7),(8) is also a solution of the linear delay differential equation,

$$\dot{x}(t) + c(t)x(t) + a(t)x(g(t)) = 0, \qquad 0 \leq t \leq \bar{t}, \qquad x(t) = N(t), \qquad t \leq 0,$$

where

$$c(t) = b(t) - \frac{r(t)}{1 + N(t)^{\gamma}}.$$

We have $c^{+}(t) \leq b(t)$, $0 \leq t \leq \bar{t}$, hence,

$$\sup_{t \geq 0} \int_{g(t)}^{t} \left[a(s) + c^{+}(s)\right] ds \leq \frac{1}{e}, \qquad 0 \leq t \leq \bar{t}.$$

Lemma 2, Part 4, implies $N(t) = x(t) > 0$, $0 \leq t \leq \bar{t}$. We have a contradiction with (31) and the theorem is proven.

REMARK. Number A_0 in Theorem 2 can be replaced by any greater number $A \geq A_0$.

The proof of Theorem 2 implies the following corollaries.

COROLLARY 2.1. *Suppose (a1)–(a4) hold. Then, (30) holds for any positive solution of (7).*

COROLLARY 2.2. *Suppose (a1)–(a4), (15), and (29) hold. Then, the solution of (7),(8) is positive.*

COROLLARY 2.3. *Suppose (a1)–(a4) hold, $r(t) = \alpha b(t)$, $\alpha > 0$, and $N(t)$ is a positive solution of (7),(8).*

(1) *If $\alpha \leq 1$ or $\alpha > 1$, $N_0 > N^* = (\alpha - 1)^{1/\gamma}$, then $N(t) \leq N_0$.*
(2) *If $\alpha > 1$, $N_0 \leq N^*$, then $N(t) \leq N^*$.*

PROOF. First, suppose $\alpha \leq 1$. Rewrite equation (24) with $B(t) = b(t)$ in the form

$$\dot{x}(t) = r(t)x(t)\frac{\alpha - 1 - [x(t)]^{\gamma}}{\alpha(1 + [x(t)]^{\gamma})}. \tag{34}$$

Hence, $\dot{x}(t) \leq 0$, $t \geq 0$. The proof of Theorem 2 implies $N(t) \leq x(t) \leq x_0 = N_0$.

Now, suppose $\alpha > 1$. Then, $N^* > 0$ is a positive equilibrium of (24) and we have two possibilities.

If $x_0 = N_0 \leq N^*$, then $N(t) \leq x(t) \leq N^*$.

If $x_0 = N_0 > N^*$, then $\dot{x}(t) < 0$.

Hence, $N(t) \leq x(t) \leq x_0 = N_0$.

REMARK. If $\alpha \leq 1$, then $r(t) \leq b(t)$. This case will be considered in the next section.

Now consider the autonomous equation

$$\dot{N}(t) = \frac{rN(t)}{1 + [N(t)]^{\gamma}} - bN(t) - aN(t - \tau), \qquad t \geq 0, \tag{35}$$

where $r > 0$, $b > 0$, $a > 0$, $\tau > 0$, $\gamma > 0$.

Denote

$$N^* = \begin{cases} \left(\frac{r}{b} - 1\right)^{1/\gamma}, & b < r < b + ae^{-\tau(r-b)}, \\ \left(\frac{r}{b + ae^{-\tau(r-b)}} - 1\right)^{1/\gamma}, & r \geq b + ae^{-\tau(r-b)}. \end{cases} \tag{36}$$

COROLLARY 2.4. *Suppose N^* is denoted by (36). Then, for a positive solution of (35),(8), we have*

$$N(t) \le A_0 = \max\{N_0, N^*\}. \tag{37}$$

If (15) holds and either

$$a\tau \exp\left\{-\left[\left(\frac{r}{1+A_0^\gamma} - b\right)\tau - 1\right]\right\} \le 1 \tag{38}$$

or $(a+b)\tau \le 1/e$, where A_0 is denoted by (37), then the solution of (35),(8) is positive.

PROOF. If N^* is defined by the first line in (36), then the first statement is a particular case of Corollary 2.3.

Now, consider the case of the second line in (36). Rewrite equation (24) with $B(t) = b + ae^{-\tau(r-b)}$ in the form

$$\dot{x}(t) = \frac{rx(t)}{1+[x(t)]^\gamma} - \left[b + ae^{-\tau(r-b)}\right] x(t). \tag{39}$$

The number N^* is a positive equilibrium of this equation. Hence, for $x_0 = N_0 > 0$, we have

$$x(t) \le \begin{cases} N_0, & N_0 > N^*, \\ N^*, & N_0 \le N^*. \end{cases}$$

Finally, the statement of Corollary 2.1 implies this corollary.

REMARK. Suppose $N_0 < N^*$. If $b < r < b + ae^{-\tau(r-b)}$, then (38) turns into

$$a\tau e \le 1. \tag{40}$$

If $r \ge b + ae^{-\tau(r-b)}$, then (38) turns into

$$a\tau e^{-\left[ae^{-\tau(r-b)} - 1\right]} \le 1. \tag{41}$$

4. EXTINCTION AND PERSISTENCE

DEFINITION 4. *We will say that $N(t)$ is an extinct solution of (7),(8) if either*

$$\lim_{t\to\infty} N(t) = 0, \tag{42}$$

or there exists $\bar{t} > 0$, such that $N(\bar{t}) = 0$.

The bounded solution is persistent if

$$\liminf_{t\to\infty} N(t) > 0. \tag{43}$$

THEOREM 3. *Assume (a1)–(a4) hold,*

$$\int_0^\infty [a(t) + b(t) - r(t)]\, dt = \infty$$

and one of the two following conditions hold,

$$b(t) \ge r(t) \tag{44}$$

or

$$a(t) + b(t) \ge r(t) \ge b(t), \qquad \limsup_{t\to\infty} [r(t) - b(t)](t - g(t)) < 1. \tag{45}$$

Then, every solution of (7),(8) is an extinct one.

PROOF. It is sufficient to prove that for every positive solution $N(t)$ of (7),(8), we have (42).

Suppose (44) holds. Since

$$N'(t) \leq -[b(t) - r(t)]\, N(t) - a(t)\, N(g(t)),$$

then, Lemma 2 implies $0 \leq N(t) \leq x(t)$, where $x(t)$ is a solution of the linear delay differential equation with positive coefficients,

$$\dot{x}(t) + [b(t) - r(t)]\, x(t) + a(t)\, x(g(t)) = 0,$$

and initial conditions (10). Again, by Lemma 2, we have $\lim_{t\to\infty} x(t) = 0$. Hence, (42) holds.

Now, suppose (45) holds. Then,

$$N'(t) \leq [r(t) - b(t)]\, N(t) - a(t)\, N(g(t)).$$

Lemma 3 implies $0 \leq N(t) \leq y(t)$, where $y(t)$ is a solution of the linear delay differential equation

$$\dot{y}(t) + a(t)\, x(g(t)) - [r(t) - b(t)]\, y(t) = 0.$$

Again, by Lemma 3, we have $\lim_{t\to\infty} y(t) = 0$. Hence, (42) holds. The theorem is proven.

COROLLARY 3.1. *Suppose $b \geq r$ or $a + b \geq r \geq b$, $(r - b)\tau < 1$. Then, every solution of (35) is an extinct one.*

THEOREM 4. *Suppose the conditions of Theorem 2 hold, the solution of the problem*

$$\dot{x}(t) = \frac{r(t)\, x(t)}{1 + x(t)^{\gamma}} - b(t)\, x(t) - a(t)\, A_0, \qquad x(0) = N_0, \tag{46}$$

is positive and persistent, where A_0 is denoted by (27).

Then, the solution of (7),(8) is persistent.

PROOF. Since $N(t) \leq A_0$, $t \geq 0$, and (15) holds, then $N(g(t)) \leq A_0$, $t \geq 0$. Hence,

$$\dot{N}(t) \geq \frac{r(t)\, N(t)}{1 + N(t)^{\gamma}} - b(t)\, N(t) - a(t)\, A_0.$$

Equation (46) is an ordinary differential equation, thus, $N(t) \geq x(t)$, which completes the proof of the theorem.

REMARK 1. The result of Theorem 4 can be applied to autonomous equations. For example, let $\gamma = 1$. If equation (46) has equilibria, then they are real solutions of the following quadratic equation

$$f(x) = rx - bx(1 + x) - aA_0(1 + x) = -bx^2 + (r - b - aA_0)\, x - aA_0 = 0.$$

Since $f(0) < 0$, then this equation has either two positive solutions or two negative solutions or no real solutions. In two latter cases, we cannot guarantee positiveness of solutions; in the first case, the solution is positive and persistent when the initial value exceeds some critical size.

REMARK 2. The statement of Theorem 4 remains true if we replace equation (46) by the following equation

$$\dot{x}(t) = \frac{r(t)\, x(t)}{1 + x(t)^{\gamma}} - [b(t) + a(t)]\, A_0, \qquad x(0) = N_0.$$

5. DISCUSSION AND NUMERICAL EXAMPLES

The problem is well posed, as justified in Theorem 1. It either has a positive solution for all t which is bounded (the size of the population cannot infinitely grow due to a negative feedback which is a typical situation in population dynamics) or becomes negative in some finite time. Lemma 4 describes the source of the possible extinction. As far as there is no harvesting or there is no delay in the harvesting term, the solution is positive for all t.

Theorem 2 provides sufficient conditions for the positiveness of solutions and presents an upper bound for a solution. Inequality (15) is vital for nonextinction in the following sense. If the harvesting rate is based on the size of the population some time ago, then for the survival of the population it is important that the field data on the population size is collected at the time when the population is not abundant. If the initial value is less than the initial function, then the harvesting based on the oversized estimation of the population can lead to the extinction at the very beginning of the history (when the influence of the prehistory is still significant). In (28), sufficient condition on harvesting, mortality and growth rates, and the delay provide the solution is positive. The greater the mortality and the harvesting rates are, the smaller should be delays providing that there is no extinction of the population in some finite time. Or else, for prescribed delays and a given natural growth rate $r(t)$ and the mortality rate $b(t)$ the harvesting rate should not exceed a certain number to avoid possible extinction. As the growth rate $r(t)$ becomes higher, the greater can be the allowed delay in harvesting. Example 1 illustrates extinction due to either greater delay or high values of prehistory compared to the initial value.

The solution estimate (30) illustrates an obvious fact that under harvesting the solution cannot exceed the solution without harvesting. Corollary 2.3 deals with the situation when a nonconstant mortality rate is proportional to the birth rate. Corollary 2.4 claims that at any point the solution does not exceed the maximal value among the equilibrium without harvesting and the initial value.

Next, Theorem 3 claims that if the total of the mortality and harvesting rates exceeds the birth rate, then the population is destined to extinct. It either equals to zero at some finite moment of time or tends to zero. Example 2 illustrates all possibilities.

Finally, Theorem 4 provides sufficient persistence conditions. Sometimes, the existence of the lower bound for a solution is not less important than positiveness of solutions. For instance, for blood diseases, the mortal rate of white or red blood cells is greater than zero. This means that for smaller values of N, the model can become irrelevant. Example 3 can demonstrate that in the presence of delay in the harvesting term, the fact that the production rate is everywhere greater than the harvesting rate still cannot guarantee the persistence of any solution. Theorem 4 not only claims the persistence, its conditions provide that the solution cannot become smaller than the solution of some nonhomogeneous ordinary differential equation (46).

We proceed to results of numerical simulations.

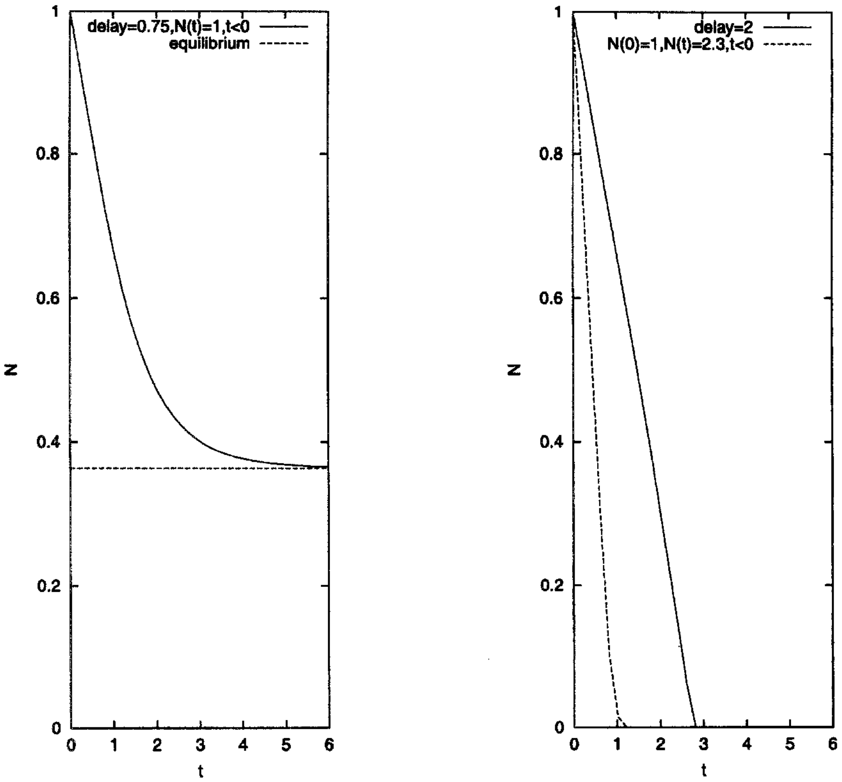
EXAMPLE 1. Let us illustrate the results of Corollary 2.4. For example, for the solution of equation (35) with $r = 1.5$, $\gamma = 1$, $b = 0.5$, $a = 0.6$, $\tau = 0.75$, and the initial conditions, $N(0) = 1$, $\varphi \equiv 1$, we have $r > a+b > b+ae^{-\tau(r-b)}$, thus, (38) takes form (41), which is satisfied,

$$a\tau e^{-\left[ae^{-\tau(r-b)}-1\right]} = 0.45e^{1-0.6e^{-0.75}} \approx 0.9213 \le 1$$

and the solution is positive (Figure 1a). If $r = 1.5$, $\gamma = 1$, $b = 0.5$, $a = 0.6$, $\tau = 2$, then condition (41) does not hold

$$a\tau e^{-\left[ae^{-\tau(r-b)}-1\right]} = 1.2e^{1-0.6e^{2}} \approx 3 > 1,$$

numerical simulations demonstrate that the solution extincts at $t \approx 2.7$ (Figure 1b). If $r = 1.5$, $\gamma = 1$, $b = 0.5$, $a = 0.6$, $\tau = 0.75$, then (38) is satisfied. However, if we take $N(0) = 1$, $\varphi \equiv 2.3$, then (15) does not hold. The solution extincts at $t \approx 1.2$ (Figure 1b).



(a) (b)

Figure 1. The behavior of the solution of (35), with $r = 1.5$, $\gamma = 1$, $b = 0.5$, $a = 0.6$, $N(0) = 1$, $\varphi \equiv 1$, $\tau = 0.75$ (a). Then, the hypotheses of Corollary 2.4 are satisfied. The solution is positive and tends to the equilibrium. The hypotheses of Corollary 2.4 are not satisfied and the solution becomes negative (b). Here, first, instead of $\tau = 0.75$, we have $\tau = 2$ and thus, (38) does not hold, or for the same parameters of the equation as in the left graph initial conditions are $N(0) = 1$, $\varphi \equiv 2.3$, thus, (15) does not hold.

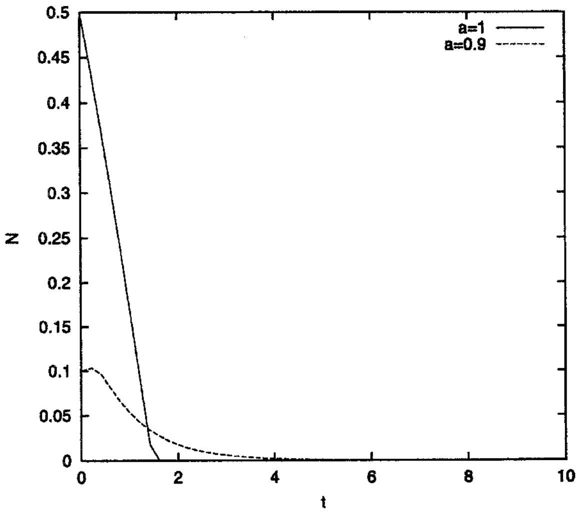


Figure 2. The behavior of solutions of (35), where $r = 1.5$, $\gamma = 1$, $b = 0.6$, with $a = 1$, $N(0) = 0.5$, $\varphi \equiv 0.5$, $g = 1$, and $a = 0.9$, $g = 0.3$, $\gamma = 0.01$, $N(0) = 0.1$, $\varphi \equiv 0$, respectively. The first solution becomes negative at about $t \approx 1.5$, while the second solution tends to zero.

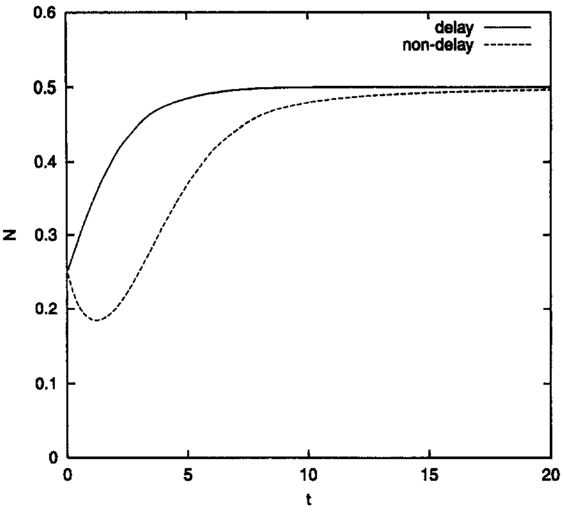


Figure 3. The comparison of solutions of the delay equation (47) with an ordinary comparison equation (48), with the same initial value 0.25. The solution of (48) is persistent (the lower bound is ≈ 1.84), by Theorem 4, so is the solution of (47) with the initial function $\equiv 0.25$. Moreover, the solution of (48) does not exceed the solution of (47).

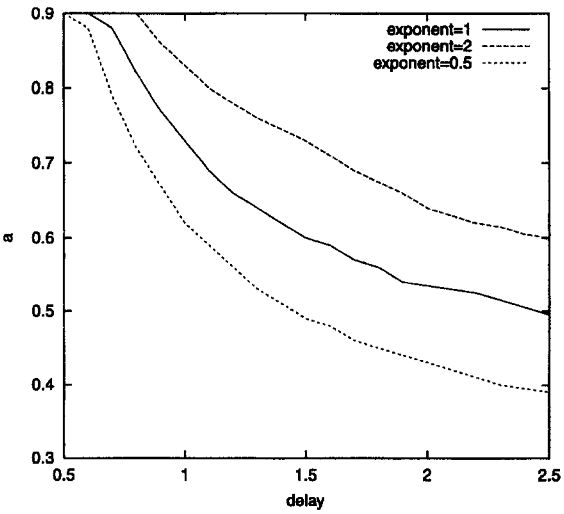


Figure 4. The largest values for coefficient a and delay τ, such that the solution of (35), with $N(0) = 0.5$, $\varphi(t) \equiv 0.5$, $r = 1.5$, $b = 0.6$, and exponential $\gamma = 0.5, 1, 2$, is still positive.

EXAMPLE 2. Figure 2 illustrates extinction when the hypotheses of Corollary 3.1 are satisfied. Here, (see Figure 2) the solutions of (35), where $r = 1.5$, $\gamma = 1$, $b = 0.6$, with $a = 1$, $N(0) = 0.5$, $\varphi \equiv 0.5$, $g = 1$, and $a = 0.9$, $g = 0.3$, $\gamma = 0.01$, $N(0) = 0.1, \varphi \equiv 0$, respectively. The first solution becomes negative at about $t \approx 1.5$, while the second solution is positive and tends to zero.

Let us illustrate Theorem 4 by the following example.

EXAMPLE 3. Consider the following equation

$$\dot{N}(t) = \frac{3N(t)}{1+N(t)} - 2N(t) - \frac{2}{(t+2)^2} N(t-1). \tag{47}$$

The equation, $\dot{x} = 3x/(1+x) - 2x$, has equilibria $x = 0, 1/2$, thus, as far as $N(0) = x(0) < 1/2$, we have $A_0 = 1/2$. The solution of (46), which takes the form

$$\dot{x} = \frac{3x}{1+x} - 2x - \frac{1}{(t+2)^2},$$

is persistent and does not exceed the solution of equation (47), with the initial conditions $\varphi(t) \equiv N(0) = 0.25$ (see Figure 3).

EXAMPLE 4. Finally, for the autonomous equation (35) consider conditions for positiveness of solutions. To this end let us fix initial conditions $N(0) = 0.5$, $\varphi(t) \equiv 0.5$, and some of coefficients, $r = 1.5$, $b = 0.6$. For various τ, $0.5 \leq \tau$, Figure 4 demonstrates such values of a that for larger a the solution becomes negative. Three graphs are presented for exponential $\gamma = 0.5, 1, 2$.

In addition, let us compare experimental results for $N(0) = 0.5$, $\varphi(t) \equiv 0.5$, $r = 1.5, b = 0.5$, $\gamma = 1$, with sufficient estimates provided by (41). For comparison, see Figure 5.

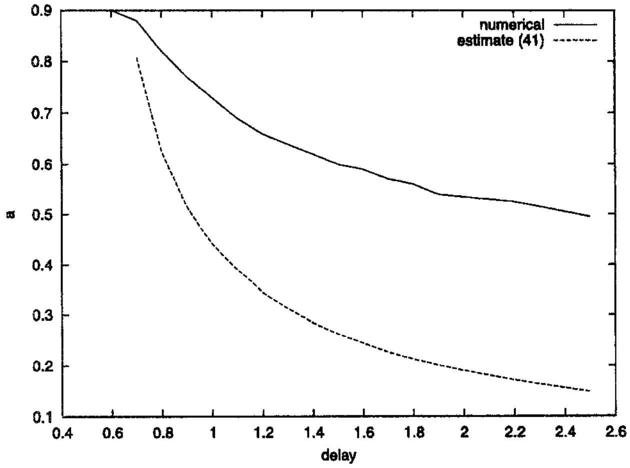


Figure 5. The largest values for coefficient a and delay τ, such that the solution of (35), with $N(0) = 0.5$, $\varphi(t) \equiv 0.5$, $r = 1.5$, $b = 0.5$, and $\gamma = 1$, is still positive, is compared to the corresponding values of a and τ obtained by estimate (41).

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