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Bifurcation of small limit cycles in Z_5-equivariant planar vector fields of order 5

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Abstract

In this paper, we consider bifurcation of small limit cycles from Hopf-type singular points in Z_5-equivariant planar vector fields of order 5. We apply normal form theory and the technique of solving coupled multivariate polynomial equations to prove that the maximal number of small limit cycles that such vector fields can have is 25. In addition, we show that no large limit cycles exist. Thus, $H(5) \geqslant 25$, where $H(n)$ denotes the Hilbert number of the nth-degree polynomial vector fields. This improves the best result of $H(5) \geqslant 24$ existing in the current literature.

Keywords: Hilbert's 16th problem; Hopf bifurcation; Focus value; Limit cycle; Normal form

1. Introduction

In 1990 Hilbert proposed the well-known 23 mathematical problems [1], which had significant impact on mathematics in the 20th century. One of the two unsolved problems is the 16th problem, which includes two parts. The second part of the problem considers the upper bound of the number of limit cycles and their relative locations in polynomial vector fields. The second part of Hilbert's 16th problem was recently chosen by Smale [2] as one of the 18 most challenging mathematical problems for the 21st century. Although the problem is still far away from

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being completely solved, the research on this problem has made great progress with significant contributions to the development of modern mathematics. The recent developments of Hilbert's 16th problem were summarized in the survey articles [3,4].

To state Hilbert's 16th problem more precisely, consider the planar vector field, described by the following polynomial differential equations:

$$\frac{dx}{dt} = P_n(x, y), \qquad \frac{dy}{dt} = Q_n(x, y), \tag{1}$$

where $P_n(x, y)$ and $Q_n(x, y)$ denote the nth-degree polynomials of x and y. The second part of Hilbert's 16th problem is to find the upper bound on the number of limit cycles that the system can have, which is denoted by $H(n)$, known as Hilbert number. In general, this is a very difficult problem. Although it has not been possible to obtain a uniform upper bound for $H(n)$, a great deal of efforts have been made in finding the maximal number of limit cycles and raising the lower bound of the Hilbert number $H(n)$ for general planar polynomial systems or for some specific systems of certain degrees, hoping to be close to the real upper bound of $H(n)$.

One direction of research on Hilbert's 16th problem is to study the weakened Hilbert's 16th problem, introduced by Arnold [5]. The weakened problem is also called the tangential or infinitesimal Hilbert's 16th problem. The basic idea of the weakened problem is to consider perturbing Hamiltonian systems so that the issue of finding the number of limit cycles is transformed to finding the roots of Abelian integrals.

If Hilbert's 16th problem is restricted to a neighborhood of isolated fixed points, then the problems becomes studying degenerate Hopf bifurcations. This gives rise to computation of focus values, which is equivalent to computing the normal form of differential equations associated with Hopf or degenerate Hopf bifurcations. The basic idea of using normal forms to consider limit cycles is as follows. Suppose the origin of system (1), $(x, y) = (0, 0)$, is a fixed point of the system, at which the eigenvalues of the Jacobian of the system is a purely imaginary pair, $\pm i\omega_c$. Then Hopf bifurcation occurs and a family of limit cycles bifurcates from a critical point. Assume that the associated normal form of the system is obtained by using, say, the method given in [6] in polar coordinates:

$$\frac{dr}{dt} = r\left(v_0 + v_1 r^2 + v_2 r^4 + \cdots + v_k r^{2k} + \cdots\right), \tag{2}$$

$$\frac{d\theta}{dt} = \omega_c + t_1 r^2 + t_2 r^4 + \cdots + t_k r^{2k} + \cdots, \tag{3}$$

where r and θ represent, respectively, the amplitude and phase of the limit cycles, and v_i, $i = 0, 1, 2, \ldots$, denote the focus values, are determined by Eq. (2).

To find k small limit cycles around the origin, first find the conditions such that $v_0 = v_1 = v_2 = \cdots = v_{k-1} = 0$, but $v_k \neq 0$, and then perform appropriate small perturbations to prove the existence of k limit cycles. For quadratic planar systems, in 1952, Bautin [7] proved that the maximal number of small limit cycles is 3. In the past few years, great progress has been achieved in obtaining better estimations of the lower bounds of $H(n)$ for $n \geqslant 3$. For cubic order systems, 12 limit cycles have been found [8–10], that is, $H(3) \geqslant 12$. In 2004, Zhang et al. [11] proved that $H(4) \geqslant 15$ by perturbing a cubic order Hamiltonian vector field with 4th-degree polynomial functions. For $n = 5$, several results have been reported, which are all based on the study of Z_q-equivariant vector fields. In 2001, Li et al. [12] proved that 5th-order planar vector fields with Z_3 symmetry could have 23 limit cycles by using the detection function method [13]. In 2002, the same authors [14] showed that 5th-order planar vector fields with Z_6 symmetry could have 24 limit cycles. The 29 limit cycles, found by Chen et al. [15] for 5th-order planer vector fields with Z_2 symmetry, was recently shown erroneous [16,17]. Therefore, the best result obtained so far for $n = 5$ is $H(5) \geqslant 24$. For $n = 6$, Wang and Yu [18] combined normal form theory with detection function method to show that $H(6) \geqslant 35$ for Z_2-equivariant vector field of degree 5 with 6th-degree polynomial perturbation. For $n = 7$, Li and Zhang [19] used the detection function method to show that $H(7) \geqslant 49$ by considering a Z_8-equivariant vector fields of degree 7. The result for $n = 9$ is $H(9) \geqslant 80$, obtained by Wang et al. [20], and that for $n = 11$ is $H(11) \geqslant 121$, proved by Wang and Yu [21].

In this paper, we consider bifurcation of small limit cycles in Z_5-equivariant vector fields of order 5. We apply local analysis to prove that such vector fields can have 25 small limit cycles bifurcating from 5 degenerate Hopf singular points. Further, we show that no large limit cycles exist in such a vector field with 25 small limit cycles. That is, $H(5) \geqslant 25$, improving the best result of $H(5) \geqslant 24$, obtained by using global bifurcation analysis.

The rest of the paper is organized as follows. In the next section, the conditions for a vector field to be Z_q-equivariant are presented. In Section 3, the existence of 25 small limit cycles in 5th-order Z_5-equivariant vector fields is proved. Section 4 is devoted to showing that there does exist large limit cycles in a 5th-order Z_5-equivariant vector field with 25 small limit cycles. Conclusion is finally drawn in Section 5.

2. Z_q-equivariant planar vector fields

In this section, for convenience, we present some existing results for planar vector fields to be Z_q-equivariant, which are needed for the next section. For more details of the results, readers are referred to [3].

Let G be a compact Lie group of transformations acting on R^n.

Definition 1. A mapping $\Phi : R^n \to R^n$ is called G-equivariant if, for all $g \in G$ and $x \in R^n$, $\Phi(gx) = g\Phi(x)$. A function $H : R^n \to R$ is called G-invariant if, for all $g \in G$ and $x \in R^n$, $H(gx) = H(x)$. If Φ is a G-equivariant mapping, then the vector field $\dot{x} = \Phi(x)$ is called a G-equivariant vector field.

Definition 2. Let q be an integer. A group Z_q-equivariant vector fields is called a cyclic group of order q if it is generated by a planar counterclockwise rotation of the vector fields through $2\pi/q$ about the origin.

To define Z_q-equivariant vector fields, introducing the transformation $z = x + iy$, $\bar{z} = x - iy$ into system (1) yields

$$\dot{z} = F(z, \bar{z}), \qquad \dot{\bar{z}} = \bar{F}(z, \bar{z}), \tag{4}$$

where $F(z, \bar{z}) = P(u, v) + i\,Q(u, v)$, $u = \frac{1}{2}(z + \bar{z})$ and $v = \frac{1}{2i}(z - \bar{z})$.

Lemma 1. *A vector field defined by* (4) *is Z_q-equivariant if and only if the function $F(z, \bar{z})$ has the following form*:

$$F(z, \bar{z}) = \sum_{\ell=1} g_\ell\big(|z|^2\big)\bar{z}^{\ell q-1} + \sum_{\ell=0} h_\ell\big(|z|^2\big)z^{\ell q+1}, \tag{5}$$

where $g_\ell(|z|^2)$ and $h_\ell(|z|^2)$ are polynomials with complex coefficients. In addition, Eq. (4) *is a Hamiltonian system having Z_q-equivariance if and only if Eq.* (5) *holds and*

$$\frac{\partial F}{\partial z}+\frac{\partial \bar{F}}{\partial \bar{z}}\equiv 0. \tag{6}$$

Lemma 2. *A Z_q-invariant function $I(z,\bar{z})$ has the following form*:

$$I(z,\bar{z})=\sum_{\ell=0} g_\ell\big(|z|^2\big)z^{\ell q}+\sum_{\ell=1} h_\ell\big(|z|^2\big)\bar{z}^{\ell q}. \tag{7}$$

Lemma 3. *The non-trivial Z_5-equivariant vector fields defined by Eq.* (4) *have the following explicit form*:

$$F(z,\bar{z})=\big(A_0+A_1|z|^2+A_2|z|^4+A_3|z|^4\big)z+A_4z^6+\big(A_5+A_6|z|^2\big)\bar{z}^4+\cdots, \tag{8}$$

where A_is are complex.

3. 25 small limit cycles in Z_5-equivariant planar vector fields of order 5

The basic idea used in this paper to find small limit cycles is to compute the normal forms associated with Hopf bifurcation and then perform appropriate perturbations on them to show the existence of the exact number of limit cycles. Before considering system (1), we first present a sufficient condition for proving the existence of small limit cycles (see [9,10] for more details).

Suppose we have obtained a normal form associated with a Hopf critical point, given by Eq. (2). Further, assume that the focus values v_i are given in terms of k system parameters:

$$v_j=v_j(a_1,a_2,\ldots,a_k),\quad j=0,1,\ldots,k. \tag{9}$$

Theorem 1. *Suppose there exist critical values a_{jc}, $j=1,2,\ldots,k$, such that*

$$v_{jc}=v_j(a_{1c},a_{2c},\ldots,a_{kc})=0,\quad j=0,1,\ldots,k-1,\quad \text{but}\quad v_{kc}\neq 0. \tag{10}$$

Further, assume that

$$\det\left[\frac{\partial(v_0,v_1,\ldots,v_{k-1})}{\partial(a_1,a_2,\ldots,a_k)}\right]_{a_j=a_{jc}}\neq 0. \tag{11}$$

Then, there exist perturbations ϵ_j, $j=1,2,\ldots,k$, satisfying

$$v_j(a_{1c}+\epsilon_1,a_{2c}+\epsilon_2,\ldots,a_{kc}+\epsilon_k)\;v_{j+1}(a_{1c}+\epsilon_1,a_{2c}+\epsilon_2,\ldots,a_{kc}+\epsilon_k)<0$$

and

$$\big|v_j(a_{1c}+\epsilon_1,a_{2c}+\epsilon_2,\ldots,a_{kc}+\epsilon_k)\big|\ll\big|v_{j+1}(a_{1c}+\epsilon_1,a_{2c}+\epsilon_2,\ldots,a_{kc}+\epsilon_k)\big|\ll 1,$$

$j=0,1,2,\ldots,k-1$, *for sufficiently small ϵ_j, which guarantees that the algebraic polynomial equation $\dot{r}=0$ has exact k positive roots for r^2, that is, system* (1) *has exact k small limit cycles in the vicinity of the origin.*

Now, we return to system (1) and assume that the vector field has Z_5 symmetry. Then applying the formula (8) yields the following function:

$$F_5(z,\bar{z})=\big(A_0+A_1|z|^2+A_2|z|^4\big)z+A_3\bar{z}^4, \tag{12}$$

where $A_j = a_j + ib_j$ are complex values (with real a_j and b_j). The real vector field corresponding to function (12) can be written as

$$\begin{aligned}\dot{x} &= a_0x - b_0y + a_1x^3 - b_1x^2y + a_1xy^2 - b_1y^3 + a_3x^4 + 4b_3x^3y - 6b_3x^2y^2 \\ &\quad + 4b_3xy^3 + a_3y^4 + a_2x^5 - b_2x^4y + 2a_2x^3y^2 - 2b_2x^2y^3 + a_2xy^4 - b_2y^5, \\ \dot{y} &= b_0x + a_0y + b_1x^3 + a_1x^2y + b_1xy^2 + a_1y^3 + b_3x^4 - 4a_3x^3y - 6b_3x^2y^2 \\ &\quad + 4a_3xy^3 + b_3y^4 + b_2x^5 + a_2x^4y + 2b_2x^3y^2 + 2a_2x^2y^3 + b_2xy^4 + a_2y^5. \end{aligned} \tag{13}$$

The two eigenvalues of the Jacobian of Eq. (13) evaluated at the origin $(x, y) = (0, 0)$ are $a_0 \pm b_0 i$, indicating that the origin $(0, 0)$ is either a focus point (when $a_0 \neq 0$) or a center ($a_0 = 0$).

Theorem 2. *The Z_5-equivariant planar vector filed of order 5, described by system (13) can have exactly a maximum of 25 small limit cycles, among which there are 5 around each of the 5 Hopf critical points of the system, and thus, $H(5) \geqslant 25$.*

Proof. Since this vector field is Z_5-equivariant, if there exists one fixed point, there are in total 5 fixed points in the system. Without loss of generality, we may assume that one of the fixed points is located on the y-axis. Further, by a simple parameter scaling, we may suppose this fixed point is $(0, 1)$. Therefore, the 5 fixed points are

$$(0, 1), \qquad \left(\pm\cos\left(\frac{\pi}{10}\right), \sin\left(\frac{\pi}{10}\right)\right), \qquad \left(\pm\sin\left(\frac{\pi}{5}\right), -\cos\left(\frac{\pi}{5}\right)\right), \tag{14}$$

which lead to the following conditions:

$$a_0 = -(a_1 + a_2 + b_3), \qquad b_0 = -(b_1 + b_2 - a_3). \tag{15}$$

Next, we want the 5 fixed points to be Hopf critical points. Thus, we set

$$b_3 = a_1 + 2a_2. \tag{16}$$

Then the eigenvalues of the Jacobian of system (13) evaluated at the 5 Hopf critical points are

$$\lambda_\pm = \pm\omega, \quad \text{where } \omega = \sqrt{5\left(4b_2a_3 + 2b_1a_3 - 5a_1^2 - 2a_1a_2 - 20a_2^2\right)} > 0. \tag{17}$$

Equation (17) shows that there are 5 free parameters which may be chosen later in perturbations. However, as shown in [10], we may use a parameter scaling and a time scaling to reduce one more parameter. In other words, one of the 5 parameters can be chosen arbitrarily, or the frequency ω can be chosen arbitrarily. To achieve this, let

$$a_1 \longrightarrow a_1\omega, \qquad a_2 \longrightarrow a_2\omega, \qquad a_3 \longrightarrow a_3\omega, \qquad b_1 \longrightarrow b_1\omega \tag{18}$$

and

$$\tau = \omega t. \tag{19}$$

Thus,

$$b_2 = \frac{1}{20a_3}\left(1 - 10b_1a_3 + 15a_3^2 + 25a_1^2 + 100a_1a_2 + 100a_2^2\right)\omega. \tag{20}$$

Since the vector field has Z_5 symmetry, we only need to consider one of the Hopf critical point, say, $(0, 1)$. Therefore, applying the following transformation

$$x = -5(a_1 + 2a_2)a_3u + v, \qquad y = 1 + 5a_3u \tag{21}$$

into system (13) yields the following canonical form of equations for the Hopf critical point up to 5th order:

$$\begin{aligned}
\dot u = {} & v - \big(A_1 + 30a_3^2A_1 - 20a_2a_3^2 + 50A_1^3\big)x^2 + \frac{1}{5}\big(1 + 75a_3^2 + 75A_1^2\big)xy - A_1y^2 \\
& - \frac{1}{2}\big(5A_1^3 + 15a_3^2A_1 + 125A_1^5 - 400a_2a_3^4 + 575a_3^4A_1 + 800a_3^2A_1^3 \\
& - 100b_1a_3^3A_1 - 200a_2a_3^2A_1^2\big)x^3 \\
& + \frac{1}{2}\big(3a_3^2 + 3A_1^2 + 75A_1^4 + 165a_3^4 - 20b_1a_3^3 + 220a_3^2A_1^2 - 80a_2a_3^2A_1\big)x^2y \\
& - \frac{1}{10}\big(+3A_1 + 75A_1^3 + 35a_3^2A_1 - 40a_2a_3^2\big)xy^2 + \frac{1}{50}\big(1 - 25a_3^2 + 25A_1^2\big)y^3 \\
& - 25\big(a_3^2 + A_1^2\big)\big(A_1 + 20A_1^3 - 5a_2A_1^2 - 10A_1a_3b_1 + 30a_3^2A_1 - 25a_2a_3^2\big)x^4 \\
& + 5a_3^2\big(a_3^2 + 3A_1^2 + 35a_3^4 + 55A_1^4 + 70a_3^2A_1^2 - 10b_1a_3^3 - 60a_2a_3^2A_1 \\
& - 30a_3b_1A_1^2 - 20a_2A_1^3\big)x^3y \\
& - 3a_3\big(A_1 + 15A_1^3 - 10a_2A_1^2 + 5a_3^2A_1 - 10a_2a_3^2 - 10A_1a_3b_1\big)x^2y^2 \\
& + \frac{1}{5}a_3^2\big(1 - 5a_3^2 + 5A_1^2 - 20A_1a_2 - 10a_3b_1\big)xy^3 + \frac{1}{5}a_3^2(a_2 + A_1)y^4 \\
& - \frac{125}{4}a_3^2\big(A_1^2 + a_3^2\big)^2\big(A_1 + 25A_1^3 - 10A_1a_3b_1 + 15a_3^2A_1 - 20a_2a_3^2\big)x^5 \\
& + \frac{25}{4}a_3^2\big(A_1^2 + a_3^2\big)\big(a_3^2 + 5A_1^2 + 125A_1^4 + 15a_3^4 + 100a_3^2A_1^2 - 50a_3b_1A_1^2 \\
& - 80a_2a_3^2A_1 - 10b_1a_3^3\big)x^4y \\
& - \frac{5}{2}a_3^2\big(5A_1^3 + 3a_3^2A_1 + 125A_1^5 - 30b_1a_3^3A_1 - 20a_2a_3^4 + 150a_3^2A_1^3 \\
& - 60a_2a_3^2A_1^2 - 50a_3b_1A_1^3 + 45a_3^4A_1\big)x^3y^2 \\
& + \frac{1}{2}a_3^2\big(a_3^2 + 5A_1^2 + 15a_3^4 + 125A_1^4 + 100a_3^2A_1^2 - 10b_1a_3^3 \\
& - 40a_2a_3^2A_1 - 50a_3b_1A_1^2\big)x^2y^3 \\
& - \frac{1}{4}a_3^2\big(A_1 + 25A_1^3 - 4a_2a_3^2 + 15a_3^2A_1 - 10A_1a_3b_1\big)xy^4 \\
& + \frac{1}{100}a_3^2\big(1 + 15a_3^2 + 25A_1^2 - 10a_3b_1\big)y^5, \\
\dot v = {} & -u - \frac{1}{2}\big(25a_3^2 + 15A_1^2 + 75a_3^4 + 625A_1^4 - 100b_1a_3^3 + 800a_3^2A_1^2 - 200a_2a_3^2A_1\big)x^2 \\
& + 2A_1\big(1 + 50a_3^2 + 50A_1^2\big)xy - \frac{1}{10}\big(1 + 75a_3^2 + 75A_1^2\big)y^2 \\
& - \frac{25}{2}\big(a_3^2 + A_1^2\big)\big(A_1^2 + 5a_3^2 + 25A_1^4 + 35a_3^4 + 260a_3^2A_1^2 - 40a_2a_3^2A_1 - 40b_1a_3^3\big)x^3 \\
& + \frac{1}{2}\big(15A_1^3 + 45a_3^2A_1 + 375A_1^5 - 400a_2a_3^2A_1^2 + 200a_2a_3^4 + 2600a_3^2A_1^3 \\
& + 1925a_3^4A_1 - 300b_1a_3^3A_1\big)x^2y
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\left(3A_1^2+3ka_3^2+75A_1^4+165a_3^4-20b_1a_3^3+260a_3^2A_1^2-40a_2a_3^2A_1\right)xy^2\\
&+\frac{1}{10}A_1\left(1+25a_3^2+25A_1^2\right)y^3\\
&-\frac{625}{4}a_3^2\left(a_3^2+A_1^2\right)\left(A_1^2+a_3^2+21A_1^4+11a_3^4-4a_2A_1^3+64a_3^2A_1^2-10a_3b_1A_1^2\right.\\
&\left.-4a_2a_3^2A_1-10b_1a_3^3\right)x^4\\
&+100a_3^2\left(a_3^2+A_1^2\right)\left(A_1+20A_1^3-5a_2A_1^2-10A_1a_3b_1+30a_3^2A_1+5a_2a_3^2\right)x^3y\\
&-\frac{15}{2}a_3^2\left(3A_1^2+a_3^2+55A_1^4+35a_3^4+70a_3^2A_1^2-10b_1a_3^3+20a_2a_3^2A_1\right.\\
&\left.-30a_3b_1A_1^2-20a_2A_1^3\right)x^2y^2\\
&+2a_3^2\left(A_1+15A_1^3-10a_2A_1^2+5a_3^2A_1+10a_2a_3^2-10A_1a_3b_1\right)xy^3\\
&-\frac{1}{20}a_3^2\left(1+5A_1^2-5a_3^2-20A_1a_2-10a_3b_1\right)y^4\\
&-\frac{625}{4}a_3^2\left(a_3^2+A_1^2\right)\left(1+25A_1^2+15a_3^2-10a_3b_1\right)x^5\\
&+\frac{625}{4}a_3^2\left(a_3^2+A_1^2\right)^2\left(A_1+25A_1^3-10A_1a_3b_1+15a_3^2A_1+4a_2a_3^2\right)x^4y\\
&-\frac{25}{2}a_3^2\left(a_3^2+A_1^2\right)\left(5A_1^2+a_3^2+125A_1^4+15a_3^4+100a_3^2A_1^2-50a_3b_1A_1^2\right.\\
&\left.+40a_2a_3^2A_1-10b_1a_3^3\right)x^3y^2\\
&+\frac{5}{2}a_3^2\left(5A_1^3+3a_3^2A_1+125A_1^5-30b_1a_3^3A_1+20a_2a_3^4+150a_3^2A_1^3\right.\\
&\left.+60a_2a_3^2A_1^2-50a_3b_1A_1^3+45a_3^4A_1\right)x^2y^3\\
&-\frac{1}{4}a_3^2\left(5A_1^2+a_3^2+125A_1^4+15a_3^4+100a_3^2A_1^2-10b_1a_3^3+80a_2a_3^2A_1\right.\\
&\left.-50a_3b_1A_1^2\right)xy^4\\
&+\frac{1}{20}a_3^2\left(A_1+25A_1^3+20a_2a_3^2+15a_3^2A_1-10A_1a_3b_1\right)y^5,
\end{aligned}
\tag{22}
$$

where $A_1=a_1+2a_2$.

Now employing the Maple program [6] to system (22) yields the focus values, v_i, explicitly expressed in terms of the system's coefficients as

$$v_0=A_1-b_3, \tag{23}$$

$$
\begin{aligned}
v_1=\frac{25}{2}a_3^2\big[&125A_1^5+5A_1^3+15a_2a_3^4+160a_3^2A_1^3+7a_3^2A_1+a_2a_3^2+15a_3^4A_1\\
&+120a_3^2a_2A_1^2+5a_2A_1^2+125a_2A_1^4-20b_1a_3^3A_1-40a_2^2a_3^2A_1-20b_1a_3^3a_2\big]
\end{aligned}
\tag{24}
$$

and

$$
\begin{aligned}
v_2&=-\frac{625}{72}a_3^2\tilde{v}_2(A_1,a_2,a_3,b_1),\\
v_3&=-\frac{25}{663552}a_3^2\tilde{v}_3(A_1,a_2,a_3,b_1),
\end{aligned}
$$

$$v_4 = -\frac{25}{95551488} a_3^2 \tilde{v}_4(A_1, a_2, a_3, b_1), \tag{25}$$

where $\tilde{v}_i$, $i = 2, 3, 4$, are lengthy polynomials of A_1, a_2, a_3 and b_1.

Note that letting $v_0 = 0$ yields $b_3 = A_1$, which is the condition given by (16) for the fixed points to have Hopf-type singularity. Then there are 4 free parameters in the expressions of v_i, $i = 1, 2, 3, 4$. Therefore, the best possibility is to choose A_1, a_2, a_3 and b_1 such that $v_i = 0$, $i = 1, 2, 3, 4$, but $v_5 \neq 0$, leading to possible existence of 5 small limit cycles in the vicinity of each of the 5 Hopf critical points. This suggests that the 5th-order Z_5-equivariant vector fields, described by Eq. (13), may have $5 \times 5 = 25$ small limit cycles.

Solving $v_1 = 0$ for b_1 yields

$$\begin{aligned} b_1 = {} & \frac{1}{20 a_3^3 (A_1 + a_2)} \big(125 A_1^5 + 5 A_1^3 + 15 a_2 a_3^4 + 160 a_3^2 A_1^3 + 7 a_3^2 A_1 + a_2 a_3^2 \\ & + 15 a_3^4 A_1 + 120 a_3^2 a_2 A_1^2 + 5 a_2 A_1^2 + 125 a_2 A_1^4 - 40 a_2^2 a_3^2 A_1\big). \end{aligned} \tag{26}$$

Then substituting the above b_1 into v_i, $i = 2, 3, 4$, results in

$$\begin{aligned} v_2 &= \frac{15(a_3^2 + A_1^2)}{2(A_1 + a_2)} \bar{v}_2(A_1, a_2, a_3), \\ v_3 &= \frac{3000(a_3^2 + A_1^2)}{(A_1 + a_2)^3} \bar{v}_3(A_1, a_2, a_3), \\ v_4 &= \frac{1500(a_3^2 + A_1^2)}{(A_1 + a_2)^5} \bar{v}_4(A_1, a_2, a_3), \end{aligned} \tag{27}$$

where $\bar{v}_i$, $i = 2, 3, 4$, are coupled polynomials of A_1, a_2, and a_3. However, we cannot solve $\bar{v}_i$, $i = 2, 3, 4$, one by one by choosing A_1, a_2 and a_3. We have to solve them simultaneously.

Noticing that $\bar{v}_i$ $(i = 2, 3, 4)$ are lower-order polynomials of a_3^2, we may eliminate a_3 from these three equations. First, eliminating a_3 from the two equations $\bar{v}_2 = 0$ and $\bar{v}_3 = 0$ yields a solution for a_3,

$$\big(a_3^{(1)}\big)^2 = \frac{a_{3n}^{(1)}(A_1, a_2)}{a_{3d}^{(1)}(A_1, a_2)}, \tag{28}$$

and a resultant equation

$$F_1 = F_1(A_1, a_2) = 0. \tag{29}$$

Here, both $a_{3n}^{(1)}(A_1, a_2)$ and $a_{3d}^{(1)}(A_1, a_2)$ are polynomials, and $F_1(A_1, a_2)$ is a 16th-degree polynomial of a_2.

Similarly, eliminating a_3 from the two equations $\bar{v}_2 = 0$ and $\bar{v}_4 = 0$ yields another solution for a_3,

$$\big(a_3^{(2)}\big)^2 = \frac{a_{3n}^{(2)}(A_1, a_2)}{a_{3d}^{(2)}(A_1, a_2)}, \tag{30}$$

and another resultant equation

$$F_2 = F_2(A_1, a_2) = 0, \tag{31}$$

where $F_2(A_1, a_2)$ is a 29th-degree polynomial of a_2. It should be noted that the final solution must satisfy $a_3^{(1)} = a_3^{(2)}$.

The remaining task is to solve the two coupled polynomial equations: $F_1 = 0$ and $F_2 = 0$. After a lengthy computation in eliminating a_2 from the two equations, we obtain an explicit (in symbolic form) solution $a_2 = a_2(A_1)$, and the following univariate polynomial equation for A_1:

$$F_3\big(A_1^2\big) = -\frac{95175}{G_0(A_1^2)} A_1^4\big(1+12A_1^2\big)^2\big(1+16A_1^2\big)^3\big(1+25A_1^2\big)^6 G_1\big(A_1^2\big)G_2\big(A_1^2\big)G_3^2\big(A_1^2\big), \tag{32}$$

where G_i, $i = 0, 1, 2, 3$, are all polynomials of A_1^2. Now, the possible solutions for $v_1 = v_2 = v_3 = v_4 = 0$ only come from the positive real solutions of $F_3(A_1^2) = 0$, i.e., from the higher-degree polynomial equations: $G_1(A_1^2) = 0$, $G_2(A_1^2) = 0$, and $G_3(A_1^2) = 0$, where G_1, G_2 and G_3 are respectively 37th-, 227th- and 257th-degree polynomials of A_1^2. Employing the built-in Maple solver *realroot* (which is based on Sturm theorem and can identify the real roots of polynomial functions within isolated intervals to an arbitrary accuracy) yields the following results: $G_1(A_1^2) = 0$, $G_2(A_1^2) = 0$ and $G_3(A_1^2) = 0$ have respectively, 1, 16 and 18 positive real solutions for A_1^2. These results are also cross-checked by using another built-in Maple solver *fsolve*. However, only the one solved from $G_1(A_1^2) = 0$ satisfies $a_3^{(1)} = a_3^{(2)}$. Therefore, the only possible solution is

$$A_1^2 = 0.21698240052718384070137839825988793921989732238598 \times 10^{-6}.$$

Note that the accuracy used in the calculation with the Maple solvers *redroot* and *fsolve* is up to 1000 decimal points. (The presentation given above is up to 50 decimal points for brevity.) Then, in backward order, we use the obtained symbolic formulas to compute the critical values of a_2, a_3, b_1, b_3 and other parameters. This unique critical solution is given by

$$\begin{aligned}
a_{0c} &= -0.00094678410409569352, & b_{0c} &= -1.80318933512810043778,\\
a_{1c} &= 0.00049612711650105944, & b_{1c} &= 2.89389952760627236198,\\
a_{2c} &= -0.00001515670963547512, & b_{2c} &= -0.96198328806045136942,\\
a_{3c} &= 0.12872690441772055479, & b_{3c} &= 0.00046581369723010920,
\end{aligned} \tag{33}$$

where the subscript c denotes critical value. With these critical parameter values, executing the Maple program [6] for calculating the focus values results in (up to 50 decimal points)

$$\begin{aligned}
v_{1c} &= -0.5 \times 10^{-54},\\
v_{2c} &= 0.5 \times 10^{-54},\\
v_{3c} &= 0.275 \times 10^{-53},\\
v_{4c} &= 0.57 \times 10^{-54},\\
v_{5c} &= -0.11700587508579960671932081080241286570118444000000 \times 10^{-13},
\end{aligned} \tag{34}$$

where v_is ($i = 2, 3, 4$) are not exactly zero due to numerical round-off errors. In fact, we have executed the program up to 1000 decimal points to obtain

$$\begin{aligned}
v_{1c} &= -0.5 \times 10^{-1003},\\
v_{2c} &= -0.162705 \times 10^{-998},\\
v_{3c} &= -0.1357285 \times 10^{-999},
\end{aligned}$$

$$v_{4c} = -0.1173415 \times 10^{-1000},$$
$$v_{5c} = -0.117005875085799606719320810802412865701184\ldots 639507814500000 \times 10^{-13},$$

from which it is noted that the first 42 decimal points of v_{5c} are identical to that given in Eq. (34). This clearly indicates that $v_{1c} = v_{2c} = v_{3c} = v_{4c} = 0$, but $v_{5c} \neq 0$. Hence, the maximal number of small limit cycles that system (13) can have is 25. To prove that the system can indeed have exactly 25 small limit cycles, we use v_2, v_3 and v_4 given in Eq. (27) to verify that determinant (11) evaluated at the critical values is non-zero:

$$\det \begin{bmatrix} \frac{\partial v_2}{\partial A_1} & \frac{\partial v_2}{\partial a_2} & \frac{\partial v_2}{\partial a_3} \\ \frac{\partial v_3}{\partial A_1} & \frac{\partial v_3}{\partial a_2} & \frac{\partial v_3}{\partial a_3} \\ \frac{\partial v_4}{\partial A_1} & \frac{\partial v_4}{\partial a_2} & \frac{\partial v_4}{\partial a_3} \end{bmatrix}_{(A_1,a_2,a_3)=(A_{1c},a_{2c},a_{3c})}$$
$$\approx \det \begin{bmatrix} -0.00003555242438938013 & -0.00109146815695580469 & 0.00000204992018225107 \\ -0.00000252901301240214 & -0.00007755867290008381 & 0.00000017099017366986 \\ -0.00000021794154213668 & -0.00000667726234527485 & 0.00000001478376687650 \end{bmatrix}$$
$$\approx 0.56511065548505834 \times 10^{-20} \neq 0,$$

which shows that the Z_5-equivariant vector fields of order 5, described by system (13) can have exactly 25 small limit cycles. Hence, $H(5) \geqslant 25$.

This completes the proof of Theorem 2. □

It should be pointed out that although the presentation for the above calculation of the determinant of the Jacobian is only up to 20 decimal points, the exact computation using Maple

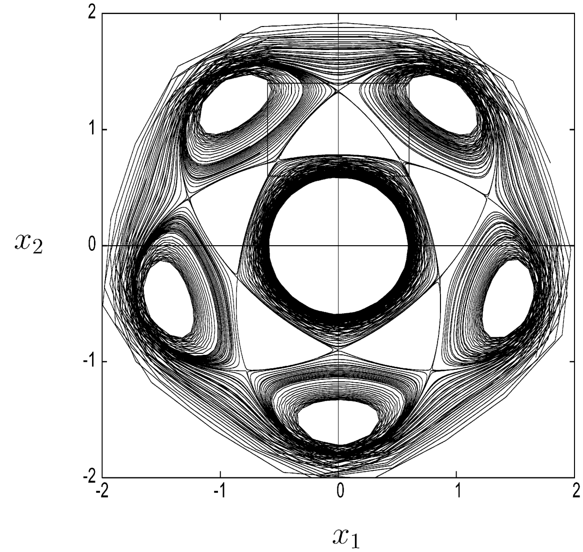

Fig. 1. The phase portrait of the unperturbed system (13) when the origin is a stable focus point for $a_{0c} = -0.0009467841$, $b_{0c} = -1.8031893351$, $a_{1c} = 0.0004961271$, $b_{1c} = 2.8938995276$, $a_{2c} = -0.0000151567$, $b_{2c} = -0.9619832881$, $a_{3c} = 0.1287269044$, $b_{3c} = 0.0004658137$.

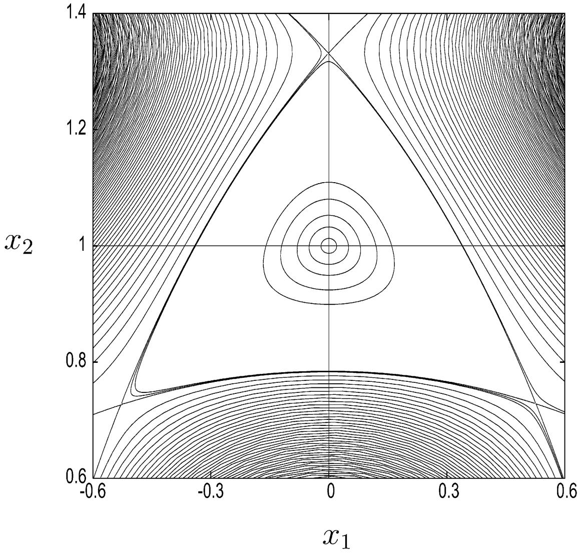

Fig. 2. The zoom in area around the point (0, 1), the box marked in Fig. 1, showing the existence of 5 small limit cycles in the vicinity of (0, 1) after an appropriate perturbation to the critical point: $a_{0c} = -0.0009467841$, $b_{0c} = -1.8031893351$, $a_{1c} = 0.0004961271$, $b_{1c} = 2.8938995276$, $a_{2c} = -0.0000151567$, $b_{2c} = -0.9619832881$, $a_{3c} = 0.1287269044$, $b_{3c} = 0.0004658137$.

has been carried out as follows: First the purely symbolic formulas for all terms of the Jacobian are explicitly obtained, and then the final numerical critical values are substituted into these symbolic expressions to find the value of the determinant. Moreover, in order to make sure that the conclusion on the determinant is correct, computational results step by step with higher and higher accuracies (up to 1000 decimal points, extremely close to the true values (A_{1c}, a_{2c}, a_{3c})) have been obtained, as if the determinant could remain around some non-zero value. Otherwise, suppose the determinant were zero, then under the 1000 decimal points accuracy, it would be at least around 10^{-980}, not 10^{-20}.

The phase portrait of the non-perturbed system (13) is shown in Fig. 1. There are 21 fixed points: 1 stable focus point at $(0, 0)$; 5 weakly stable focus points at the locations given in Eq. (14); 5 unstable focus points and 10 saddle points are symmetrically located, as shown in Fig. 1. After proper perturbations, 5 small limit cycles exist in the vicinity of each of the 5 fine focus points. The zoom-in neighborhood of the point (0, 1) is depicted in Fig. 2, which shows the existence of 5 limit cycles. Since $v_5 < 0$ and so $v_0 > 0$, indicating that the stable fine focus points become unstable under perturbations. Thus, the smallest limit cycles is unstable, the next one is stable, and so on. The largest one is unstable.

4. No large limit cycles exist in the Z_5-equivariant planar vector fields of order 5

In the previous section, we have proved that the 5th-order Z_5-equivariant system (13) can have 25 small limit cycles. In this section, we want to investigate the possible existence of large limit cycles that system (13) may have.

We again consider system (13) with the parameter values given in Eq. (33), for which we have shown that the system has 25 small limit cycles under appropriate perturbations. Now we want to employ numerical simulation to show that no large limit cycles exist in system (13). It is easy

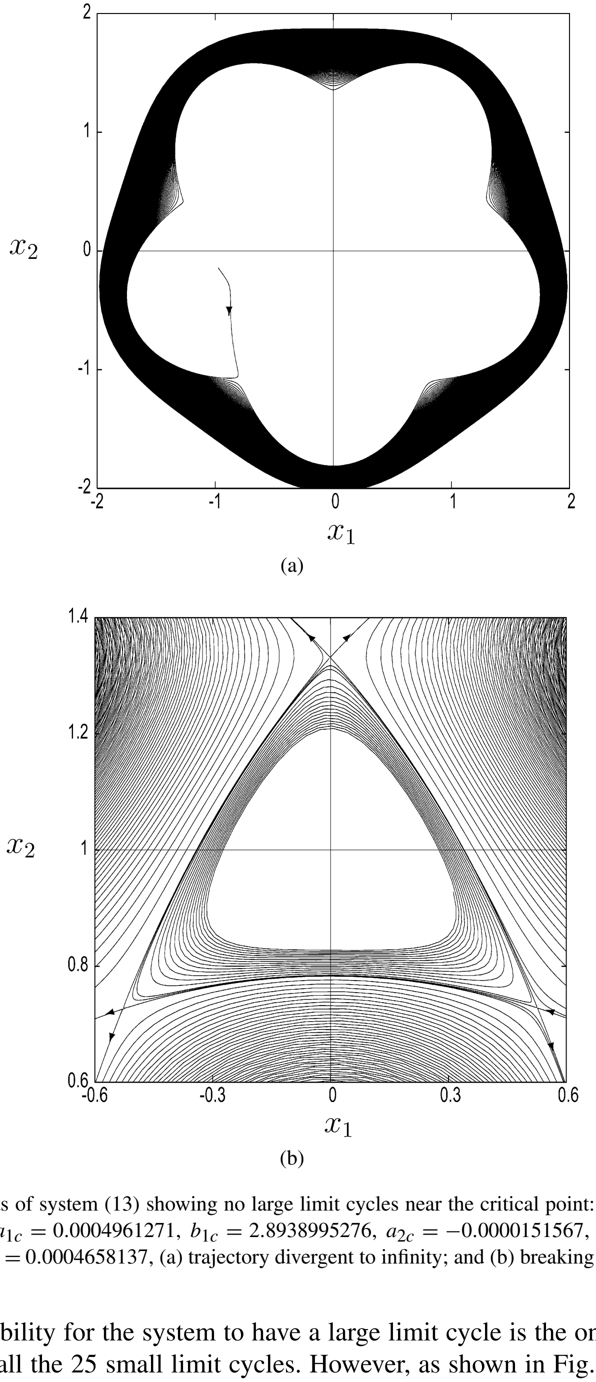

Fig. 3. The phase portraits of system (13) showing no large limit cycles near the critical point: $a_{0c} = -0.0009467841$, $b_{0c} = -1.8031893351$, $a_{1c} = 0.0004961271$, $b_{1c} = 2.8938995276$, $a_{2c} = -0.0000151567$, $b_{2c} = -0.9619832881$, $a_{3c} = 0.1287269044$, $b_{3c} = 0.0004658137$, (a) trajectory divergent to infinity; and (b) breaking of heteroclinic orbits.

to see that one possibility for the system to have a large limit cycle is the one to enclose all the 21 fixed points and all the 25 small limit cycles. However, as shown in Fig. 3(a), the trajectory starting from a region enclosing one of the fine focus point diverges to infinity, which excludes the existence of this possible large limit cycle.

Another possible existence to have a large limit cycle may be obtained via the breaking of heteroclinic orbits. It can be seen from Fig. 1 that the five regions in the neighborhoods of the fine focus points are actually enclosed by the heteroclinic trajectories. However, when the heteroclinic orbits break under perturbations, it does not give rise to large limit cycles, as shown in Fig. 3(b).

In summary, the 5th-order Z_5-equivariant system (13) cannot have large limit cycles. Therefore, the maximal number of limit cycles obtained in such a vector field is 25, that is, $H(5) \geqslant 25$.

5. Conclusion

A detailed study has been given to bifurcation of limit cycles, which exist in Z_5-equivariant planar vector fields of order 5. Based on the normal form computation for degenerate Hopf bifurcations, it has been shown that such a 5th-order Z_5-equivariant planar vector field can have a maximum of 25 small limit cycles. Further, numerical simulation shows that no large limit cycles exist in such vector fields. In conclusion, we have proved that $H(5) \geqslant 25$.

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