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Walks and the spectral radius of graphs

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Abstract

Given a graph G, write $\mu(G)$ for the largest eigenvalue of its adjacency matrix, $\omega(G)$ for its clique number, and $w_k(G)$ for the number of its k-walks. We prove that the inequalities

$$\frac{w_{q+r}(G)}{w_q(G)} \leqslant \mu^r(G) \leqslant \frac{\omega(G)-1}{\omega(G)} w_r(G)$$

hold for all $r > 0$ and odd $q > 0$. We also generalize a number of other bounds on $\mu(G)$ and characterize semiregular and pseudo-regular graphs in spectral terms.

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1. Introduction

Our graph–theoretic notation is standard (e.g., see [2]); in particular, we assume that graphs are defined on the vertex set $\{1, 2, \ldots, n\} = [n]$. Given a graph G, a *k-walk* is a sequence of vertices $v_1, \ldots, v_k$ of G such that v_i is adjacent to v_{i+1} for all $i = 1, \ldots, k-1$; we write $w_k(G)$ for the number of k-walks in G. The eigenvalues of the adjacency matrix $A(G)$ of G are ordered as $\mu(G) = \mu_1 \geqslant \cdots \geqslant \mu_n$.

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Various bounds of $\mu(G)$ in terms of $w_k(G)$ are known; the earliest one, due to Collatz and Sinogowitz [4], reads as

$$\mu(G) \geqslant \frac{2e(G)}{v(G)} = \frac{w_2(G)}{w_1(G)}. \tag{1}$$

This inequality was strengthened by Hofmeister [9,10] to

$$\mu^2(G) \geqslant \frac{1}{v(G)} \sum_{u \in V(G)} d^2(u) = \frac{w_3(G)}{w_1(G)}, \tag{2}$$

in turn, improved by Yu et al. [18] to

$$\mu^2(G) \geqslant \frac{w_5(G)}{w_3(G)},$$

and by Hong and Zhang [13] to

$$\mu^2(G) \geqslant \frac{w_7(G)}{w_5(G)}.$$

In this note we prove that, in fact, the inequality

$$\mu^r(G) \geqslant \frac{w_{q+r}(G)}{w_q(G)}$$

holds for all $r > 0$ and odd $q > 0$.

Let $\omega(G)$ be the clique number of G. Wilf [17] gave the bound

$$\mu(G) \leqslant \frac{\omega(G)-1}{\omega(G)} v(G) = \frac{\omega(G)-1}{\omega(G)} w_1(G),$$

and Nikiforov [15] showed that

$$\mu^2(G) \leqslant 2\frac{\omega(G)-1}{\omega(G)} e(G) = \frac{\omega(G)-1}{\omega(G)} w_2(G),$$

generalizing earlier results in [5,7,12,16,17].

In this note we prove that, in fact, the inequality

$$\mu^r(G) \leqslant \frac{\omega(G)-1}{\omega(G)} w_r(G)$$

holds for every $r \geqslant 1$.

We generalize also a number of other upper and lower bounds on $\mu(G)$ in terms of walks and characterize pseudo-regular and semiregular graphs in terms of their eigenvectors.

The rest of the paper is organized as follows. In Section 2 we recall some basic notions used further, in Section 3 we investigate lower bounds on $\mu(G)$, and in Section 4 we investigate upper bounds on $\mu(G)$.

2. Some preliminary results

Given a graph G and a vertex $u \in V(G)$, write $\Gamma(u)$ for the set of neighbors of u and $w_k(u)$ for the number of the k-walks starting with u; for every two vertices $u, v \in V(G)$, write $w_k(u, v)$ for the number of the k-walks starting with u and ending with v.

We state below some basic results related to walks in graphs.

2.1. The number of k-walks in a graph

Let G be a graph of order n with eigenvalues $\mu_1 \geqslant \cdots \geqslant \mu_n$ and $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be corresponding orthogonal unit eigenvectors. For every $i \in [n]$, let $\mathbf{u}_i = (u_{i1}, \ldots, u_{in})$ and set $c_i = (\sum_{j=1}^{n} u_{ij})^2$.

The number of k-walks in G (see, e.g., [3], p. 44, Theorem 1.10) is given as follows.

Theorem 1. *For every $k \geqslant 1$, $w_k(G) = c_1\mu_1^{k-1} + \cdots + c_n\mu_n^{k-1}$.*

In particular, for $k = 1$,

$$\sum_{i=1}^{n} c_i = n. \tag{3}$$

We also list several equalities that we will use later without reference.

$$\begin{array}{ll} \sum_{u \in V(G)} d^2(u) = w_3(G); & \sum_{uv \in E(G)} d(u)d(v) = w_4(G); \\ \sum_{u \in V(G)} w_p^2(u) = w_{2p-1}(G); & \sum_{u \in V(G)} w_p(u)w_q(u) = w_{p+q-1}(G); \\ \sum_{v \in V(G)} w_r(u,v)w_p(v) = w_{p+r}(u); & \sum_{u,v \in V(G)} w_r(u,v)w_p(u)w_q(v) = w_{p+q+r-2}(G). \end{array}$$

2.2. The inequality of Motzkin and Straus

The following result of Motzkin and Straus [14] will be used in Section 4.

Theorem 2. *For any graph G of order n and real numbers $x_1, \ldots, x_n$ with $x_i \geqslant 0$, $(1 \leqslant i \leqslant n)$, and $x_1 + \cdots + x_n = 1$,*

$$\sum_{ij \in E(G)} x_i x_j \leqslant \frac{\omega(G) - 1}{\omega(G)}. \tag{4}$$

Equality holds iff the subgraph induced by the vertices corresponding to nonzero entries of $\mathbf{x}$ is a complete $\omega(G)$-partite graph such that the sum of the x_i's in each part is the same.

Wilf [17] was the first to apply inequality (4) to graph spectra, obtaining, in particular, the following result.

Theorem 3. *Let $\mathbf{x} = (x_1, \ldots, x_n)$ be an eigenvector to $\mu(G)$ with $\|\mathbf{x}\| = 1$. Then*

$$\mu(G) = \sum_{ij \in E(G)} x_i x_j \leqslant \frac{\omega(G) - 1}{\omega(G)} \left(\sum_{i=1}^{n} x_i \right)^2. \tag{5}$$

It is rather entertaining to find the connected graphs for which equality holds in (5). We note without a proof that for $G = K_{4n,4n,n}$ equality holds in (4)—it is enough to consider the vector $\mathbf{x} = (x_1, \ldots, x_{9n})$ defined as

$$x_i = \begin{cases} (12n)^{-1/2}, & 1 \leqslant i \leqslant 8n, \\ (3n)^{-1/2}, & 8n < i \leqslant 9n. \end{cases}$$

Here we state a partial result.

Theorem 4. *Let G be a connected graph and $\mathbf{x} = (x_1, \ldots, x_n)$ be a unit eigenvector to $\mu(G)$ such that*

$$\mu(G) = \frac{\omega(G) - 1}{\omega(G)} \left(\sum_{i=1}^{n} x_i \right)^2 .$$

Then G is a complete $\omega(G)$-partite graph.

Proof. Since G is connected, $x_i > 0$ for every $i \in [n]$. The assertion follows from the case of equality in (4). □

3. Lower bounds on $\mu(G)$

Given a graph with no isolated vertices and a vertex v, call the value $\sum_{v \in \Gamma(u)} d(v)/d(u)$ the *average degree* of u. A graph G with no isolated vertices is called *pseudo-regular* if its vertices have the same average degree. A graph is called *semiregular* if it is bipartite and vertices belonging to the same part have the same degree.

In this section we first prove Theorem 5 and then show that its hypothesis cannot be relaxed. Next we describe pseudo-regular and semiregular graphs in terms of their eigenvectors, and finally we extend two other lower bounds on $\mu(G)$.

The following theorem generalizes results stated in [18] and [13]. Note, that the case of equality stated in [18], Theorem 4, is incorrect.

Theorem 5. *For every graph G,*

$$\mu^r(G) \geqslant \frac{w_{q+r}(G)}{w_q(G)} \tag{6}$$

for all $r > 0$ and odd $q > 0$. If equality holds in (6), then each component of G is pseudo-regular or semiregular.

Proof. Let $v(G) = n$. Theorem 1 implies (6) by

$$\frac{w_{q+r}(G)}{w_q(G)} = \frac{\sum_{i=1}^{n} c_i \mu_i^{q+r-1}}{\sum_{i=1}^{n} c_i \mu_i^{q-1}} = \mu^r(G) \frac{\sum_{i=1}^{n} c_i \left(\frac{\mu_i}{\mu_1}\right)^{q+r-1}}{\sum_{i=1}^{n} c_i \left(\frac{\mu_i}{\mu_1}\right)^{q-1}} \leqslant \mu^r(G). \tag{7}$$

Suppose now that

$$\mu^r(G) = \frac{w_{q+r}(G)}{w_q(G)}. \tag{8}$$

Assume first that G is connected and let M be the set of all $i \in [2, n]$ such that $c_i \neq 0$ and $\mu_i \neq 0$. We shall show that if G is nonbipartite, then $M = \varnothing$. From (7) we find that

$$\sum_{i=2}^{n} c_i \left(\frac{\mu_i}{\mu_1}\right)^{q+r-1} = \sum_{i=2}^{n} c_i \left(\frac{\mu_i}{\mu_1}\right)^{q-1},$$

and so, $|\mu_i| = \mu_1$ for every $i \in M$, contradicting that G connected and nonbipartite. Hence, $w_k(G) = c_1\mu_1^{k-1}$ for every $k > 0$. In particular, $w_4(G) = \sqrt{w_3(G)w_5(G)}$, and so

$$\sum_{u\in V(G)} d(u)w_3(u) = w_4(G) = \sqrt{w_3(G)w_5(G)} = \sqrt{\sum_{u\in V(G)} d^2(u) \sum_{u\in V(G)} w_3^2(u)}.$$

The condition for equality in Cauchy–Schwarz inequality implies that $w_3(u)/d(u)$ is constant for all vertices u, i.e., that G is pseudo-regular.

Let now G be bipartite. Since the spectrum of G is symmetric with respect to 0, it follows that either $M = \varnothing$ or $M = \{n\}$. If $M = \varnothing$ (i.e., $c_n = 0$), the case reduces to the previous one. If $c_n > 0$, then, by (3), we have

$$\sum_{u\in V(G)} d^2(u) = w_3(G) = c_1\mu_1^2 + c_n\mu_1^2 = n\mu_1^2,$$
$$\sum_{u\in V(G)} w_3^2(u) = w_5(G) = c_1\mu_1^4 + c_n\mu_1^4 = n\mu_1^4.$$

We see that

$$\left(\frac{1}{n}\sum_{u\in V(G)} d^2(u)\right)^2 = \frac{1}{n}\sum_{u\in V(G)} w_3^2(u) \geqslant \left(\frac{1}{n}\sum_{u\in V(G)} w_3(u)\right)^2 = \left(\frac{1}{n}\sum_{u\in V(G)} d^2(u)\right)^2,$$

implying that $w_3(u)$ is constant for every u. The following argument is borrowed from [8]. Letting u to be a vertex of minimum degree $\delta(G) = \delta$ and v be a vertex of maximum degree $\Delta(G) = \Delta$, we see that

$$w_3(u) \leqslant \delta\Delta \leqslant w_3(v) = w_3(u);$$

thus, every vertex of degree δ is adjacent only to vertices of degree Δ and vice versa. Since G is connected, it follows that it is semiregular.

If the graph is not connected, say let $G_1, \ldots, G_k$ be its components, we have

$$\mu^r(G) = \frac{\sum_{i=1}^k w_{q+r}(G_i)}{\sum_{i=1}^k w_q(G_i)} \leqslant \frac{\mu^r(G_i)\sum_{i=1}^k w_q(G_i)}{\sum_{i=1}^k w_q(G_i)} \leqslant \mu^r(G).$$

Thus, (8) implies that $\mu^r(G_i) = w_{q+r}(G_i)/w_q(G_i)$ for each component of G_i, completing the proof. □

3.1. The case of even q

Observe that if G is connected and nonbipartite, then the ratio $w_{q+r}(G)/w_q(G)$ tends to $\mu^r(G)$ as q tends to infinity. Indeed, from (7) and $|\mu_i|/\mu_1 < 1$ holding for every $i = 2, \ldots, n$, we obtain the following theorem.

Theorem 6. *For every connected nonbipartite graph G and every $\varepsilon > 0$, there exists $q_0(\varepsilon)$ such that if $q > q_0(\varepsilon)$ then*

$$(1-\varepsilon)\frac{w_{q+r}(G)}{w_q(G)} \leqslant \mu^r(G) \leqslant (1+\varepsilon)\frac{w_{q+r}(G)}{w_q(G)}$$

for every $r > 0$.

Inequality (6) may fail for q even as shown by the following example for $q = 2k$ and odd r. Let $0 < a < b$ be integers and $G = K_{a,b}$ be the complete bipartite graph with parts of size a and b. We see that

$$w_{2k}(G) = 2a^k b^k,$$
$$w_{2k+r}(G) = a(ba)^{k+(r-1)/2} + b(ba)^{k+(r-1)/2},$$
$$\frac{w_{2k+r}(G)}{w_{2k}(G)} = \frac{a+b}{2}(ba)^{(r-1)/2} > (ab)^{r/2} = \mu^r(G).$$

Therefore, for bipartite G, q even and r odd, $\mu^r(G)$ may differ considerably from $w_{q+r}(G)/w_q(G)$, no matter how large q is. We are not able to answer the following natural question.

Problem 7. Let G be a connected bipartite graph. Is it true that

$$\mu^r(G) \geqslant \frac{w_{q+r}(G)}{w_q(G)}$$

for every even $q \geqslant 2$ and $r \geqslant 2$?

We also note without a proof that the graph $G = K_{2t,2t,t}$ satisfies $\mu^2(G) < w_4(G)/w_2(G)$.

3.2. Spectral characterization of pseudo-regular graphs

Write $\mathbf{i}$ for the vector $(1, \ldots, 1) \in \mathbb{R}^n$. As a by-product of the proof of Theorem 5 we obtain the following condition for pseudo-regular graphs.

Theorem 8. *If G is a pseudo-regular graph and μ_s is an eigenvalue of G such that $0 < |\mu_s| < \mu(G)$, then every eigenvector to μ_s is orthogonal to* $\mathbf{i}$. *If G has no bipartite component, then this condition is also sufficient.*

Proof. Let $v(G) = n$; suppose $0 < |\mu_s| < \mu_1$ and $\mathbf{u}_s = (u_{s1}, \ldots, u_{sn})$ is a unit eigenvector to μ_s. Let $\mathbf{u}_1, \ldots, \mathbf{u}_s, \ldots, \mathbf{u}_n$ be orthogonal unit eigenvectors of G to $\mu_1, \ldots, \mu_n$. If G is pseudo-regular, then $w_4(G) = \sqrt{w_3(G)w_5(G)}$; using the definitions of Section 2.1, we see that

$$\sum_{i=1}^{n} c_i \left(\frac{\mu_i}{\mu_1}\right)^3 = \sqrt{\sum_{i=1}^{n} c_i \left(\frac{\mu_i}{\mu_1}\right)^2 \sum_{i=1}^{n} c_i \left(\frac{\mu_i}{\mu_1}\right)^4}.$$

The condition for equality in Cauchy–Schwarz's inequality implies that $|\mu_i/\mu_1| = \mu_1/\mu_1 = 1$ whenever $|\mu_i| > 0$ and $c_i > 0$. Hence, $c_s = (\sum_{i=1}^{n} u_{si})^2 = 0$, i.e., $\mathbf{u}_s$ is orthogonal to $\mathbf{i}$.

Now assume that G has no bipartite component and $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are orthogonal unit eigenvectors to $\mu_1, \ldots, \mu_n$. Using the definitions of Section 2.1, we see that

$$w_k(G) = \mu^{k-1}(G) \sum_{\mu_i = \mu(G)} c_i$$

for every $k \geqslant 1$. In particular, $w_4(G) = \sqrt{w_3(G)w_5(G)}$, and, as in the proof of Theorem 5, we see that G is pseudo-regular. $\square$

3.3. Spectral characterization of semiregular graphs

In Theorem 11 below we characterize connected semiregular graphs in terms of their eigenvectors. We shall need two simple lemmas.

Lemma 9. *If a graph $G = G(n)$ is semiregular, then $w_3(G) = \mu^2(G)n$.*

Proof. Indeed, since G is semiregular, $w_3(u) = w_3(G)/n$ for every $u \in V(G)$, implying that

$$w_{k+2}(G) = \sum_{u \in V(G)} w_3(u) w_k(u) = \frac{w_3(G)}{n} w_k(G).$$

Since

$$\lim_{k \to \infty} \frac{w_{k+2}(G)}{w_k(G)} = \mu^2(G)$$

we see that $w_3(G) = \mu^2(G)n$. $\square$

Lemma 10. *If G is a connected bipartite graph with $w_3(G) = \mu^2(G)n$, then G is semiregular.*

Proof. Note that $\mu^2(G)$ is the maximum eigenvalue of $A^2(G)$. Since

$$\mu^2(G) > \frac{1}{n}\|A^2\|_1 = \frac{1}{n} w_3(G),$$

unless all row sums of $A^2(G)$ are equal, we deduce that $w_3(u)$ is constant for every u. As in the proof of Theorem 5, it follows that G is semiregular. $\square$

Theorem 11. *Let $G = G(n)$ be a connected bipartite graph with eigenvalues $\mu_1 \geqslant \cdots \geqslant \mu_n$. Then G is semiregular iff for all s such that $0 < |\mu_s| < \mu(G)$ every eigenvector to μ_s is orthogonal to* $\mathbf{i}$.

Proof. Assume $0 < |\mu_s| < \mu_1$ and $\mathbf{u}_s = (u_{s1}, \ldots, u_{sn})$ is a unit eigenvector to μ_s. Let $\mathbf{u}_1, \ldots, \mathbf{u}_s, \ldots, \mathbf{u}_n$ be orthogonal unit eigenvectors to $\mu_1, \ldots, \mu_n$, and $c_1, \ldots, c_n$ be as defined in Section 2.1. If every eigenvector to μ_s is orthogonal to $\mathbf{i}$, then

$$n = c_1 + c_n,$$
$$\sum_{i=1}^{n} d^2(i) = w_3(G) = (c_1 + c_n)\mu^2 = n\mu_1^2,$$

so G is semiregular by Lemma 10.

If G is semiregular, then by Lemma 9,

$$\sum_{i=1}^{n} c_i \left(\frac{\mu_i}{\mu_1}\right)^2 = \frac{w_3(G)}{\mu^2(G)} = n = \sum_{i=1}^{n} c_i.$$

This implies that $|\mu_i/\mu_1| = 1$ whenever $|\mu_i| > 0$ and $c_i > 0$. Hence, $c_s = 0$, i.e., $\mathbf{u}_s$ is orthogonal to $\mathbf{i}$. $\square$

3.4. More lower bounds

A common device for finding lower bounds on $\mu(G)$ is the Rayleigh principle applied with carefully chosen vectors.

Let $p \geqslant 0$, $r \geqslant 1$ be integers and G be a graph of order n with no isolated vertices. Setting $x_i = w_p(i)/\sqrt{w_{2p-1}(G)}$ for all $i \in [n]$ and letting $\mathbf{x} = (x_1, \dots, x_n)$, the Rayleigh principle gives another proof of inequality (6) by

$$\mu^r(G) \geqslant \langle A^r(G)\mathbf{x}, \mathbf{x}\rangle = \frac{1}{w_{2p-1}(G)} \sum_{u,v\in V(G)} w_{r+1}(u, v) w_p(u) w_p(v) = \frac{w_{2p+r-1}(G)}{w_{2p-1}(G)}.$$

Set $y_i = \sqrt{w_p(i)/w_p(G)}$ for all $i \in [n]$ and let $\mathbf{y} = (y_1, \dots, y_n)$. By the Rayleigh principle we obtain the following general bound

$$\mu^r(G) \geqslant \langle A^r(G)\mathbf{y}, \mathbf{y}\rangle = \frac{1}{w_p(G)} \sum_{u,v\in V(G)} w_{r+1}(u, v)\sqrt{w_p(u) w_p(v)}. \tag{9}$$

Since by Cauchy–Schwarz's inequality we have

$$\sum_{u,v\in V(G)} w_{r+1}(u, v)\sqrt{w_p(u) w_p(v)} \sum_{u,v\in V(G)} \frac{w_{r+1}(u, v)}{\sqrt{w_p(u) w_p(v)}} \geqslant \left(\sum_{u,v\in V(G)} w_{r+1}(u, v)\right)^2 = w_{r+1}^2(G),$$

inequality (9) implies also that

$$\mu^r(G) \sum_{u,v\in V(G)} \frac{w_{r+1}(u, v)}{\sqrt{w_p(u) w_p(v)}} \geqslant \frac{w_{r+1}^2(G)}{w_p(G)}. \tag{10}$$

Setting $p = 2$, $r = 1$, we obtain the following inequalities proved by Favaron, Mahéo, and Saclé [8], and in a wider context also by Hoffman et al. [11],

$$\mu(G) \geqslant \frac{1}{2e(G)} \sum_{uv\in E(G)} \sqrt{d(u)d(v)}, \tag{11}$$

$$\mu(G) \geqslant \frac{2e(G)}{\sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)d(v)}}}. \tag{12}$$

As shown in [8] and [11] equality holds in (11) and (12) iff G is regular or semiregular. The case of equality in (9) and (10) is an open question.

4. Upper bounds on $\mu(G)$

In this section we present two general upper bounds on $\mu(G)$. Theorem 14 below gives the first bound in terms of the clique number and the number of walks. The bound of the second type is given in Section 4.1. The proof of Theorem 14 relies on two simple preliminary results.

Lemma 12. *For every $r > 0$ and every graph G,*

$$w_{2r}(G) \leqslant \frac{\omega(G) - 1}{\omega(G)} w_r^2(G).$$

Proof. Indeed, we have

$$w_{2r}(G) = \sum_{uv\in E(G)} w_r(u)w_r(v) \leqslant \frac{\omega(G)-1}{\omega(G)}\left(\sum_{u\in V(G)} w_r(u)\right)^2 = \frac{\omega(G)-1}{\omega(G)} w_r^2(G).$$

□

Applying Lemma 12 several times, we generalize it as follows.

Corollary 13. *For every graph G and $k, r > 0$,*

$$\frac{\omega(G)-1}{\omega(G)} w_{2^k r}(G) \leqslant \left(\frac{\omega(G)-1}{\omega(G)} w_r(G)\right)^{2^k}.$$

We are ready now to prove the main result of this section.

Theorem 14. *For every graph G and $r \geqslant 1$,*

$$\mu^r(G) \leqslant \frac{\omega(G)-1}{\omega(G)} w_r(G) \tag{13}$$

Proof. Clearly it suffices to prove inequality (13) for connected graphs. We shall assume first that G is nonbipartite. Assume that (13) fails, i.e.,

$$\mu^r(G) > (1+c)\frac{\omega(G)-1}{\omega(G)} w_r(G)$$

for some $G, r > 0, c > 0$. Then, by Corollary 13, for every $k > 0$,

$$\mu^{2^k r}(G) > \left((1+c)\frac{\omega(G)-1}{\omega(G)} w_r(G)\right)^{2^k} \geqslant (1+c)^{2^k}\frac{\omega(G)-1}{\omega(G)} w_{2^k r}(G). \tag{14}$$

Note that Theorem 1 implies that for every ε,

$$c_1\mu^{q-1}(G) < (1+\varepsilon) w_q(G) \tag{15}$$

for all sufficiently large q. Hence, for $q = 2^k r$ and k sufficiently large, Theorem 3 and inequality (15) imply that

$$\mu^{2^k r}(G) \leqslant \frac{\omega(G)-1}{\omega(G)} c_1\mu^{2^k r-1}(G) < (1+\varepsilon)\frac{\omega(G)-1}{\omega(G)} w_{2^k r}(G),$$

contradicting (14).

Finally we have to prove (13) for bipartite G. Then $\omega(G) = 2$, so we have to prove that $\mu^r(G) \leqslant w_r(G)/2$ for every $r \geqslant 2$. If r is odd, Theorems 3 and 1 imply

$$\mu^r(G) \leqslant \frac{1}{2} c_1\mu_1^{r-1} \leqslant \frac{1}{2} w_r(G).$$

Let now r be even. Write $cw_k(G)$ for the number of *closed walks* on k vertices in G (i.e., k-walks with the same start and end vertex.) It is known that

$$cw_{k+1}(G) = tr(A^k(G)) = \mu_1^k + \cdots + \mu_n^k. \tag{16}$$

The spectrum of bipartite graphs is symmetric with respect to 0, thus $2\mu^r(G) \leqslant cw_{r+1}(G) \leqslant w_r(G)$, completing the proof. □

Theorem 15. *Suppose that G is graph such that equality holds in (13) for some $r \geqslant 1$. If $r = 1$, then G is a regular complete $\omega(G)$-partite graph. If $r > 1$, then G has a single nontrivial component G_1. If $\omega(G) > 2$, then G_1 is a regular complete $\omega(G)$-partite graph. If $\omega(G) = 2$, then G_1 is a complete bipartite graph, and if r is odd, then G_1 is regular.*

Proof. Assume

$$\mu^r(G) = \frac{\omega(G) - 1}{\omega(G)} w_r(G) \tag{17}$$

and let c_i be defined as in Section 2.1.

If $r = 1$ then

$$\frac{\omega(G) - 1}{\omega(G)} v(G) = \mu(G) \leqslant \sqrt{2\frac{\omega(G) - 1}{\omega(G)} e(G)};$$

from the case of equality in Turán's theorem (see, e.g., [2]) it follows that G is regular complete $\omega(G)$-partite graph.

Assume now $r \geqslant 2$; let G_1 be a component of G with $\mu(G) = \mu(G_1)$. If G_2 is another nontrivial component of G, then

$$\mu^r(G_1) = \mu^r(G) = \frac{\omega(G) - 1}{\omega(G)}(w_r(G_2) + w_r(G_1)) > \frac{\omega(G) - 1}{\omega(G)} w_r(G_1),$$

a contradiction; thus G_1 is the only nontrivial component of G. We also see that the equality (17) holds for G_1, so for simplicity we shall assume that G is connected. From Corollary 13 and (17) we deduce that

$$\mu^{2^k r}(G) = \frac{\omega(G) - 1}{\omega(G)} w_{2^k r}(G) = \frac{\omega(G) - 1}{\omega(G)} \mu^{2^k r - 1}(G) \sum_{i=1}^{n} c_i \left(\frac{\mu_i}{\mu_1}\right)^{2^k r - 1} \tag{18}$$

for every integer $k > 0$. Assume G is nonbipartite; therefore, $|\mu_n(G)| < \mu(G)$ and, letting k tend to infinity, we find that

$$\mu(G) = \frac{\omega(G) - 1}{\omega(G)} c_1.$$

From Theorem 4 it follows that G is a complete $\omega(G)$-partite graph, and thus G has no positive eigenvalues other than $\mu(G)$. Hence, from (18), any c_i corresponding to a negative eigenvalue must be 0. Therefore,

$$n = w_1(G) = c_1\mu(G) = \frac{\omega(G)}{\omega(G) - 1}\mu(G),$$

a case that is settled above.

Let now G be bipartite. If r is odd, we have

$$\mu^r(G) = \frac{1}{2} w_r(G) = \frac{1}{2}\sum_{i=1}^{n} c_i (\mu_i)^{r-1};$$

so, by Theorem 3 $c_1 = 1/2$. Moreover, either $c_i = 0$ or $\mu_i = 0$ for $i = 2, \ldots, n$. We have again

$$n = w_1(G) = c_1\mu(G) = 2\mu(G),$$

implying that G is a regular complete bipartite graph.

For even r we have

$$2\mu^r(G) \leqslant cw_{r+1}(G) \leqslant w_r(G) = 2\mu^r(G),$$

and, in view of (16), we conclude that G has only two nonzero eigenvalues—μ_1 and μ_n. Hence, in our case, Smith's theorem implies that G is a complete bipartite graph. $\square$

4.1. More upper bounds

It is known that the Perron root of a nonnegative matrix does not exceed its maximal row sum. This idea has been exploited to obtain the following bounds

$$\mu(G) \leqslant \max_{u\in V(G)} \sqrt{w_3(u)}, \tag{19}$$

$$\mu(G) \leqslant \max_{u\in V(G)} \frac{w_3(u)}{d(u)}, \tag{20}$$

$$\mu(G) \leqslant \max_{uv\in E(G)} \sqrt{d(u)d(v)}, \tag{21}$$

$$\mu(G) \leqslant \max_{uv\in E(G)} \sqrt{\frac{w_3(u)w_3(v)}{d(u)d(v)}}. \tag{22}$$

Inequalities (19) and (20) are proved in [8], inequality (21) is proved in [1], and inequality (22) in [6]. As an attempt to interrupt this monotonic sequence we propose the following general result.

Theorem 16. *For every integers $p \geqslant 1$, $r \geqslant 1$ and any graph G,*

$$\mu^r(G) \leqslant \max_{u\in V(G)} \frac{w_{r+p}(u)}{w_p(u)}.$$

Proof. Set $b_{ii} = w_p(i)$ for each $i \in [n]$ and let B be the diagonal matrix with main diagonal $(b_{11}, \dots, b_{nn})$. Since $B^{-1}A^r(G)B$ has the same spectrum as $A^r(G)$, $\mu^r(G)$ is bounded from above by the maximum row sum of $B^{-1}A^r(G)B$—say the sum of the kth row—and so,

$$\mu^r(G) \leqslant \sum_{v\in V(G)} w_r(k,v)\frac{w_p(v)}{w_p(k)} = \frac{w_{r+p}(k)}{w_p(k)} \leqslant \max_{u\in V(G)} \frac{w_{r+p}(u)}{w_p(u)}. \qquad \square$$

Setting $p = 1$, $r = 2$, we obtain (19); the case $p = 2$, $r = 1$ implies (20). Furthermore, (21) follows from (19) by

$$\mu^2(G) \leqslant \max_{u\in V(G)} w_3(u) = \max_{uv\in E(G)} d(u)\left(\frac{1}{d(u)}\sum_{v\in\Gamma(u)} d(v)\right) \leqslant \max_{uv\in E(G)} d(u)d(v),$$

and (22) follows by

$$\mu^2(G) \leqslant \max_{u\in V(G)} \frac{w_4(u)}{d(u)} = \max_{u\in V(G)} \frac{w_3(u)}{d(u)}\frac{w_4(u)}{w_3(u)} \leqslant \max_{u\in V(G)} \frac{w_3(u)}{d(u)}\left(\frac{\sum_{v\in\Gamma(u)} w_3(v)}{\sum_{v\in\Gamma(u)} d(v)}\right)$$

$$\leqslant \max_{u\in V(G)} \frac{w_3(u)}{d(u)} \left(\frac{1}{d(u)} \sum_{v\in\Gamma(u)} \frac{w_3(v)}{d(v)} \right) \leqslant \max_{uv\in E(G)} \frac{w_3(u)}{d(u)} \frac{w_3(v)}{d(v)}$$

with plenty of room.

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