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A new symplectic surgery: the 3-fold sum

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Abstract

We define a symplectic surgery along positively intersecting symplectic surfaces in 4-manifolds, the generalized symplectic sum, and prove an existence theorem for the special case of a 3-fold sum. With a slight generalization of the 3-fold sum, we show how to sum along immersed surfaces and indicate a relation between the sums and algebraic desingularization. We use images of the moment map for a torus acting in the neighborhood of intersection points to illustrate when it is possible to perform the proposed sums. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

A smooth manifold M is *symplectic* if it is equipped with a nondegenerate closed 2-form ω. The nondegeneracy of the form means that its top exterior power is a volume form. The existence of a nondegenerate 2-form implies the existence of an almost complex structure (a complex structure on the tangent bundle). While any open almost complex manifold admits a symplectic structure [8], the only way to show that a closed manifold is symplectic is to actually produce a symplectic structure on it. Among the methods for constructing symplectic manifolds, some are analogs of complex algebraic operations (blowing up and down, branched covers), some are motivated by topological constructions (the symplectic sum, symplectic fibrations) and some are more closely tied to symplectic geometry (symplectization and symplectic reduction).

Since a Kähler form is closed and nondegenerate, any Kähler manifold is symplectic. In 1976 Thurston [23] showed how to put a symplectic structure on a fiber bundle with

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symplectic fiber and base, in particular on a twisted torus bundle that has b_1 odd and therefore admits no Kähler structure. (It was later discovered that this example was previously known to Kodaira [13].) The first simply-connected example was a 10-dimensional manifold constructed by McDuff [16] by blowing up $\mathbb{C}P^5$ along a symplectically embedded copy of Thurston's manifold (the above-mentioned torus bundle). Recently Gompf [6] applied the symplectic sum to produce a myriad of interesting examples, among them a simply-connected symplectic 4-manifold that, by calculations of Fintushel and Stern [5], is not diffeomorphic to any complex surface. Gompf also used the symplectic sum to prove that any finitely presented group can be realized as the fundamental group of a symplectic 4-manifold.

Gromov [10] introduced the notion of a symplectic sum, which topologically is a connected sum along codimension 2 submanifolds with anti-isomorphic normal bundles. Proofs that the symplectic sum does in fact yield a symplectic manifold were provided independently by Gompf [6] and McCarthy and Wolfson [15]. Note that the usual (pointwise) topological connected sum of two symplectic manifolds does not in general admit a symplectic structure induced from the summands (except in dimension 2 where it is a trivial case of the symplectic sum). In particular, in dimension 4 the connected sum of two symplectic manifolds admits no almost complex structure, and consequently no symplectic structure.

Gromov [10] had also suggested the possibility of performing a symplectic sum along immersed submanifolds with orthogonal intersections. The author defined and studied such a generalization for dimension 4 in [22]. In this paper we focus primarily on a special case, the 3-fold sum, proving in Section 4 that it is indeed a symplectic operation. (Another special case, the 4-fold sum was used by McDuff and the author [19] to show the symplectic equivalence of certain 4-manifolds.)

The motivation for defining the 3-fold sum came from an effort to answer a question of Bogomolov [3] as to whether one can define a symplectic analog of algebraic desingularization. By algebraic desingularization we mean the passage from a degenerate complex algebraic hypersurface to a smooth one by a perturbation of its defining equation. We show in Section 6.2 that the 3-fold sum does realize the desingularization of the degenerate cubic $z_1 z_2 z_3 = 0$ in $\mathbb{C}P^3$. This degenerate cubic provides a (topological) model for a generic triple intersection of algebraic surfaces in an algebraic 3-fold. We conjecture that the generalized symplectic sum of the type considered in this paper, together with the symplectic sum, can be used to realize the algebraic desingularization of any degenerate degree d hypersurface in $\mathbb{C}P^3$ (or in fact any algebraic 3-fold).

In the next section we define the generalized symplectic sum, and as a warm-up define the symplectic sum in such a way that it is clearly a special case of a generalized symplectic sum. We also define the 3-fold sum and state the existence theorem we prove in Section 4. In Section 3 we collect some useful facts about Hamiltonian torus actions and moment maps and describe the model neighborhoods of submanifolds and boundaries that we use in our proofs. While torus actions and moment maps are not necessary for any of the proofs, the geometric picture one can extract using these notions is quite helpful as a guide in these constructions. In Section 6.1 we show how to perform a symplectic surgery along two pairs of symplectic surfaces among which one surface is immersed. When the immersed surface is a sphere, the surfaces of the other pair cannot have the same area and we are forced to modify slightly the definition of the 3-fold sum to one which can increase the total volume. We discuss this modification of the sum in Section 5.

The constructions described in this paper could be generalized to higher dimensions, but we concentrate here on the 4-dimensional case where the triviality of any complex line bundle over a punctured surface simplifies matters. Throughout this paper, unless otherwise specified, the reader should assume that surfaces and manifolds are without boundary and submanifolds are embedded.

2. Background and results

2.1. The symplectic sum

Consider a pair of codimension 2 symplectomorphic submanifolds $\Sigma \subset (M,\omega)$ and $\Sigma' \subset (M',\omega')$. Recall that Σ, Σ' being *symplectomorphic* means there exists a diffeomorphism $\psi:(\Sigma,\omega_\Sigma) \to (\Sigma',\omega_{\Sigma'})$ such that $\psi^*(\omega_{\Sigma'}) = \omega_\Sigma$. Here the symplectic forms ω_Σ, $\omega_{\Sigma'}$ are induced by inclusion in M, M'. If M, M' are 4-manifolds, then the surfaces Σ, Σ' are symplectomorphic if they have the same genus and area.

The symplectic sum of M, M' along Σ, Σ' is the union of two manifolds with boundary. Topologically, it is the union of $M - N(\Sigma)$ and $M' - N(\Sigma')$ glued along their boundaries, where N denotes neighborhood. The identified boundaries constitute the *gluing locus* $X \subset \widetilde{M}$, a circle bundle over $\Sigma = \Sigma'$. See Fig. 1 where $\overline{M}$, $\overline{M}'$ are diffeomorphic to $M - N(\Sigma)$, $M' - N(\Sigma')$.

We define the symplectic sum so as to facilitate generalization. The key to our definition is to notice that the open manifold $(M - \Sigma) \sqcup (M' - \Sigma')$ is symplectomorphic to the interior of a manifold $\overline{M} \sqcup \overline{M}'$, well defined up to symplectomorphism, whose boundary is diffeomorphic to the union of the unit normal bundles of Σ and Σ'. We call $\overline{M} \sqcup \overline{M}'$ the *associated manifold with boundary*. Note that henceforth we do not require that Σ, Σ' lie in different manifolds.

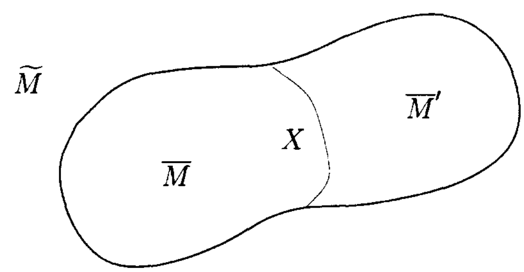

Fig. 1. A symplectic sum $\widetilde{M}$ and its gluing locus X.

Definition 2.1. Let $\mathcal{C} = \{\Sigma, \Sigma'\}$ be a pair of disjoint embedded symplectic surfaces in a symplectic 4-manifold (M, ω) (which need not be connected). Let $\overline{M}$ be the associated manifold with boundary. A closed symplectic manifold $(\widetilde{M}, \widetilde{\omega})$ is the *symplectic sum* of M along Σ, Σ' if there exists a symplectic embedding $\varphi: (M - \mathcal{C}) \to \widetilde{M}$ which extends to a surjective symplectic immersion $\overline{\varphi}: \overline{M} \to \widetilde{M}$.

The kernel of the symplectic structure restricted to the boundary $\partial\overline{M}$ defines a line field whose integral is the *characteristic foliation*, which in this case is a circle fibration. Forming the symplectic sum amounts to identifying the boundary components of $\overline{M}$ via a diffeomorphism (induced by the immersion $\overline{\varphi}$) that respects the characteristic foliations on the boundaries and covers a symplectomorphism of the surfaces. Such a diffeomorphism can be found whenever Σ, Σ' are symplectomorphic and have anti-isomorphic normal bundles. Restricting to dimension 4, this existence statement can be phrased as follows:

Theorem 2.2 (Gromov [10], McCarthy and Wolfson [15], Gompf [6]). *A symplectic sum of M along symplectic surfaces Σ, Σ' exists if and only if Σ, Σ' have equal genus and area, and $k + k' = 0$ where k, k' are the self-intersection numbers of Σ, Σ'.*

That the symplectic sum exists when the normal bundles of the submanifolds are trivial is clear. The neighborhoods of the submanifolds admit split symplectic structures, and the sum can be done fibrewise thanks to the facts that in dimension 2 any annuli of the same area (including the punctured disk) are symplectomorphic and that there is an area preserving diffeomorphism of any annulus to itself that interchanges the inner and outer boundaries. When the normal bundles are twisted a little more care needs to be taken, but the topological requirement that the normal bundles be anti-isomorphic still suffices for the symplectic sum to exist.

Because the complement $\widetilde{M} - X$ is symplectomorphic to $M - (\Sigma \sqcup \Sigma')$, cutting $\widetilde{M}$ along X and collapsing the circle fibers on each resulting boundary component yields M. This inverse of the symplectic sum is the process of symplectic cutting defined by Lerman [14]. The collapsing of the circle fibers is a special case of symplectic reduction. The necessity of the symplectic equivalence of Σ, Σ' is clear since they both must be symplectic reductions of the same hypersurface, the gluing locus $X = \overline{\varphi}(\partial\overline{M})$.

Remark 2.3. The ability to sum along submanifolds that are not symplectomorphic depends upon the ability to deform the symplectic structure on M to make them symplectomorphic. We discuss the appropriate deformation for dimension 4 in Section 5.1.

We choose not to form symplectic sums by removing tubular neighborhoods of symplectomorphic surfaces and identifying collar neighborhoods since the construction would then require a deformation of the symplectic structures unless the normal bundles were trivial.

2.2. *The generalized symplectic and 3-fold sums*

We now define the generalized symplectic sum along surfaces that have orthogonal intersections. Note that it suffices for the intersections to be positive, since the surfaces can then be perturbed to have orthogonal intersections. The definition is analogous to that of the symplectic sum, except now the boundary of the associated manifold $\overline{M}$ has corners along Lagrangian tori. There is one Lagrangian torus for each intersection point, hence we call them *intersection tori*. To form a generalized symplectic sum we identify corresponding smooth components of the connected boundary components of $\overline{M}$. These smooth components are circle bundles over punctured surfaces—the surfaces along which we glue, minus any intersection points. Thus the gluing locus X is a singular hypersurface whose singular set $X_0 \subset X$ is a union of Lagrangian tori. See Fig. 2. Recall that a submanifold $L \subset (M, \omega)$ is *Lagrangian* if the dimension of L is half the dimension of M and $j_L{}^*\omega = 0$ for the inclusion map j_L. We make precise in Definition 3.10 the notion of the manifold with boundary $\overline{M}$ associated to $M - (\Sigma_1 \cup \Sigma_2')$ where Σ_1, Σ_2' are surfaces that intersect orthogonally at one point. It is a trivial matter to extend this definition to the manifold with boundary associated to $M - C$ for a general collection of surfaces with orthogonal intersections.

Definition 2.4. Let C be a collection of intersecting immersed symplectic surfaces in a symplectic 4-manifold (M, ω) (which need not be connected). Assume that any intersections of the surfaces are symplectically orthogonal. Let $\overline{M}$ be the associated manifold with boundary. A closed symplectic manifold $(\widetilde{M}, \widetilde{\omega})$ is a *generalized symplectic sum of M along C* if there exists a symplectic embedding $\varphi: (M - C) \to \widetilde{M}$ which extends to a surjective symplectic immersion $\overline{\varphi}: \overline{M} \to \widetilde{M}$.

The immersion $\overline{\varphi}$ defines the diffeomorphisms between smooth components of the boundary of $\overline{M}$. If the collection of surfaces consists of a pair of disjoint symplectomorphic surfaces then the generalized symplectic sum is just a symplectic sum.

In this paper we restrict our attention to generalized symplectic sums in which X_0 consists of one Lagrangian torus and locally there are three smooth components of X meeting along the one torus of X_0 (or equivalently, such that the preimage $\overline{\varphi}^{-1}(X_0)$ is a disjoint union of three Lagrangian tori). Fig. 3(a) represents a 3-fold sum, a sum of three manifolds (M_i, ω_i), $i = 1, 2, 3$, each of which contains a pair of surfaces

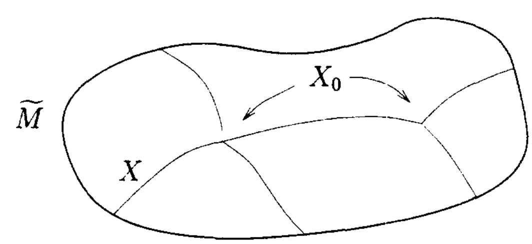

Fig. 2. A generalized symplectic sum and its gluing locus.

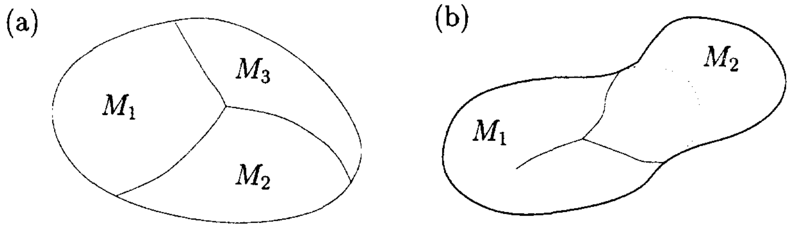

Fig. 3. (a) A 3-fold sum. (b) A generalized symplectic sum.

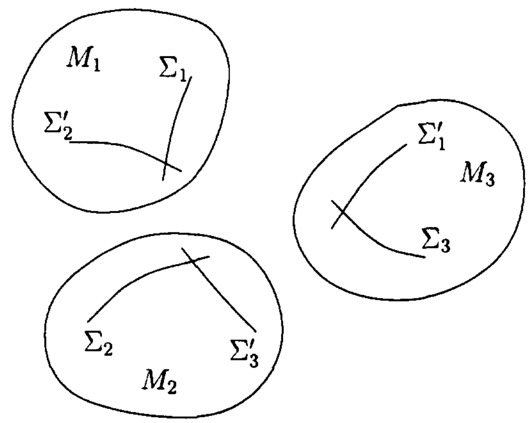

Fig. 4. Admissible surfaces for a 3-fold sum.

Σ_i, Σ'_{i+1} that intersect positively at exactly one point and such that for each i (understood mod 3), Σ_i and Σ'_i are symplectomorphic. See Fig. 4. Fig. 3(b) represents a generalized symplectic sum of two manifolds, one of which contains an immersed surface, such as in Example 6.1.

We define the 3-fold sum as a special case of the generalized symplectic sum which applies only to collections of three pairs of surfaces:

Definition 2.5. Let $\mathcal{C}$ be a collection of three pairs of symplectic surfaces in a symplectic 4-manifold (M,ω) (which need not be connected). A manifold $\widetilde{M}$ is a 3-fold sum of M along $\mathcal{C}$ if it is a generalized symplectic sum such that $\overline{\varphi}^{-1}(X_0)$ is the disjoint union of three intersection tori.

In Section 4 we examine the configurations of surfaces for which a 3-fold sum exists. The following characterization is convenient:

Definition 2.6. A collection of symplectic surfaces $\mathcal{C}$ is *admissible* for a 3-fold if they can be labeled so that $\mathcal{C} = \{\Sigma_i, \Sigma'_i\}_{i=1}^3$ where

- Σ_i, Σ'_i are disjoint and diffeomorphic for each i, and
- there are exactly three (symplectically orthogonal) intersections $x_1 = \Sigma_1 \cap \Sigma'_2$, $x_2 = \Sigma_2 \cap \Sigma'_3$, $x_3 = \Sigma_3 \cap \Sigma'_1$.

Then the content of Theorem 4.1 is that for a 3-fold sum along a collection of surfaces to exist, the surfaces must be admissible and the pairs Σ_i, Σ_i' must be symplectomorphic with $k_i + k_i' = -1$ for each i, where k_i, k_i' are the self-intersection numbers of Σ_i, Σ_i'. The existence theorem is the converse:

Theorem 2.7 (Symington [22]). *Consider a collection $\mathcal{C} = \{\Sigma_i, \Sigma_i'\}_{i=1}^3$ of symplectic surfaces in (M, ω) which are admissible for a 3-fold sum. If for each i the surfaces Σ_i, Σ_i' are symplectomorphic and $k_i + k_i' = -1$, there exists a generalized symplectic sum $(\widetilde{M}, \widetilde{\omega})$ of M along $\mathcal{C}$, in this case called a 3-fold sum.*

The following is the simplest example of a 3-fold sum. More interesting examples are given in Section 6. Recall that a *proper transform* of a surface $\Sigma \subset M$ is a surface in the class $[\Sigma] - [E]$ once a point of M on Σ has been blown up and E is the resulting exceptional sphere.

Example 2.8. Let $M = M_1 \sqcup M_2 \sqcup M_3$ where each $M_i = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Consider the collection of surfaces $\mathcal{C} = \{\widetilde{L}_i, E_{i-1}\}_{i=1}^3$ where $E_i \subset M_i$ is an exceptional curve and $\widetilde{L}_i \in [L_i] - [E_i]$ is the proper transform of a line in M_i. This collection is admissible for a 3-fold sum and $\widetilde{L}_i \cdot \widetilde{L}_i + E_{i-1} \cdot E_{i-1} = -1$. If we choose the areas of all the lines L_i to be the same and of all the exceptional spheres E_i to be half that area, then the pairs $\widetilde{L}_i$, E_{i-1} are symplectomorphic for each i and we can form the 3-fold sum along $\mathcal{C}$. Because these manifolds are toric it is easy to see, using images of the moment map, that the 3-fold sum along this collection is just a copy of $\mathbb{C}P^2$ whose lines have area $3/2$ times the area of the original complex lines. We explain this in the next section in Example 3.7.

The 3-fold sum is distinct from other symplectic constructions when it is performed along surfaces that do not intersect any exceptional spheres (embedded spheres of self-intersection -1). It remains to be seen whether it will yield new examples of symplectic 4-manifolds.

3. Preliminaries

3.1. Hamiltonian actions and moment maps

While very few smooth 4-manifolds admit a global Hamiltonian T^2 action (only blow-ups of $\mathbb{C}P^2$ or $S^2 \times S^2$), the images of these manifolds under the moment map give rise to a nice way to represent the symplectic neighborhood of a pair of orthogonally intersecting symplectic surfaces in any symplectic 4-manifold.

Guillemin and Sternberg [11] and Atiyah [1] proved that the image of any closed connected symplectic manifold under the moment map for a torus action is a convex polytope. Delzant [4] studied the special case when the dimension of the torus is half the dimension of the manifold. He showed that if the action is *effective*, i.e., if every element of the group other than the identity moves at least one point, then the symplectic manifold is determined up to equivariant symplectomorphism by its image under the moment map. We recall the necessary definitions.

Definition 3.1. A T^n action on a symplectic manifold (M, ω) is *Hamiltonian* if for any element ξ of the Lie algebra $\mathfrak{t} = \mathbb{R}^n$ there is a function $f_\xi : M \to \mathbb{R}$ such that $\omega(X_\xi, \cdot) = -\mathrm{d}f_\xi$ where X_ξ is the fundamental vector field for ξ. The function f_ξ, defined up to a constant, is called a Hamiltonian for the vector field X_ξ.

Definition 3.2. The *moment map* for a Hamiltonian T^n action on a symplectic manifold (M, ω) is a map $\mu : M \to \mathfrak{t}^* = \mathbb{R}^n$ such that for $x \in M$, $\langle \mu(x), \xi \rangle = f_\xi(x)$ defines a Lie algebra homomorphism from $\mathfrak{t}$ to the Lie algebra of smooth functions (with respect to the Poisson bracket) which takes ξ to f_ξ.

Example 3.3. The standard Hamiltonian torus action on $\mathbb{C}^2$ is $t \cdot (z_1, z_2) = (\mathrm{e}^{\mathrm{i}t_1} z_1, \mathrm{e}^{\mathrm{i}t_2} z_2)$. The standard moment map associated to this action is $\mu : \mathbb{C}^2 \to \mathbb{R}^2$ given by $\mu(z_1, z_2) = (|z_1|^2/2, |z_2|^2/2)$.

Remark 3.4. If a symplectic manifold admits a torus action $T^n \times M \to M$ given by $t \cdot x = t(x)$ then it also admits a torus action $t \cdot x = (Bt)(x)$ for any B that is an automorphism of the torus. If μ is a moment map for the original action then $B^T \mu$ is a moment map for the new one. Thus if the moment map images of two closed connected manifolds with a half-dimensional effective torus action are equivalent up to unimodular transformation, then the manifolds are symplectomorphic.

Example 3.5. The images of moment maps for standard torus actions on $\mathbb{C}P^2$, $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ are shown in Fig. 5(a)–(c). In the image of a moment map, the preimage of a vertex is a point, the preimage of a point on the interior of an edge is a circle and the preimage of a point on the interior of the polytope is a torus. Thus the edges of the polytope represent intersecting spheres and the preimage of a neighborhood of a vertex is a ball—a neighborhood of an intersection of two spheres. We use thick lines to indicate the orbit in the preimage of these points is not the full torus. The images of $\mathbb{C}P^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ illustrate that blowing up a point in the symplectic category amounts to removing a ball and collapsing the circles of the characteristic foliation on the resulting boundary to form an exceptional sphere.

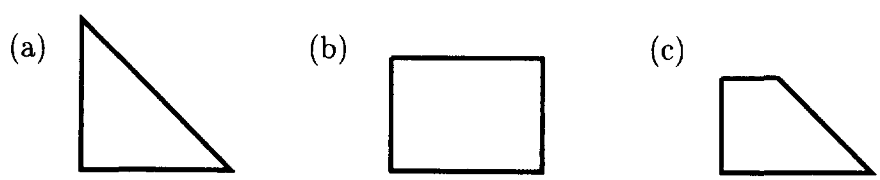

Fig. 5. (a) $\mathbb{C}P^2$; (b) $S^2 \times S^2$; (c) $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

A nice feature of the moment map is that we can read off of the image both the self-intersection numbers and the areas of the spheres that are preimages of an edge. For instance, choosing the torus action appropriately, the image lies in the first quadrant of $\mathbb{R}^2$ with one corner at the origin and two spheres S_1, S_2 whose images lie along the p_1, p_2 axes of $\mathbb{R}^2$ with endpoints at the origin and $(a_1, 0)$, $(0, a_2)$, respectively. Let m_1, m_2 be the slopes of the adjacent edges with vertices at $(a_1, 0)$, $(0, a_2)$. Then the areas and self-intersection numbers of the spheres S_1, S_2 are $2\pi a_1$, $2\pi a_2$ and $-1/m_1$, $-m_2$. This can be verified by looking at the image of $\mathbb{P}(L_k \oplus \mathbb{C})$, the projectivized line bundle over a sphere with Chern class k, under two different torus actions. These are the Hirzebruch surfaces [12], i.e., rational ruled surfaces.

Example 3.6. The rational ruled surfaces can be equipped with a Hamiltonian torus action by embedding them into $\mathbb{C}P^1 \times \mathbb{C}P^2$ as

$$\left\{\left([z_0 : z_1], [x_0 : x_1 : x_2]\right) \middle| z_0^k x_1 - z_1^k x_0 = 0\right\}$$

and taking the restriction of the Hamiltonian torus action

$$t \cdot \left([z_0 : z_1], [x_0 : x_1 : x_2]\right) = \left([\mathrm{e}^{\mathrm{i}t_1} z_0 : z_1], [x_0 : \mathrm{e}^{\mathrm{i}kt_1} x_1 : \mathrm{e}^{\mathrm{i}t_2} x_2]\right)$$

(cf. Audin [2]). Letting $\xi = ([x : y], [y^{-k}u : x^{-k}u : v])$ be a point on the ruled surface, the moment map $\mu = (\mu_1, \mu_2)$ for this action is

$$\mu_1(\xi) = \frac{1}{2}\left(\frac{|x|^2}{|x|^2 + |y|^2} + \frac{-k|x|^{-2k}|u|^2}{(|x|^{-2k} + |y|^{-2k})|u|^2 + |v|^2}\right),$$

$$\mu_2(\xi) = \frac{1}{2}\left(\frac{|v|^2}{(|x|^{-2k} + |y|^{-2k})|u|^2 + |v|^2}\right).$$

In the image of $\mathbb{P}(L_k \oplus \mathbb{C})$ under this moment map, S_1 is a section and S_2 is a fiber. For $k = -1$ the image is shown in Fig. 5(c). To interchange the positions of the images of the section and the fiber, modify the action by taking

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where B is applied as in Remark 3.4.

In the examples shown in Fig. 5, letting M denote the 4-manifold, the preimage of $\mu(M) \cap (\{p_2 < \varepsilon_1\} \cup \{p_1 < \varepsilon_2\})$ is a symplectic neighborhood of $S_1 \cup S_2$ consisting of plumbed disk bundles. This neighborhood is invariant under the torus action.

Example 3.7. Using the images of moment maps we can see that the 3-fold sum in Example 2.8 is just $\mathbb{C}P^2$. To do this we apply to the images linear maps that are compositions of unimodular transformations and translations. We denote these as the sum of a matrix and a vector, so $(B + b)p = Bp + b$ where B is unimodular and p, b are vectors in $\mathbb{R}^2$. Recall that in this example M is a disjoint union of three copies M_1, M_2, M_3 of $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$. Let $\overline{M}_i$ be the manifold with boundary associated to $M_i - (E_i \cup \widetilde{L}_i)$. Suppose the area of a line in any of these is a and the fibers $\widetilde{L}_i$ have area $a/2$. If

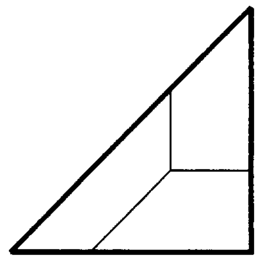

Fig. 6. Example of a 3-fold sum yielding $\mathbb{C}P^2$.

μ is the moment map for the torus action described in Section 3.1 then the union $(B_1 + b_1)(\mu(\overline{M}_1)) \cup (B_2 + b_2)(\mu(\overline{M}_2) \cup B_3(\overline{M}_3))$ is, up to translation, an image of $\mathbb{C}P^2$ if we choose

$$B_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$b_1 = \begin{pmatrix} 0 \\ -\frac{1}{2}a \end{pmatrix}, \quad b_2 = \begin{pmatrix} -\frac{3}{2}a \\ -\frac{1}{2}a \end{pmatrix}.$$

See Fig. 6 where we distinguish the image of the boundary of $\overline{M}_i$ from the images of the punctured spheres invariant under the torus action by using thin lines. The new projective plane contains projective lines that are the connected sum of a line and a fiber and so have area $3a/2$. Note that translations of these images amounts to varying by a constant the Hamiltonians defined by the torus actions, but these functions are defined only up to a constant to begin with.

3.2. Model neighborhoods

The possibility of summing along codimension 2 symplectomorphic submanifolds depends only on the isomorphism class (symplectic, or unitary) of the normal bundle thanks to the symplectic neighborhood theorem which states that the submanifold and its normal bundle determine the symplectic structure of the neighborhood. In particular, in a symplectic 4-manifold, if two embedded symplectic surfaces have the same genus, area and self-intersection number then they have symplectomorphic neighborhoods. This implies that the symplectic sum can be described in terms of model neighborhoods that are easy to classify.

In this section we construct model neighborhoods for orthogonally intersecting symplectic surfaces (Σ_1, ω_1), (Σ_2', ω_2'). (The primes are used here to be consistent with the notation for the 3-fold sum.) Fortunately we have a symplectic neighborhood theorem for such pairs of surfaces: the symplectic structure of the neighborhood depends only on the self-intersection numbers, areas and genera of the surfaces (cf. [20,22]). Therefore, given two symplectic surfaces, we can build model neighborhoods that are classified by two integers: the Euler classes of the two disk bundles being plumbed (i.e., the self-intersection numbers of the surfaces in the model neighborhood). Then given any two symplectic surfaces that intersect orthogonally at one point in a 4-manifold, their union has a tubular neighborhood symplectomorphic to one of the model neighborhoods.

Because any symplectic 2-plane bundle over a surface minus a disk is trivial, we can construct model neighborhoods of $\Sigma_1 \cup \Sigma_2'$ that are built out of three simple pieces: a neighborhood of the intersection point and product neighborhoods of the surfaces minus closed 2-disks centered at $x_1 = \Sigma_1 \cap \Sigma_2'$. By Darboux's theorem (the symplectic neighborhood theorem applied to a point), the first piece is symplectomorphic to a neighborhood of the origin in $(\mathbb{C}^2, \omega_o \oplus \omega_o)$ where $\omega_o \oplus \omega_o = \frac{1}{2}i(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2)$ is the standard symplectic structure on $\mathbb{C}^2$. Note that in higher dimensions we would not have such a simple decomposition of the neighborhood since the normal bundle of a punctured submanifold would in general have nontrivial invariants.

We choose the neighborhood of the origin so that its image under the moment map encodes the self-intersection numbers in the same way as for closed toric 4-manifolds. Thus, although a closed symplectic surface with genus greater than zero has no neighborhood that admits a Hamiltonian T^2 action, we can still represent the data of area and self-intersection number in a simple diagram. We call the chosen neighborhood of the origin a "hinge" $H \subset (\mathbb{C}^2, \omega_o \oplus \omega_o)$ because of the shape of its image under the moment map. We define

$$H = W_1 \cup \tau W_2 \quad \text{with}$$
$$W_1 = \left\{(z_1, z_2) \,\middle|\, \tfrac{1}{2}|z_1|^2 < \gamma_1 - \tfrac{1}{2}k_1|z_2|^2,\ \tfrac{1}{2}|z_2|^2 < \varepsilon_1\right\},$$
$$W_2' = \left\{(z_1, z_2) \,\middle|\, \tfrac{1}{2}|z_1|^2 < \gamma_2' - \tfrac{1}{2}k_2'|z_2|^2,\ \tfrac{1}{2}|z_2|^2 < \varepsilon_2'\right\},$$

where $\tau(z_1, z_2) = (z_2, z_1)$, $\gamma_1, \gamma_2' > 0$ and the thickness parameters ε_1, ε_2' are chosen small enough that $\gamma_1 - k_1\varepsilon_1 > \varepsilon_2'$ and $\gamma_2' - k_2'\varepsilon_2' > \varepsilon_1$.

To complete the neighborhood we define "strips" $S_1 \subset (\Sigma_1 \times \mathbb{C}, \omega_1 \oplus \omega_o)$, $S_2' \subset (\Sigma_2' \times \mathbb{C}, \omega_2' \oplus \omega_o)$. Let $D_1^- \subset D_1 \subset (\mathbb{C}, \omega_o)$ be a standard 2-disk of area $2\pi\gamma_1^- < 2\pi\gamma_1$ and choose a symplectic embedding $h_1 : (D_1, \omega_o) \to (\Sigma_1, \omega_1)$, $h_1((0,0)) = x_1$. Define D_2', $D_2'^-$ and h_2' similarly. Denote the closure of D_1^- by $\overline{D}_1^-$. Then

$$S_1 = \left\{(x, z_2) \,\middle|\, x \in \Sigma_1 - h_1(\overline{D}_1^-),\ \tfrac{1}{2}|z_2|^2 < \varepsilon_1\right\}$$

and S_2' is defined similarly. We paste these together using the gluing maps ϕ_1, $\tau\phi_2'$ where

$$\phi_1\big(h_1(z_1), z_2\big) = \left(z_1\left(1 - k_1\frac{|z_2|^2}{|z_1|^2}\right)^{1/2}, z_2\left(\frac{z_1}{|z_1|}\right)^{k_1}\right)$$

is defined on the domain $U_1 \subset S_1$,

$$U_1 = \left\{(x, z_2) \,\middle|\, x \in h_1(D_1 - \overline{D}_1^-),\ \tfrac{1}{2}|z_2|^2 < \varepsilon_1\right\},$$

and ϕ_2' is defined similarly on U_2'. Because the maps ϕ_1, ϕ_2' and τ are symplectic, the space

$$N = H \cup_{\phi_1} S_1 \cup_{\tau\phi_2'} S_2'$$

is a smooth symplectic manifold. By construction, the normal bundles of Σ_1, Σ_2' in this model neighborhood have Euler classes k_1, k_2'.

Remark 3.8. Note that $N_1 = W_1 \cup_{\phi_1} S_1$ and $N_2' = W_2' \cup_{\phi_2'} S_2'$ are model symplectic neighborhoods of Σ_1, Σ_2', the union of which is N. We call ε_1, ε_2' the *thicknesses* of N_1, N_2'.

Now look at the image of the hinge H under the moment map for the standard action on $\mathbb{C}^2$, namely $\mu(z_1, z_2) = (\frac{1}{2}|z_1|^2, \frac{1}{2}|z_2|^2)$. Choosing coordinates (p_1, p_2) on $\mathbb{R}^2$, the image is

$$\mu(H) = \{(p_1,p_2) \mid 0 \leqslant p_1 < \gamma_1 - k_1 p_2,\ 0 \leqslant p_2 < \varepsilon_1\}$$
$$\cup \{(p_1,p_2) \mid 0 \leqslant p_2 < \gamma_2' - k_2' p_1,\ 0 \leqslant p_1 < \varepsilon_2'\}.$$

Fig. 7 shows the image of a hinge for surfaces with self-intersection numbers $k_1 = 2$, $k_2' = -1$. By design we can read the self-intersection numbers off of the diagram as for spheres in a closed toric manifold. Furthermore, the disks D_1, D_2' of areas $2\pi\gamma_1$, $2\pi\gamma_2'$ have images that are line segments of lengths γ_1, γ_2'.

Note that each factor of the action of the torus $t = (t_1, t_2)$ on H extends to a circle action on one of the strips, e.g., $t_1 \cdot (x, z_2) = (x, \mathrm{e}^{\mathrm{i}t_1} z_2)$ for points in S_2'. This is a well defined action since the gluing maps ϕ_i are equivariant with respect to the circle action.

A model neighborhood of an immersed surface or a network of intersecting surfaces is now easy to construct. The model neighborhood consists of a hinge for each intersection point and a strip for each surface (minus disks centered at any intersection points on the surface). The Euler class of the normal bundle of a surface is then the sum of the contributions from each of the gluing maps that attach the strip to the hinges that correspond to intersection points on the surface.

3.3. Associated manifolds and boundaries

We now construct $(\overline{M}, \overline{\omega})$, the symplectic manifold with boundary associated to $M - (\Sigma_1 \cup \Sigma_2')$ where Σ_1, Σ_2' are a pair of symplectic surfaces with one orthogonal intersection.

We use the adjective "associated" to indicate that $M - (\Sigma_1 \cup \Sigma_2')$ and the interior of $\overline{M}$ are symplectomorphic, and that the boundary reduction of $(\overline{M}, \overline{\omega})$ is (M, ω). By

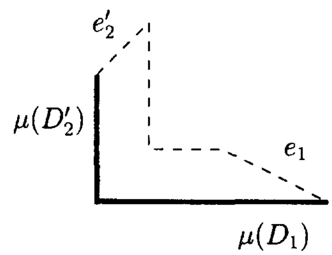

Fig. 7. Moment map image of a hinge.

the *boundary reduction* of $(\overline{M},\overline{\omega})$ we mean the manifold obtained by collapsing the circle fibers of the characteristic foliation on each smooth component of $\partial\overline{M}$. This has the effect of collapsing the intersection torus to a point. Thus the boundary reduction collapses $\partial\overline{M}$ to $\Sigma_1\cup\Sigma_2'$, and restricted to any smooth component of the boundary it is just symplectic reduction. In general, the *symplectic reduction* of a coisotropic submanifold $\Lambda\subset(M,\omega)$ is the symplectic manifold that is the quotient of Λ by the leaves of the foliation defined by $\ker(\omega|_\Lambda)$. A *coisotropic* submanifold is one whose tangent bundle contains its own symplectic orthogonal complement, $T\Lambda^{\perp_\omega}\subset T\Lambda$. The closedness of ω implies the integrability of the plane field $\ker(\omega|_\Lambda)=T\Lambda^{\perp_\omega}$.

To build $\overline{M}$ we start with a model neighborhood N that is symplectomorphic to a neighborhood of $\Sigma_1\cup\Sigma_2'\subset(M,\omega)$, so $\omega_1=j_1^*(\omega)$ where $j_1:\Sigma_1\to M$ is the inclusion map, and similarly for ω_2'. From N we will construct $\overline{N}$, a model collar neighborhood for the boundary of $\overline{M}$. Using the symplectic equivalence of the interior of $\overline{N}$ and $N-(\Sigma_1\cup\Sigma_2)$, we define

$$\overline{M}=\bigl(M-(\Sigma_1\cup\Sigma_2')\bigr)\cup_{f\pi|_{\operatorname{int}\overline{N}}}\overline{N},$$

where $f:N\to M$ is a symplectic embedding and $\pi:\overline{N}\to N$ is a symplectomorphism on the interior of $\overline{N}$ and symplectic reduction on $\partial\overline{N}$.

To define $\overline{N}$ all we need to do is make a symplectic change of coordinates on $N-(\Sigma_1\cup\Sigma_2')$, essentially replacing punctured disks with annuli and then taking the closure of the "inside" boundaries of these. Indeed, on S_1 and S_2' make the following change of coordinates in the fibers: $p_2=\frac{1}{2}|z_2|^2$ and $q_2=\arg z_2=z_2/|z_2|$. On H make that same change of coordinates in both complex planes yielding coordinates (p,q) where $p=(p_1,p_2)$, $q=(q_1,q_2)$. Then $N-(\Sigma_1\cup\Sigma_2')$ is symplectomorphic to the interior of

$$\overline{N}=\overline{H}\cup_{\overline{\phi}_1,\tau\overline{\phi}_2}(\overline{S}_1\cup\overline{S}_2'),$$

where

$$\begin{aligned}\overline{H}&=\bigl\{(p,q)\,\big|\,p_1<\gamma_1-k_1p_2,\ 0\leqslant p_2<\varepsilon_1\bigr\}\\&\quad\cup\bigl\{(p,q)\,\big|\,p_2<\gamma_2'-k_2'p_1,\ 0\leqslant p_1<\varepsilon_2'\bigr\},\\ \overline{S}_1&=\bigl\{(x,p_2,q_2)\,\big|\,x\in\Sigma_1-h_1(\overline{D}_1^-),\ 0\leqslant p_2<\varepsilon_1\bigr\}\end{aligned}$$

and $\overline{S}_2'$ is defined similarly. The symplectic map $\overline{\phi}_1$ is defined by

$$\overline{\phi}_1\bigl(\overline{h}_1(p_1,q_1),p_2,q_2\bigr)=(p_1-k_1p_2,q_1,p_2,q_2+k_1q_1)$$

with $\overline{h}_1(p_1,q_1)=h_1(z_1)$ and $\tau(p,q)=(Tp,Tq)$ where

$$T=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

Again, $\overline{\phi}_2'$ is defined analogously. The symplectic structure $\omega_{\overline{N}}$ agrees with $\omega_1\oplus\omega_{\rm std}$, $\omega_2'\oplus\omega_{\rm std}$ on $\overline{S}_1$, $\overline{S}_2'$ and with $\omega_{\rm std}\oplus\omega_{\rm std}=\mathrm{d}p_1\wedge\mathrm{d}q_1+\mathrm{d}p_2\wedge\mathrm{d}q_2$ on $\overline{H}$. The boundary of $\overline{N}$ has two smooth components joined along a Lagrangian torus T defined by $\{p_1=p_2=0\}$. The inverse of this change of coordinates extends to a smooth surjective submersion,

$\pi : \overline{N} \to N$. The intersection torus T is the preimage under π of the intersection point: $T = \pi^{-1}((0,0)) = \pi^{-1}f^{-1}(x_1)$ where $x_1 = \Sigma_1 \cap \Sigma_2'$.

Remark 3.9. Notice that the projection of $\overline{H}$ onto the p-coordinates is the same as the image of H under the moment map for the standard torus action on $\mathbb{C}^2 \supset H$.

Definition 3.10. Consider a pair of embedded symplectic surfaces $\mathcal{C} = \{\Sigma_1, \Sigma_2'\}$ that intersect orthogonally in one point in (M, ω) and let $f : N \to M$ be a symplectic embedding that realizes N, a model neighborhood of $\Sigma_1 \cup \Sigma_2'$ as a neighborhood of $\mathcal{C}$ in M. The *associated manifold with boundary* for $M - \mathcal{C}$ is the manifold $(\overline{M}, \overline{\omega})$ given by

$$\overline{M} = (M - \mathcal{C}) \cup_{f\pi|_{\operatorname{int}\overline{N}}} \overline{N}.$$

The symplectic structure $\overline{\omega}$ is defined by $\overline{\omega}|_{M-\mathcal{C}} = \omega$ and $\overline{\omega}|_{\overline{N}} = \omega_{\overline{N}}$.

The symplectic structure $\overline{\omega}$ is well defined since $f\pi$ is a symplectic map. It is not hard to show that $\overline{M}$ is well defined up to symplectomorphism (cf. [22]).

By construction $\partial\overline{M}$ has one connected boundary component with two smooth components which we call the associated boundaries (to Σ_1, Σ_2') and label $\partial_{\overline{\Sigma}_1}$, $\partial_{\overline{\Sigma}_2'}$. By $\overline{\Sigma}_1$ we mean the surface with boundary obtained by removing $x_1 = \Sigma_1 \cap \Sigma_2'$ from Σ_1, making a symplectic change of coordinates $z_1 = (p_1, q_1)$ on a coordinate chart centered at x_1 and taking the closure, adding in the boundary given by $\{p_1 = 0\}$. The intersection $\partial_{\overline{\Sigma}_1} \cap \partial_{\overline{\Sigma}_2'}$ is the intersection torus $\pi^{-1}f^{-1}(x_1)$. Furthermore, the characteristic foliations on the smooth components of the boundary $\partial_{\overline{\Sigma}_1}$, $\partial_{\overline{\Sigma}_2'}$ are circle fibrations over $\overline{\Sigma}_1$, $\overline{\Sigma}_2'$.

In the more general situation where we have intersecting, possibly immersed surfaces, the manifold $\overline{M}$ is constructed in the same way, with the same changes of coordinates on the strips and hinges as in the above case.

4. Existence of the 3-fold sum

In this section we prove the existence of the 3-fold sum. We also rephrase the theorem in terms of the image of the moment map, providing a diagrammatic way of checking for the existence of a 3-fold sum.

We begin with the following result on the structure of a 3-fold sum:

Theorem 4.1. *Suppose $(\widetilde{M}, \widetilde{\omega})$ is a 3-fold sum of (M, ω) along a collection $\mathcal{C}$ of three pairs of symplectic surfaces. Then the surfaces of $\mathcal{C}$ are admissible for a 3-fold sum and for each $i = 1, 2, 3$ the surfaces Σ_i, Σ_i' are symplectomorphic and their self-intersection numbers satisfy $k_i + k_i' = -1$.*

Proof. In the definition of the 3-fold sum, the fact that $\overline{\varphi} : \overline{M} \to \widetilde{M}$ must be an immersion implies that the image of each intersection torus, arising in $\overline{M}$ from an orthogonal intersection of surfaces in $\mathcal{C}$, must belong to the singular part of the gluing locus $X_0 \subset X$. Thus, because $\overline{\varphi}^{-1}(X_0)$ is the disjoint union of 3 intersection tori, there are exactly 3 intersection points among the surfaces. Without loss of generality we choose to label the surfaces so that the pairs Σ_i, Σ_i' give rise to smooth boundary components that are identified in $\widetilde{M}$. Then the Σ_i, Σ_i' must be disjoint, for otherwise the gluing would collapse the intersection torus to a circle by identifying transverse circle fibrations, contradicting the assumption that $\overline{\phi}$ is an immersion. So in order to have 3 intersection points, each surface must intersect one from another pair and we can choose the labeling of admissible surfaces. The surfaces Σ_i, Σ_i' must be symplectomorphic because $\Sigma_i - x_1$ and $\Sigma_i' - x_{i-1}$ are both symplectic reductions of the same hypersurface.

To see the necessity of the condition $k_i + k_i' = -1$, notice that because X_0 is a Lagrangian torus, it has a neighborhood symplectomorphic to a neighborhood of the zero section of $T^*T^2 = T^2 \times \mathbb{R}^2$. Because $\overline{\varphi}^{-1}(X_0)$ is the union of three intersection tori, its neighborhood must be symplectomorphic to a union of three hinges $\overline{H}_i$. The boundary identifications induce maps on the 1-dimensional homology of the $\overline{H}_i$ given by

$$\Omega_i = \begin{pmatrix} k_i + k_i' & -1 \\ 1 & 0 \end{pmatrix},$$

where we have chosen the canonical generators for the homology of the $\overline{H}_i \subset T^*T^2$, namely $\{p_1 = p_2 = \delta, q_2 = \alpha\}$ and $\{p_1 = p_2 = \delta, q_1 = \alpha\}$ for constants δ, α. For the neighborhood of X_0 to be trivial, we must have $\Omega_1\Omega_2\Omega_3 = \mathrm{Id}$. This can happen only if $k_i + k_i' = -1$ for $i = 1, 2, 3$. $\square$

We now prove the existence theorem:

Theorem 4.2. *Consider a collection $\mathcal{C} = \{\Sigma_i, \Sigma_i'\}_{i=1}^3$ of symplectic surfaces in (M, ω) which are admissible for a 3-fold sum. Suppose for each i the surfaces Σ_i, Σ_i' are symplectomorphic and $k_i + k_i' = -1$ where k_i, k_i' are the self-intersection numbers of Σ_i, Σ_i'. Let $(\overline{M}, \overline{\omega})$ be the manifold with boundary associated to $M - \mathcal{C}$. If $k_i + k_i' = -1$ for each $i = 1, 2, 3$ then there is a symplectic manifold $(\widetilde{M}, \widetilde{\omega})$ and a symplectic embedding $\varphi : (M - \mathcal{C}) \to \widetilde{M}$ that extends to a surjective symplectic immersion $\overline{\varphi} : \overline{M} \to \widetilde{M}$.*

Recall that the interior of $\overline{M}$ is symplectomorphic to $M - \mathcal{C}$.

To demonstrate existence we define diffeomorphisms that identify the corresponding smooth components $\partial_{\overline{\Sigma}_i}$, $\partial_{\overline{\Sigma}_i'}$ of $\partial\overline{M}$ and yield a smooth symplectic manifold. In particular, we make sure that the diffeomorphisms (boundary identifications) respect the characteristic foliations on the corresponding smooth boundary components and that we do not loose smoothness or agreement of the symplectic structures of the summands near the images of the intersection tori.

Before starting the proof, we note that any map $\mathcal{B} : T^*T^2 \to T^*T^2$ of the form $\mathcal{B}(p, q) = (Bp, B^{-T}q)$ with B unimodular is a symplectomorphism with respect to the standard symplectic structure $\omega_{\mathrm{std}} \oplus \omega_{\mathrm{std}} = \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \mathrm{d}p_2 \wedge \mathrm{d}q_2$. We use the notation (p, q) to represent a point (p_1, q_1, p_2, q_2) in T^*T^2 with $p = (p_1, p_2)$ and $q = (q_1, q_2)$.

Proof. We define boundary identification maps ϑ_i between the corresponding smooth boundary components of model collar neighborhoods $\overline{N}_i$ associated to model symplectic neighborhoods N_i of $\Sigma_i \cup \Sigma'_{i+1}$. We can always do this because the corresponding boundary components are trivial circle bundles that cover symplectomorphic punctured surfaces. Here the subscripts should be understood mod 3, so Σ'_4 denotes the surface Σ'_1.

Once we establish that

$$N_X = \bigcup_{i=1}^{3} \overline{N}_i / \vartheta_1, \vartheta_2, \vartheta_3$$

defines a smooth open symplectic manifold, it follows that the manifold

$$\widetilde{M} = \overline{M} / \vartheta_1, \vartheta_2, \vartheta_3$$

is a 3-fold sum of M along $\mathcal{C}$ with N_X a model neighborhood of the gluing locus $X \subset \widetilde{M}$ since

$$\overline{M} = (M - \mathcal{C}) \cup_{f_1\pi_1, f_2\pi_2, f_3\pi_3} \left(\bigcup_{i=1}^{3} \overline{N}_i \right),$$

implying

$$\widetilde{M} = (M - \mathcal{C}) \cup_{f_1\pi_1, f_2\pi_2, f_3\pi_3} (N_X).$$

Here the f_i, π_i are as in Section 3.3 and each map π_i is restricted to the interior of $\overline{N}_i$. The symplectic structure $\widetilde{\omega}$ is induced from ω via the changes of coordinates.

We begin by defining, for each i, the restriction of the map ϑ_i to the part of the boundary component $\partial_{\overline{\Sigma}_i}$ that belongs to $\overline{S}_i$:

$$\vartheta_i(x, 0, q_2) = (\psi_i(x), 0, -q_2),$$

where $\psi_i : \Sigma_i \to \Sigma'_i$ is a symplectomorphism that maps the intersection point in Σ_i to the intersection point in Σ'_i, i.e., $\psi_i(x_i) = x_{i-1}$. It is easy to see that this makes $\overline{S}_i \cup \overline{S}'_i / \vartheta_i$ into an open symplectic manifold with a product symplectic structure. Indeed, the maps ϑ_i are the boundary identifications induced from embeddings $\overline{S}_i, \overline{S}'_i \to (\Sigma_i \times T^*T^1, \omega_i \oplus \omega_{\text{std}})$ that are the identity and $\psi_i^{-1} \times A$ where $A : T^*T^1 \to T^*T^1$ is the symplectic map $A(p_2, q_2) = (-p_2, -q_2)$. (We assume that we have chosen the parameters γ_i, γ'_i to be equal in the definitions of $\overline{H}_i$, $\overline{H}_{i-1}$ and $\overline{S}_i$, $\overline{S}'_i$.)

The gluing maps $\overline{\phi}_i : \overline{U}_i \to \overline{H}_i$ and $\tau\overline{\phi}'_i : \overline{U}'_i \to \overline{H}_{i-1}$ that attach the strips $\overline{S}_i$, $\overline{S}'_i$ to the hinges $\overline{H}_i$, $\overline{H}_{i-1}$ are defined as in Section 3.3. They induce continuations of the ϑ_i to the boundaries of the hinges. For each i, the map induced on the boundary in the overlap is the restriction to $\partial(\overline{\phi}_i(\overline{U}_i))$ of the map $\overline{\phi}'_i \vartheta_i (\overline{\phi}_i)^{-1}$. In the local coordinates (p, q) of the hinges $\overline{H}_i$, these maps are the restrictions to $\{\gamma_i^- < p_1 < \gamma_i,\ p_2 = 0\}$ of the symplectic map

$$\mathcal{K}(p, q) = (Kp, K^{-T}q), \quad \text{where } K = \begin{pmatrix} 0 & -1 \\ 1 & k_i + k'_i \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

by hypothesis. We extend the maps ϑ_i to $\{0 \leqslant p_1 < \gamma_i,\ p_2 = 0\}$, using the same definition with respect to these coordinates. Then

$$N_{X_0} = \bigcup_{i=1}^{3} \overline{H}_i/\vartheta_1, \vartheta_2, \vartheta_3$$

is a space with open boundary.

The space N_{X_0} is a smooth symplectic manifold because it is symplectomorphic to a neighborhood of the zero section $T_0 \subset T^*T^2$. To see this, define a symplectic embedding $\mathcal{Z} : N_{X_0} \to T^*T^2$ onto a neighborhood of T_0 by specifying $\mathcal{Z}|_{\overline{H}_i} = \mathcal{Z}_i$ where

$$\mathcal{Z}_1 = \mathcal{K}, \qquad \mathcal{Z}_2 = \mathcal{K}^2, \qquad \mathcal{Z}_3 = \mathcal{K}^3 = \mathrm{Id}.$$

The map $\mathcal{Z}$ is well defined because for each i we have $\mathcal{Z}_i|_{\{p_2=0\}} = \mathcal{Z}_{i-1}|_{\{p_1=0\}}\vartheta_i$ where the subscripts are understood mod 3. Therefore

$$\begin{aligned} N_X &= N_{X_0} \cup_{\phi_1, \tau\phi_1'} \left(S_1 \cup S_1'/\vartheta_i\right) \cup_{\phi_2, \tau\phi_2'} \left(S_2 \cup S_2'/\vartheta_2\right) \cup_{\phi_3, \tau\phi_3'} \left(S_3 \cup S_3'/\vartheta_3\right) \\ &= \bigcup_{i=1}^{3} \overline{N}_i/\vartheta_1, \vartheta_2, \vartheta_3 \end{aligned}$$

is also a smooth symplectic manifold. □

In the neighborhood of each intersection point x_i we define a local moment map to be the composition $\mu f_i^{-1}|_{f_i(H_i)}$ where $f_i : N_i \to M$ is a symplectic embedding realizing N_i as a neighborhood of $\Sigma_i \cup \Sigma_{i+1}' \subset M$ and $\mu(z_1, z_2) = \frac{1}{2}(|z_1|^2, |z_2|^2) \subset \mathbb{R}^2$ is the moment map for the standard torus action on $\mathbb{C}^2$. The following proposition rephrases the existence of a 3-fold sum in terms of the geometric properties of the images of local moment maps. We assume that $\gamma_i = \gamma_i'$ in the definitions of the H_i, H_{i-1}, reflecting that Σ_i, Σ_i' have equal area. Recall that

$$K = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Proposition 4.3. *A* 3*-fold sum along an admissible collection* $\mathcal{C}$ *of symplectic surfaces exists if and only if the union of images* $K(\mu(H_1)) \cup K^2(\mu(H_2)) \cup K^3(\mu(H_3))$ *in* $\mathbb{R}^2$ *is an open* (*nonconvex*) 9*-gon in which the images of corresponding surfaces coincide and the images do not overlap on any open set.* (Fig. 8 shows such a 9-gon.)

Proof. By Theorems 4.1 and 4.2 we know that a 3-fold sum along an admissible collection $\mathcal{C}$ exists if and only if the corresponding surfaces Σ_i, Σ_i' have the same area and $k_i + k_i' = -1$. We simply need to equate this criteria with the geometric statements about the images of local moment maps.

Consider the union of images $K(\mu(H_1)) \cup K^2(\mu(H_2)) \cup K^3(\mu(H_3))$. The corresponding (closed) sides lie along the same ray, so the union of images can be open if and only if these coincide exactly, in which case the areas of the surfaces agree since $\gamma_i = \gamma_i'$ by choice.

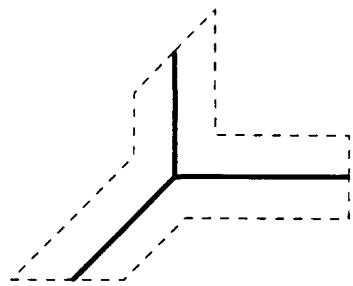

Fig. 8. Local moment map image for a 3-fold sum.

The open edges e_i, e'_{i+1} of $\mu(H_i)$ that are adjacent to $\mu(\Sigma_i)$, $\mu(\Sigma'_{i+1})$ have slopes $-1/k_i$, $-k'_{i+1}$, respectively. See Fig. 7. The unimodular transformation K maps the vector $\binom{-k_i}{1}$ to $\binom{-1}{-k_i-1}$. Thus the slopes of e'_i and Ke_i agree if and only if $k_i + k'_i = -1$, in which case the union of images has 9 open sides. □

5. Mismatched areas

5.1. Thickening and thinning

The possibility of forming a symplectic sum or a 3-fold sum depends upon the symplectic equivalence of the pairs of surfaces along which one plans to glue. In this section we define a useful way to deform a symplectic structure in the neighborhood of a symplectic submanifold Σ_i, namely by thickening or thinning along Σ_i. This type of deformation was described in [19] using the model neighborhood for a single surface. Here we give a description in terms of the model for intersecting surfaces.

Thickening or thinning along Σ_i consists of replacing a neighborhood of Σ_i by a neighborhood of a different thickness. Unless the normal bundle of Σ_i is trivial this process changes the area of Σ_i, and it always changes the area of the intersecting surface Σ'_{i+1}. Thickening or thinning along Σ'_{i+1} is defined analogously.

We first work with model neighborhoods, defining deformations $\mathbb{T}_t(\overline{N}_i)$ of $\overline{N}_i$ by replacing the minimum value of 0 on the p_2 coordinate by a minimum value of t, so $\mathbb{T}_0(\overline{N}_i) = \overline{N}_i$. More specifically,

$$\mathbb{T}_t(\overline{N}_i) = \mathbb{T}_t(\overline{H}_i) \cup_{(\overline{\phi}_i)_t} \mathbb{T}_t(\overline{S}_i) \cup_{\overline{\phi}'_{i+1}} \overline{S}_{i+1},$$

where

$$\begin{aligned}\mathbb{T}_t(\overline{H}_i) &= \bigl\{(p,q) \bigm| 0 \leqslant p_1 < \gamma_i - k_i p_2,\ t \leqslant p_2 < \varepsilon_i\bigr\}\\ &\quad \cup \bigl\{(p,q) \bigm| t \leqslant p_2 < \gamma'_{i+1} - k'_{i+1} p_1,\ 0 \leqslant p_1 < \varepsilon'_{i+1}\bigr\},\\ \mathbb{T}_t(\overline{S}_i) &= \bigl\{(x,p_2,q_2) \bigm| x \in \Sigma_i - h_i(\overline{D}_i^-),\ t \leqslant p_2 < \varepsilon_i\bigr\},\end{aligned}$$

and $(\overline{\phi}_i)_t$ is given by the same formula as $\overline{\phi}_i$ but defined on the domain

$$\mathbb{T}_t(\overline{U}_i) = \bigl\{(x,p_2,q_2) \bigm| x \in h_i(D_i - \overline{D}_i^-),\ t \leqslant p_2 < \varepsilon_i\bigr\}.$$

Now let $\mathbb{T}_t(N_i)$ be the symplectic boundary reduction of $\mathbb{T}_t(\overline{N}_i)$. The symplectic form $\omega_{\mathbb{T}_t(N_i)}$ is induced via changes of coordinates from the forms $\omega_{\text{std}} \oplus \omega_{\text{std}}$ on $\mathbb{T}_t(\overline{H}_i)$ and $\omega_{\Sigma_i} \oplus \omega_{\text{std}}$ on $\mathbb{T}_t(\overline{S}_i)$. Note, for the map $(\overline{\phi}_i)_t$ to be defined on the domain $\mathbb{T}_t(\overline{U}_i)$, the parameter t must lie in the interval $(\gamma_i/k_i, \varepsilon_i)$ if $k_i < 0$. If $k_i \geqslant 0$ then t can be anywhere in $(-\infty, \varepsilon_i)$.

To thicken or thin M along Σ_i means to replace $f_i(N_i)$ with a copy of $\mathbb{T}_t(N_i)$, yielding a closed symplectic manifold. We denote the thickened or thinned manifolds by $\mathbb{T}^+_{\Sigma_i}(M)$, $\mathbb{T}^-_{\Sigma_i}(M)$ and the new surfaces by Σ_i^+, Σ_i^-, respectively. After thickening $(t < 0)$ or thinning $(t > 0)$ the area of Σ_i has changed from A_i to $A_i - 2\pi k_i t$ and the area A'_{i+1} has changed to $A'_{i+1} - 2\pi t$. Note that if the normal bundle of Σ_i is trivial then the ambient manifold can be thickened arbitrarily along Σ_i and the area of Σ_i remains unchanged.

The procedure of thinning gives a nice way to see the convexity (or concavity) of a symplectic neighborhood as dictated by the isomorphism class of the normal bundle, or equivalently, the self-intersection number of the surface. Namely, the convexity of a neighborhood can be thought of as a measure of the rate of growth of the area of Σ_i^- with respect to thinning.

5.2. *Patching*

In this section we introduce another way to accommodate mismatches in the areas of corresponding surfaces when trying to perform generalized symplectic sums and 3-fold sums. In addition to allowing ourselves to deform the symplectic structure of M before performing a sum, we use neighborhoods of the zero section of T^*T^2 as "patches" and allow $\widetilde{M}$ to have larger volume than M. This added flexibility is sometimes necessary to form the sum along an immersed surface, as we shall see in Section 6.1.

Specifically, we define *domains in* T^*T^2 to be open symplectic manifolds of the form $U \times T^2 \subset (T^*T^2, \omega_{\text{std}} \oplus \omega_{\text{std}})$ where U is a domain in $\mathbb{R}^2$. Then we modify the definition of the generalized symplectic sum by letting $\overline{\varphi}$ be not surjective, yet asking that $\widetilde{M} - \overline{\varphi}(\overline{M})$ be properly contained in a union of open manifolds, each one symplectomorphic to a domain in T^*T^2.

We use these modified definitions and the language of Proposition 4.3 to express the existence theorem for the 3-fold sum in this setting. Recall that maps of the form $B + b$ are compositions of unimodular transformations and translations with $(B+b)p = Bp+b$.

Proposition 5.1. *Suppose $C \subset (M, \omega)$ is a collection of surfaces admissible for a 3-fold sum. A 3-fold sum along the collection exists if there exist vectors b_1, b_2 such that the union of images $(K + b_1)(\mu(H_1)) \cup (K^2 + b_2)(\mu(H_2)) \cup K^3(\mu(H_3))$, together with the domain bounded by the union, is an open 9-gon.*

We assume here that the parameters γ_i, γ'_{i+1} of the hinges H_i are chosen such that $\int_{\Sigma_i} \omega - 2\pi\gamma_i = \int_{\Sigma'_i} \omega - 2\pi\gamma'_i$ for each i, so that differences in the lengths of the sides of the hinges reflect differences in the areas of corresponding surfaces.

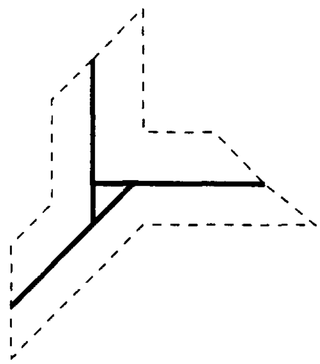

Fig. 9. Local moment map images for a 3-fold sum with mismatched areas.

Proof. Suppose that we have vectors b_1, b_2 such that the union of images satisfies the hypotheses. We construct $\widetilde{M}$ as follows. For each i choose Darboux neighborhoods $h_i(D_i) \subset \Sigma_i$, $h'_{i+1}(D'_{i+1}) \subset \Sigma'_{i+1}$ of x_i where D_i, $D'_{i+1} \subset \mathbb{R}^2$ are disks of radius γ_i, γ'_{i+1} and h_i, h'_{i+1} are symplectic embeddings, and choose symplectomorphisms $\psi_i : (\Sigma_i - h_i(\overline{D}_i^-)) \to (\Sigma'_i - h'_i(\overline{D'_i}^-))$. Choose model neighborhoods N_i as in Section 3.2 and boundary identifications $\vartheta_i : \overline{S}_i \to \overline{S}'_i$ as in the proof of Theorem 4.2. Use the maps $K + b_1$, $K^2 + b_2$, K^3 to construct $N_{X_0} \subset T^*T^2$ as the domain in T^*T^2 that is the union of the images and the manifold they bound (see Fig. 9). We can then glue all the strips $\overline{S}_i$, $\overline{S}'_i$ to N_{X_0} via the gluing maps $\overline{\phi}_i$, $\tau\overline{\phi}'_i$ as defined in Section 3.2. The compatibility of these maps with the ϑ_i follows because $k_i + k'_i = -1$ for each i by the arguments of Proposition 4.3. □

It is not hard to work out the exact constraints Proposition 5.1 places on the areas. When trying to sum along surfaces with mismatched areas it is helpful to first thin along the surfaces in question and then allow oneself the possibility of thickening in order to satisfy the hypotheses of the proposition. To quantify how much one can thin, recall that in the definition of the model neighborhoods N_i there are parameters ε_i, ε'_{i+1} that determine the size of each neighborhood. Consider $\mathcal{C} = \{\Sigma_i, \Sigma'_i\}_{i=1}^3$, a collection of surfaces in (M, ω) that is admissible for the 3-fold sum and assume $k_i + k'_i = -1$ for $i = 1, 2, 3$. Suppose $\{\varepsilon_i, \varepsilon'_i\}_{i=1}^3$ are such that the model neighborhoods $\{N_i\}_{i=1}^3$ can be disjointly and symplectically embedded in M as neighborhoods of the $\{\Sigma_i, \Sigma'_{i+1}\}_{i=1}^3$. Thinning along Σ_i by an amount δ_i and along Σ'_i by δ'_{i+1} changes the area of Σ_i from $\int_{\Sigma_i} \omega$ to $A_i = \int_{\Sigma_i} \omega - 2\pi k_i \delta_i - 2\pi\delta'_{i+1}$ and changes the area of Σ'_{i+1} to $A'_{i+1} = \int_{\Sigma'_{i+1}} \omega - 2\pi\delta_i - 2\pi k'_{i+1}\delta'_{i+1}$. One can thin by any amounts $\delta_i < \varepsilon_i$ and $\delta'_{i+1} < \varepsilon'_{i+1}$. Using these modified areas, we can thicken and then apply a 3-fold sum whenever the hypotheses of the following proposition are satisfied.

Proposition 5.2. *Let $\mathcal{C} = \{\Sigma_i, \Sigma'_i\}_{i=1}^3$ be a collection of surfaces in (M, ω) admissible for the 3-fold sum. Suppose $k_i + k'_i = -1$ for $i = 1, 2, 3$ and define*

$$\Delta A_i = A_i - A'_i.$$

Then the 3*-fold sum exists after thickening along the surfaces if there exist nonnegative constants* d_1, d_2, d_3 *such that*

$$\Delta A_1 - \Delta A_2 = k_1' d_1 + k_2 d_2 + d_3,$$
$$\Delta A_2 - \Delta A_3 = d_1 + k_2' d_2 + k_3 d_3.$$

Proof. To prove the proposition it suffices to show that we can choose the vectors b_1, b_2 of Proposition 5.1. Recalling that $K^3 = \mathrm{Id}$, the only domain that can be bounded by the union of images

$$(K + b_1)(\mu(H_1)) \cup (K^2 + b_2)(\mu(H_2)) \cup K^3(\mu(H_3)),$$

is a triangle with vertices at $b_1 = (a,0)$, $b_2 = (0,-a)$, $(0,0)$ for some $a \in \mathbb{R}$. To achieve this we need that after thickening all of the mismatches in the corresponding areas become equal. (They will all be equal to a.)

Consider thickening along the surfaces Σ_i, Σ_i' by amounts $\delta_i, \delta_i' > 0$ (not necessarily the same as the amounts by which we thinned!). Then the symplectic area A_i changes to $A_i + 2\pi\delta_{i+1}' + 2\pi k_i \delta_i$ and A_i' changes similarly. Letting $d_i = 2\pi(\delta_i + \delta_i')$ and equating the differences in the corresponding areas of the surfaces after thickening yields $\Delta A_1 - \Delta A_2 = k_1' d_1 + k_2 d_2 + d_3$ and $\Delta A_2 - \Delta A_3 = d_1 + k_2' d_2 + k_3 d_3$.

It only remains to check that we can thicken by appropriate amounts δ_i, δ_i' such that $\delta_i + \delta_i' = d_i$ and that the modified areas are all positive. To do this we can choose $d_i = \delta_i$ whenever $k_i \geqslant 0$ and $d_i = \delta_i'$ whenever $k_i < 0$. □

6. Applications

In this section we consider applications of the generalized symplectic sum to the problems of summing along immersed submanifolds and of defining a symplectic analog of algebraic desingularization. We consider only generalized symplectic sums in which X_0, the singular part of the gluing locus X, is a disjoint union of Lagrangian tori whose preimages are disjoint unions of three intersection tori. Then in the neighborhood of each connected component of X_0, the sum has the structure of a 3-fold sum: the neighborhood is symplectomorphic to three hinges glued together. The gluing construction on the complement of these neighborhoods is just the matching of trivial circle fibrations over punctured surfaces that have the same area and the same number of punctures.

The 3-fold sum itself may be a source for new examples of symplectic 4-manifolds since it is distinct from previously understood symplectic constructions when performed along a collection $\mathcal{C} \subset (M, \omega)$ such that $\mathcal{C}$ does not contain any surfaces that are proper transforms or exceptional spheres. When a 3-fold sum is preceded by blowing up a point, the same manifold can be constructed via a sequence of symplectic sums. The equivalence up to diffeomorphism of the two procedures was shown by Gompf [7] and the relations between the induced symplectic structures was examined by the author [22].

6.1. Immersed surfaces

We consider here the possibility of performing a symplectic surgery along a symplectic surface Σ_1 that has one orthogonal self-intersection, say at the point x_1. Then a neighborhood of $x_1 \subset \Sigma_1$ consists of a two disks $D_1, D_2' \subset \Sigma_1$ that intersect orthogonally at x_1.

Notice that we cannot perform a generalized symplectic sum along a collection that consists of a pair of surfaces, each of which has one orthogonal self-intersection. The reason once again is that the map $\overline{\varphi}: \overline{M} \to \widetilde{M}$ is an immersion. However, the connected boundary component $\partial_{\overline{\Sigma}_1}$ must be glued to a boundary component $\partial_{\overline{\Sigma}_1'}$ that contains two intersection tori. Therefore the corresponding surface Σ_1' must intersect two other surfaces at points x_2, x_3. If we choose these two surfaces to be symplectomorphic, say Σ_2, Σ_2' then we can apply a generalized symplectic sum which near X_0 looks like a 3-fold sum. Indeed, let $D_2 \supset x_2$, $D_1' \supset x_3$ be neighborhoods of x_2, x_3 in Σ_1' and let $D_3' \supset x_2$, $D_3 \supset x_3$ be their neighborhoods in Σ_2', Σ_2. See Fig. 10 in which both the configuration of surfaces and their correspondences in the 3-fold sum are shown.

To analyze the sum, we work with two model neighborhoods which are symplectically embedded in M via maps f_1, f_2. The model neighborhood of Σ_1 is built out of a strip S_1 which is a product neighborhood of $\Sigma_1 - (\overline{D}_1^- \cup \overline{D'}_2^-)$, and a hinge H_1 with $f_1(H_1) \supset D_1, D_2'$. Meanwhile the model neighborhood of $\Sigma_1' \cup \Sigma_2 \cup \Sigma_2'$ is a union of three strips S_1', S_2, S_2' joined by two hinges H_2, H_3 such that $f_2(H_2) \supset D_2, D_3'$ and $f_2(H_3) \supset D_3, D_1'$.

The Euler classes of the normal bundles of Σ_1, Σ_1' are determined by the twists introduced by the transition maps on collar neighborhoods of D_1, D_2' and D_2, D_1'. The transition maps for each pair of disks D_i, D_i' should have contributions that sum to -1, as for the 3-fold sum. The self-intersection number of Σ_1 is 2 greater than the Euler class of the normal bundle because of the double point. Therefore, the conditions on the self-intersection numbers for performing this generalized symplectic sum are $k_1 + k_1' = 0$ and $k_2 + k_2' = -1$.

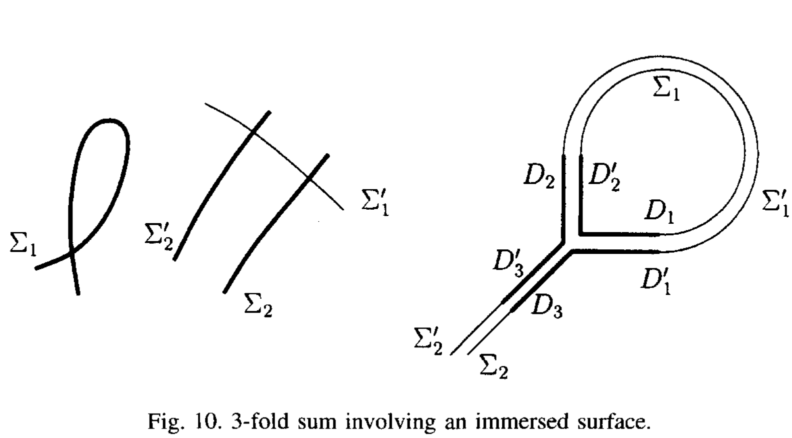

Fig. 10. 3-fold sum involving an immersed surface.

Example 6.1. Let the manifold (M, ω) be a disjoint union $M_1 \sqcup M_2$ in which M_1 is an elliptic surface with positive Euler characteristic and $M_2 = (\Sigma \times S^2)\#\overline{\mathbb{C}P^2}$ where Σ is a surface. The homology of M_2 is generated by $[\Sigma]$, $[S^2]$, $[E]$ where E is the exceptional sphere. In the class of the fiber of M_1, choose an immersed sphere S having one transverse double point (a fish-tail fiber). Also choose a section Σ, a proper transform $\widetilde{\Sigma}$ in the class $[\Sigma] - [E]$, and a fiber $F = S^2$ in M_2. (In the notation used to describe this type of sum, this means choosing $\Sigma_1 = S$, $\Sigma_1' = F$, $\Sigma_2 = \Sigma$, and $\Sigma_2' = \widetilde{\Sigma}$.) The self-intersections of these surfaces are such that we can try to sum M_1, M_2 along S, F, Σ, $\widetilde{\Sigma}$ using a generalized symplectic sum.

Notice however that no matter how we deform the symplectic structure of M_2, the area of $\widetilde{\Sigma}$ is less than the area of Σ. But we can glue thanks to Proposition 5.1, choosing vectors $b_1 = (0, a)$, $b_2 = (-a, 0)$ where $2\pi a = \Delta A_1 = \Delta A_2 = \Delta A_3$. In this context $\Delta A_1 - \Delta A_2 = 0$ is the difference in area between S and F. The difference ΔA_3 is determined by the size of the blow up.

If we choose Σ to be a sphere, the result of the construction is to recover the original manifold with a deformation of the symplectic structure in the neighborhood of the immersed sphere.

Our choice for M_2 in the above example is dictated by the following theorem:

Theorem 6.2 (Gromov [9], McDuff [17]). *Suppose (M, ω) is a closed symplectic 4-manifold which contains an embedded symplectic 2-sphere of nonnegative self-intersection. Then (M, ω) is symplectomorphic to a blow up at n points ($n \geqslant 0$) of either $\mathbb{C}P^2$ with its standard Kähler structure or a ruled symplectic 4-manifold.*

Indeed, to perform a generalized symplectic sum along an immersed sphere, the corresponding surface in the collection must be an embedded sphere. If the immersed sphere has self-intersection zero and we want the sum to have a gluing locus with only one singular component that looks locally like a 3-fold sum, then the embedded sphere must also have self-intersection zero. This forces M_2 to be a blowup of a ruled surface.

6.2. *Algebraic desingularization*

We return now to the question that motivated the definition of the 3-fold sum: whether one can define a symplectic surgery which replicates algebraic desingularization when performed along algebraic hypersurfaces. The simplest example of algebraic desingularization in dimension 4 is passage from the locus of $z_1 z_2 = 0$ in $\mathbb{C}P^3$, two intersecting planes, to the locus of $z_1 z_2 = \varepsilon z_0 z_3$, a quadric surface. This looks like a good candidate for the symplectic sum: considering the original planes abstractly, they each contain a line along which we need to perform the surgery. However, the sum of the self-intersection numbers of these lines is 2, so we first need to blow up two points on these lines. Up to a deformation of the symplectic forms, it does not matter on which of the lines one blows up [22]. (That this choice does not affect the diffeomorphism type of the result is due to Gompf [6].) The result of the gluing is $S^2 \times S^2$. Any symplectic form on $S^2 \times S^2$ is isotopic to a Kähler form [9,17], so up to isotopy we have realized the algebraic desingularization of two complex planes in $\mathbb{C}P^3$. Recall that two symplectic forms ω_0, ω_1 are *isotopic* if there is a path of symplectic forms ω_t such that for every $0 \leqslant t \leqslant 1$ the cohomology class of ω_t equals that of ω_0. Note that if two symplectic forms ω_0, ω_1 are isotopic, then by Moser's argument [21] there is a path of diffeomorphisms $\psi_t : M \to M$ with $\psi_0 = \mathrm{Id}$ such that $\psi_t^{-1}(\omega_t) = \omega_0$; in particular (M, ω_0) and (M, ω_1) are symplectomorphic.

The next example brings us to the 3-fold sum. Consider the locus of $z_1 z_2 z_3 = 0$ in $\mathbb{C}P^3$, namely three intersecting planes. Considering the three planes abstractly as $\{P_i\}_{i=1}^3$, in each plane P_i there is a pair of orthogonal lines L_i, L'_{i+1} such that $L_i \subset P_i$, $L'_i \subset P_{i-1}$ coincide when the planes are embedded in $\mathbb{C}P^3$.

This collection of lines is admissible for a 3-fold sum. In order to make the corresponding self-intersection numbers sum to -1, we need to blow up three times on each pair of lines. After doing so, does performing a 3-fold sum yield a cubic surface with a Kähler structure? The answer is yes, up to symplectomorphism.

To apply the 3-fold sum we let $M = M_1 \sqcup M_2 \sqcup M_3$ where each $M_i = \mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$ with three exceptional spheres $E_{i,1}$, $E_{i,2}$, $E_{i,3}$. Let $\mathcal{C} = \{\Sigma_i, \Sigma'_i\}_{i=1}^3$ where Σ_i, Σ'_i are the proper transforms of L_i, L'_i with $[\Sigma_i] = [L_i] - [E_{i,1}]$ and $[\Sigma'_i] = [L'_i] - [E_{i-1,2}] - [E_{i-1,3}]$. Choosing the areas of the exceptional curves so that Σ_i, Σ'_i have the same area, we can glue.

To relate the symplectic structure to a Kähler structure we appeal to a recent result of McDuff:

Theorem 6.3 (McDuff [18]). *Let (M, ω) be a symplectic 4-manifold which does not have simple* SW *type. Then any deformation between two cohomologous symplectic forms on M may be homotoped through deformations with fixed endpoints to an isotopy.*

A deformation between two symplectic forms is a path of symplectic forms connecting them.

Proposition 6.4. *Let M be a disjoint union of three complex planes P_i, $i = 1, 2, 3$, and consider a pair of lines L_i, L'_{i+1} in each. (As usual, the subscripts are understood* mod 3.*) Blowing up three points on each pair L_i, L'_i and taking the 3-fold sum along the collection of surfaces yields a manifold $(\widetilde{M}, \widetilde{\omega})$ symplectomorphic to $\mathbb{C}P^2 \# 6\overline{\mathbb{C}P^2}$ with a Kähler form.*

Proof. The manifold $\widetilde{M}$ has no 1-dimensional homology and has second Betti number $b_2 = 7$. By Lemma 6.5, $\widetilde{M}$ contains a symplectically embedded sphere of self-intersection zero. Therefore, by Theorem 6.2 the 3-fold sum $\widetilde{M}$ is symplectomorphic to $\mathbb{C}P^2 \# 6\overline{\mathbb{C}P^2}$ (and also $S^2 \times S^2 \# 5\overline{\mathbb{C}P^2}$). Since there is a Kähler form in each cohomology class on $\mathbb{C}P^2 \# 6\overline{\mathbb{C}P^2}$, $\widetilde{\omega}$ is cohomologous to a Kähler form. Because $\mathbb{C}P^2 \# 6\overline{\mathbb{C}P^2}$ is of simple SW type, the proposition follows from Theorem 6.3 and Moser's argument. □

Lemma 6.5. *The 3-fold sum $\widetilde{M}$ contains an embedded symplectic sphere of self-intersection zero.*

Proof. We begin by being more explicit in the construction of the $\widetilde{M}$. Let $E_{i,1}$, $E_{i,2}$, $E_{i,3}$ be exceptional spheres introduced by blowing up three points on each pair of corresponding lines $L_i \subset P_i$, $L_i' \subset P_{i-1}$. Let Σ_i, Σ_{i+1}' be the proper transforms of the lines L_i, L_{i+1}', with $\Sigma_i = L_i$ if L_i is not blown up. Choose symplectomorphisms $\psi_i : \Sigma_i \to \Sigma_i'$ such that they do not identify any two points that both lie on exceptional spheres. (Of course, we also require $\psi_i(\Sigma_i \cap \Sigma_{i+1}') = \Sigma_{i-1} \cap \Sigma_i'$.) Then there are six points $\{x_{i,1}, x_{i,2}, x_{i,3}\} \subset \Sigma_i$, $\{x_{i,1}', x_{i,2}', x_{i,3}'\} \subset \Sigma_i'$ such that $\psi_i(x_{i,j}) = x_{i,j}'$ and either $x_{i,j}$ or $x_{i,j}'$ belongs to $E_{i,j}$. Construct $\widetilde{M}$ by identifying $\partial_{\overline{\Sigma}_i}$, $\partial_{\overline{\Sigma}_i'}$ via diffeomorphisms that cover the symplectomorphisms ψ_i.

Let $\{y_{i,j}\}_{i=1}^3$ be points in the lines L_i such that if $x_{i,j} = \Sigma_i \cap E_i$ then $y_{i,j}$ is the point that was blown up to create E_i and otherwise $y_{i,j} = x_{i,j}$. Define points $y_{i,j}'$ similarly. Choose other points $x, x' \in \Sigma_1, \Sigma_1'$ such that $\psi_1(x) = x'$ and let y, y' be the corresponding points in L_1, L_1'. Let S be a symplectic sphere in P_3 through $y_{3,1}$ and y' such that S intersects L_3 and L_1' orthogonally. (For instance, choose S to be a perturbation of a line through those two points.) Let S' be a similar sphere in P_1 through the points $y_{2,2}'$, y. In $\widetilde{M}$ we find one of the following four symplectic spheres of self-intersection zero (depending upon which surfaces we chose to blow up): $\widetilde{S}\#\widetilde{S}'$, $\widetilde{S}\#(S'\#E_{2,2})$, $(S\#E_{3,1})\#\widetilde{S}'$, or $(S\#E_{3,1})\#(S'\#E_{2,2})$ where $\widetilde{S}$, $\widetilde{S}'$ are the proper transforms of S, S'. Fig. 11 shows the first possibility. □

Note that in both Figs. 11 and 12 the open circles represent the points $x_{i,j}$.

Because the manifold $\widetilde{M}$ is a cubic surface, it must contain 27 spheres of self-intersection -1. Using the same description of lines and exceptional spheres as in the above proof, it is not hard to see these 27 spheres.

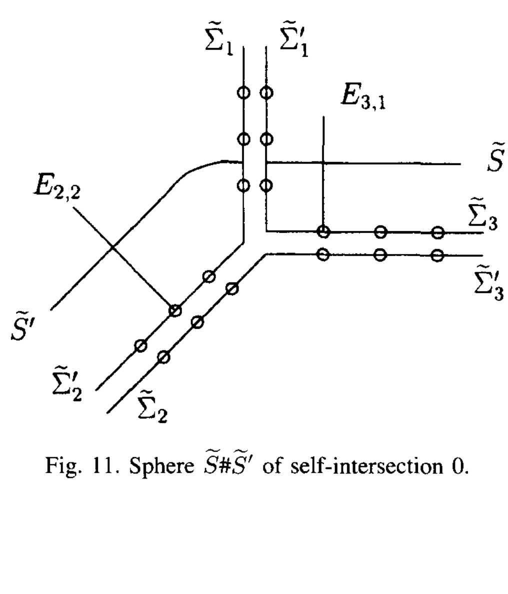

Fig. 11. Sphere $\widetilde{S}\#\widetilde{S}'$ of self-intersection 0.

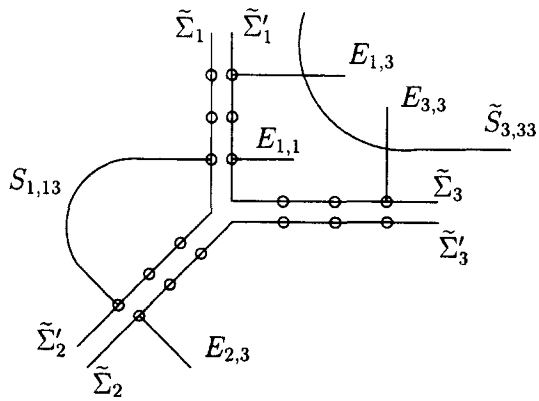

Fig. 12. Spheres $\widetilde{S}_{3,33}$ and $S_{1,13}\#E_{1,1}\#E_{3,3}$ of self-intersection -1.

Let $S_{i,jk}$, $1 \leqslant i,j,k \leqslant 3$, be the 27 lines in the three planes P_i that pass through the points $y_{i,j}$ and $y'_{i+1,k}$, perturbed so that their intersections with L_i, L'_{i+1} are orthogonal. Blowing up nine points as prescribed, each $S_{i,jk}$ will either be blown up or summed to an exceptional curve at the points $y'_{i,j}$, $y_{i+1,k}$, yielding -1 spheres. For instance, Fig. 12 shows the sphere yielded by the line $S_{3,33}$ if $y_{3,3}$ and $y'_{1,3}$ are blown up, and the sphere yielded by $S_{1,13}$ if $y'_{1,1}$ and $y_{2,3}$ are blown up.

Moving on to more planes, we could ask whether a symplectic surgery along d planes in $\mathbb{C}P^3$ yields a symplectic manifold that is symplectomorphic to a Kähler degree d algebraic surface. On each plane the intersection locus is a union of $(d-1)$ lines which intersect in $(d-1)(d-2)/2$ points. Each of these intersection points belongs to a set of three that are identified under the embedding in $\mathbb{C}P^3$ as the mutual intersection of three planes. The lines in the planes are admissible for a generalized symplectic sum in which the structure of the sum near any singular part of the gluing locus would be that of a 3-fold sum.

Conjecture 6.6. Given d planes and an arrangement of lines in each that corresponds to the intersection locus for d planes in $\mathbb{C}P^3$, blow up d points on each pair of lines and make the intersections orthogonal. Construct $(\widetilde{M}, \widetilde{\omega})$ by performing a generalized symplectic sum in which each connected component of X_0 is the image of three intersection tori. Then $(\widetilde{M}, \widetilde{\omega})$ is symplectomorphic to a degree d complex surface with a Kähler form.

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