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Nordhaus–Gaddum inequalities for the fractional and circular chromatic numbers

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A B S T R A C T

For a graph G on n vertices with chromatic number $\chi(G)$, the Nordhaus–Gaddum inequalities state that $\lceil 2\sqrt{n}\rceil \leq \chi(G)+\chi(\overline{G}) \leq n+1$, and $n \leq \chi(G)\cdot\chi(\overline{G}) \leq \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor$. Much analysis has been done to derive similar inequalities for other graph parameters, all of which are integer-valued. We determine here the optimal Nordhaus–Gaddum inequalities for the circular chromatic number and the fractional chromatic number, the first examples of Nordhaus–Gaddum inequalities where the graph parameters are *rational-valued*.

1. Introduction

In [19], Nordhaus and Gaddum determined bounds for the sum and product of the chromatic numbers of a graph and its complement.

Theorem 1.1 (*[19]*). *Let G be a graph on n vertices. Then,*

$$\lceil 2\sqrt{n}\rceil \leq \chi(G)+\chi(\overline{G}) \leq n+1,$$
$$n \leq \chi(G)\cdot\chi(\overline{G}) \leq \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor.$$

Nordhaus and Gaddum also showed that these bounds are optimal by finding examples of graphs for which equality is reached. Since then, various papers have been published on determining optimal bounds for $\pi(G)+\pi(\overline{G})$ and $\pi(G)\cdot\pi(\overline{G})$, for other graph parameters π. In the literature, these results are known as *Nordhaus–Gaddum inequalities*.

We say that the function $f(n)$ is an *optimal lower bound* for $\pi(G)+\pi(\overline{G})$ if for every integer n, $f(n) \leq \pi(G)+\pi(\overline{G})$ for any graph G on n vertices, and the value $f(n)$ cannot be replaced by any larger real number (an *optimal upper bound* is defined analogously). Since there are only finitely many graphs on n vertices, the optimal bound $f(n)$ is simply the minimum value of $\pi(G)+\pi(\overline{G})$ over all possible graphs G on n vertices. Thus, for every n there must be at least one graph G with n vertices for which equality is attained. As a specific example, $f(n)=\lceil 2\sqrt{n}\rceil$ is *the* optimal lower bound for $\chi(G)+\chi(\overline{G})$, as shown in [19]. In some papers, it is written that $2\sqrt{n} \leq \pi(G)+\pi(\overline{G})$ is the optimal lower bound; by our definition, that will not be the case.

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Nordhaus–Gaddum inequalities have been established for numerous other graph parameters, such as the independence and edge-independence number [3,8], list-colouring number [7,10], diameter, girth, circumference, and edge-covering number [25], connectivity and edge-connectivity number [6], achromatic and pseudoachromatic number [1,26], and arboricity [18,23]. In some cases, bounds are found, yet it is unknown if they are optimal. A survey of known theorems (pre-1971) is given in [2]. As an example, two such results are as follows:

Let $\alpha_1(G)$ be the edge-independence number of G. Then, it is shown [3] that

$$\left\lfloor \frac{n}{2} \right\rfloor \leq \alpha_1(G) + \alpha_1(\overline{G}) \leq 2 \cdot \left\lfloor \frac{n}{2} \right\rfloor,$$
$$0 \leq \alpha_1(G) \cdot \alpha_1(\overline{G}) \leq \left\lfloor \frac{n}{2} \right\rfloor^2.$$

Let $\beta_1(G)$ be the edge-covering number of G. Then, it is shown [25] that

$$2 \cdot \left\lceil \frac{n}{2} \right\rceil \leq \beta_1(G) + \beta_1(\overline{G}) \leq 2n - 2 - \left\lfloor \frac{n}{2} \right\rfloor,$$
$$\left\lfloor \frac{n}{2} \right\rfloor^2 \leq \beta_1(G) \cdot \beta_1(\overline{G}) \leq \frac{n(n-1)}{2}.$$

In all of the known examples, the parameter $\pi(G)$ is *integer*-valued. In this paper, we provide the first instances of Nordhaus–Gaddum inequalities where the parameters are *rational*-valued, and our optimal bounds are non-integers. We will determine the optimal bounds for $\pi(G)+\pi(\overline{G})$ and $\pi(G)\cdot\pi(\overline{G})$, when $\pi(G)$ is the *fractional chromatic number* of G (denoted by $\chi_f(G)$), and when $\pi(G)$ is the *circular chromatic number* of G (denoted by $\chi_c(G)$). We will establish these Nordhaus–Gaddum inequalities using a generalization of the well-known Ramsey function, motivated by a technique in [3].

2. Definitions

For any graph G, the *clique number* $\omega(G)$ is the cardinality of the largest clique in G, and the *independence number* $\alpha(G)$ is the cardinality of the largest independent set in G. The *chromatic number* of a graph, $\chi(G)$, is the smallest size of a cover of the vertices of G by independent sets. We can alternatively define $\chi(G)$ using an integer program (IP) [5]. Let M denote the vertex-independent set incidence matrix of G. The rows are indexed by the vertices $\{v_1, v_2, \dots, v_n\}$, and the columns are indexed by the independent subsets of the vertices, $\{I_1, I_2, \dots, I_m\}$. The (i, j) entry of M is 1 when $v_i \in I_j$, and is 0 otherwise. Then $\chi(G) = \min \mathbf{1}^{\mathrm{T}}\mathbf{x}$, where $M\mathbf{x} \geq \mathbf{1}$, $\mathbf{x} \geq \mathbf{0}$, and $\mathbf{x} \in \mathbb{Z}^m$ (where $\mathbf{1}$ denotes the m by 1 vector of all 1's).

Definition 2.1 (*[21]*). Let M be the vertex-independent set incidence matrix of G. Then, the *fractional chromatic number* $\chi_f(G)$ is the relaxation of the integer program for $\chi(G)$ into a linear program:

$$\chi_f(G) = \min \mathbf{1}^{\mathrm{T}}\mathbf{x}, \quad \text{where } M\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0}, \text{ and } \mathbf{x} \in \mathbb{R}^m.$$

Note that by definition, $\chi_f(G) \leq \chi(G)$, for all graphs G. By taking the integer program of a graph parameter and relaxing the IP into a linear program, we may define a corresponding *fractional* analogue (see [21]). This enables us to define parameters such as the fractional clique number, fractional domination number, fractional matching number, among many others. It is known [21] that each of these fractional parameters takes on only rational values, hence the name. Much recent research has been conducted on the properties of these fractional graph parameters (for more information on the uses and applications of fractional graph theory, we refer the reader to [21]).

The following theorem will be important in our analysis.

Theorem 2.2 (*[14]*). *For any vertex-transitive graph G, $\chi_f(G) = \frac{|V(G)|}{\alpha(G)}$.*

Now we define the *circular chromatic number* $\chi_c(G)$.

Definition 2.3 (*[22,27]*). Let k and d be positive integers with $k \geq 2d$. A (k, d)-*colouring* of a graph $G = (V, E)$ on n vertices is a mapping $C : V \to \{0, 1, \dots, k-1\}$ such that $d \leq |C(x) - C(y)| \leq k - d$ for any $xy \in E(G)$. Then, the *circular chromatic number* $\chi_c(G)$ is the infimum of $\frac{k}{d}$ for which there exists a (k, d)-colouring of G.

Note that $\chi(G)$ is just the smallest k for which there exists a $(k, 1)$-colouring of G. So $\chi_c(G)$ is a generalization of $\chi(G)$, where $\chi_c(G) \leq \chi(G)$ for all G. The circular chromatic number is sometimes referred to as the *star chromatic number* [22,27]. An extensive survey of important results and applications of circular chromatic numbers is found in [28].

The following theorems are well known and straightforward to show.

Theorem 2.4 (*[22]*). *For any graph G, $\chi(G) = \lceil \chi_c(G) \rceil$.*

Theorem 2.5 (*[17]*). *Let G be any graph on n vertices. Then,*

$$\max\left\{\omega(G), \frac{n}{\alpha(G)}\right\} \le \chi_f(G) \le \chi_c(G) \le \chi(G).$$

In other words, knowing $\chi_c(G)$ immediately determines the chromatic number (though not vice versa). Also, if $\omega(G) = \chi(G) = k$ for some integer k, then this implies that $\chi_f(G) = \chi_c(G) = k$ as well.

If $\chi_f(G) = \chi_c(G)$, we say that G is *star extremal*. The notion of star extremality in graphs was first introduced in the study of the chromatic number and the circular chromatic number of the lexicographic product of graphs [11]. At the conclusion of this paper, we verify the optimality of each of our bounds by determining an extremal graph attaining the desired value. In the most difficult of our cases, the optimality will be established by finding a star extremal *circulant* graph G.

3. The main theorem

We now state the main theorem of the paper, which determines the optimal Nordhaus–Gaddum inequalities for both $\chi_f(G)$ and $\chi_c(G)$. We note that this question was partially answered by Wang and Zhou [24], who proved that $\chi_c(G)+\chi_c(\overline{G}) \le n+1$ and $\chi_c(G)\cdot\chi_c(\overline{G}) \ge n$. We now provide all of the correct optimal bounds.

To simplify the proof, we split the main result into two separate theorems; first we establish our desired bounds, and then we prove the optimality of these bounds by constructing for each n, a graph G of order n for which equality is attained.

Theorem 3.1. *Let G be a graph of order n. Set*

$$t(n) = \min\left\{\lceil 2\sqrt{n}\rceil, \frac{n}{\lfloor\sqrt{n}\rfloor} + \frac{n}{\left\lfloor\sqrt{n+\sqrt{n}}\right\rfloor}\right\}.$$

Then, for each $n \ge 1$,

$$t(n) \le \chi_f(G) + \chi_f(\overline{G}) \le \chi_c(G) + \chi_c(\overline{G}) \le n+1,$$
$$n \le \chi_f(G)\cdot\chi_f(\overline{G}) \le \chi_c(G)\cdot\chi_c(\overline{G}) \le \left\lfloor\left(\frac{n+1}{2}\right)^2\right\rfloor.$$

Theorem 3.2. *All bounds given in the statement of Theorem 3.1 are optimal.*

Note the similarity of Theorem 3.1 to Theorem 1.1: in three of the four cases, the bounds are identical. However, the lower bound for $\chi_f(G) + \chi_f(\overline{G})$ is different. For example, if $n = 7$, then Theorem 3.1 implies that $\chi_f(G) + \chi_f(\overline{G}) \ge \min\{6, \frac{35}{6}\} = \frac{35}{6}$, whereas Theorem 1.1 shows that $\chi(G) + \chi(\overline{G}) \ge \lceil 2\sqrt{7}\rceil = 6$.

In Section 4, we introduce the π*-Ramsey function*, a generalization of the well-known Ramsey function. We prove that the optimal lower bound on $\chi_f(G) + \chi_f(\overline{G})$ can be represented in terms of this π-Ramsey function. In Section 5, we use this result to prove Theorems 3.1 and 3.2.

4. The π-Ramsey function

Given positive integers $a_1, a_2, \ldots, a_k$, the *Ramsey number* $r(a_1, a_2, \ldots, a_k)$ is the smallest n such that in any k-edge decomposition $G_1 \oplus G_2 \oplus \cdots \oplus G_k$ of K_n, $\omega(G_i) \ge a_i$ for at least one index i. Ramsey's celebrated theorem [20] states that $r(a_1, a_2, \ldots, a_k)$ is well defined for each choice of the a_i's. A comprehensive survey of important results and applications of Ramsey theory appears in [12].

To prove Theorem 3.1, we introduce a generalized class of Ramsey numbers, which we will call π-Ramsey functions. This definition first appeared in the literature as f-Ramsey functions in [4], and was developed further in [15].

Definition 4.1 (*[4]*). Let $(a_1, a_2, \ldots, a_k)$ be a k-tuple of positive real numbers. Then for any parameter π, the ***π**-Ramsey function* $r_\pi(a_1, a_2, \ldots, a_k)$ is the smallest integer n such that in any k-edge decomposition $G_1 \oplus G_2 \oplus \cdots \oplus G_k$ of K_n, $\pi(G_i) \ge a_i$ for at least one index i.

The ω-Ramsey function is just the standard Ramsey function $r(a_1, a_2, \ldots, a_k)$. We note that for some graph parameters π, the π-Ramsey function is not well defined. For example, if we let $\pi(G)$ be the number of components of G, then $r_\pi(2, 2)$ does not exist. However, by a result in [15], if $\lim_{n\to\infty}\pi(K_n) = \infty$ and $\pi(H) \le \pi(G)$ whenever $H \subseteq G$, then $r_\pi(a_1, a_2, \ldots, a_k)$ is well defined for all G. Note that $r_\pi(a_1, a_2, \ldots, a_k)$ is a non-decreasing function in each coordinate.

The analysis of π-Ramsey functions is motivated by the following theorem, which provides the optimal lower bound for *any* generalized Nordhaus–Gaddum inequality. Note that in the following theorem, $\pi(G)$ is not restricted to be an integer; in fact, $\pi(G)$ can be any positive real number.

Theorem 4.2. *Let π be a graph parameter, with $\lim_{n\to\infty}\pi(K_n)=\infty$ and $\pi(H)\le\pi(G)$ whenever $H\subseteq G$. Then, for any k-edge decomposition $G_1\oplus G_2\oplus\cdots\oplus G_k$ of the complete graph K_n,*

$$\sum_{i=1}^{k}\pi(G_i)\ge\inf\left\{\sum_{i=1}^{k}a_i \mid \text{ each } a_i>0 \text{ and } r_\pi(a_1+\varepsilon,\ldots,a_k+\varepsilon)>n\;\forall\varepsilon>0\right\}.$$

Moreover, this lower bound is optimal.

Proof. Let S be the set of real numbers t for which there exists a k-tuple of positive real numbers $(a_1,a_2,\ldots,a_k)$ such that $t=\sum_{i=1}^{k}a_i$ and $r_\pi(a_1+\varepsilon,a_2+\varepsilon,\ldots,a_k+\varepsilon)>n$, for all $\varepsilon>0$.

First we justify that S is non-empty. Let r be the smallest number for which $\pi(H)\le r$ for every subgraph $H\subseteq K_n$. Then for any k-edge decomposition $G_1\oplus G_2\oplus\cdots\oplus G_k$ of K_n, we must have $\pi(G_i)\le r$. Then $(r,r,\ldots,r)$ is a k-tuple satisfying the above conditions, and so $kr\in S$.

Thus S is non-empty and it must have a finite-valued infimum (since $\inf S\ge 0$). In fact, it is straightforward to see that $S=(m,\infty)$ or $S=[m,\infty)$, where $m=\inf S$. We wish to prove that $\sum_{i=1}^{k}\pi(G_i)\ge m$.

On the contrary, suppose that there exists a k-edge decomposition $G_1\oplus G_2\oplus\cdots\oplus G_k$ of K_n for which $\sum_{i=1}^{k}\pi(G_i)=m'<m$. Let $\pi(G_i)=b_i$ for each i. Now consider the π-Ramsey number $r_\pi(b_1+\varepsilon,b_2+\varepsilon,\ldots,b_k+\varepsilon)$.

If there exists an $\varepsilon>0$ such that $r_\pi(b_1+\varepsilon,b_2+\varepsilon,\ldots,b_k+\varepsilon)\le n$, then by definition, there must be an index i such that $b_i=\pi(G_i)\ge b_i+\varepsilon$, a contradiction. Therefore, we must have $r_\pi(b_1+\varepsilon,b_2+\varepsilon,\ldots,b_k+\varepsilon)>n$ for all $\varepsilon>0$, so $\sum_{i=1}^{k}b_i=m'<m\le\sum_{i=1}^{k}a_i$, contradicting the minimality of m. Hence, no such m' exists, and we conclude that $\sum_{i=1}^{k}\pi(G_i)\ge m$ for all k-edge decompositions $G_1\oplus G_2\oplus\cdots\oplus G_k$ of K_n.

By the definition of the π-Ramsey function, $r_\pi(a_1+\varepsilon,a_2+\varepsilon,\ldots,a_k+\varepsilon)>n$ implies the existence of a k-edge decomposition $G_1\oplus G_2\oplus\cdots\oplus G_k$ of K_n with $\pi(G_i)<a_i+\varepsilon$ for each i. So in this decomposition, $\sum_{i=1}^{k}\pi(G_i)<m+k\varepsilon$. Since such a decomposition exists for any $\varepsilon>0$, we conclude that $\sum_{i=1}^{k}\pi(G_i)$ can be made as close to m as we wish. $\square$

By determining explicit formulas for π-Ramsey functions, we can determine optimal lower bounds for various generalized Nordhaus–Gaddum inequalities. For some parameters (such as the clique number $\omega(G)$), it seems intractable to determine values for $r_\pi(a_1,a_2,\ldots,a_k)$, even for the case $k=2$. That is why the optimal lower bound for the Nordhaus–Gaddum inequality $\alpha(G)+\alpha(\overline{G})=\omega(G)+\omega(\overline{G})$ is a formula in terms of Ramsey functions [3].

In [16], the *fractional Ramsey function* is introduced, which is just the π-Ramsey function for the parameter $\chi_f(G)$ (which is equivalent to the fractional clique number $\omega_f(G)$, as explained in [21]). By our notation, we will write this function as $r_{\chi_f}(a_1,a_2,\ldots,a_k)$. By Theorem 2.5, $\chi_f(G)\ge\omega(G)$, and so $r_{\chi_f}(a_1,a_2,\ldots,a_k)$ is well defined, since it is bounded by $r(a_1,a_2,\ldots,a_k)$. The following theorem describes an explicit formula for the fractional Ramsey function in the two-variable case.

Theorem 4.3 (*[14,16]*). *Let $r_{\chi_f}(x,y)$ be the χ_f-Ramsey function for two variables. Let $x,y\ge 2$ be any real numbers. Then,*

$$r_{\chi_f}(x,y)=\min\{\lceil(\lceil x\rceil-1)y\rceil,\lceil(\lceil y\rceil-1)x\rceil\}.$$

Knowing this formula for $r_{\chi_f}(x,y)$ is the key to proving Theorem 3.1, since Theorem 4.2 provides the optimal lower bound for $\chi_f(G)+\chi_f(\overline{G})$ in terms of this Ramsey function. We have the following corollary.

Corollary 4.4. *Let G be a graph on n vertices. Then*

$$\chi_f(G)+\chi_f(\overline{G})\ge\inf\left\{a_1+a_2 \mid r_{\chi_f}(a_1+\varepsilon,a_2+\varepsilon)>n\;\forall\varepsilon>0\right\}.$$

We now find the minimum value of a_1+a_2 for which $r_{\chi_f}(a_1+\varepsilon,a_2+\varepsilon)>n$ for all $\varepsilon>0$. This will establish the optimal lower bound for $\chi_f(G)+\chi_f(\overline{G})$, which in turn will give us the optimal lower bound for $\chi_c(G)+\chi_c(\overline{G})$.

5. Proof of the main theorem

Before we proceed with the proof of Theorem 3.1, we require a definition and several lemmas. To simplify the notation, we introduce the function $t(n)$.

Definition 5.1. For each integer $n\ge 1$, set

$$t(n)=\min\left\{\lceil 2\sqrt{n}\rceil,\ \frac{n}{\lfloor\sqrt{n}\rfloor}+\frac{n}{\left\lfloor\sqrt{n+\sqrt{n}}\right\rfloor}\right\}.$$

The following three lemmas will all include this definition of $t(n)$. Our main theorem, Theorem 3.1, will follow quickly from these three results.

Lemma 5.2. *Let $p = \lfloor\sqrt{n}\rfloor$. Then $n = p^2 + q$ for some $0 \leq q \leq 2p$. Then, $t(n)$ can be represented as the following piecewise function.*

$$t(n) = \begin{cases} \dfrac{n}{\lfloor\sqrt{n}\rfloor} + \dfrac{n}{\left\lfloor\sqrt{n+\sqrt{n}}\right\rfloor} = \dfrac{2n}{p} & \text{if } p^2 \leq n < p^2 + \dfrac{p}{2} \\ \lceil 2\sqrt{n}\rceil = 2p+1 & \text{if } p^2 + \dfrac{p}{2} \leq n \leq p^2 + p \\ \dfrac{n}{\lfloor\sqrt{n}\rfloor} + \dfrac{n}{\left\lfloor\sqrt{n+\sqrt{n}}\right\rfloor} = \dfrac{n(2p+1)}{p^2+p} & \text{if } p^2 + p + 1 \leq n \leq p^2 + \dfrac{3p}{2} \\ \lceil 2\sqrt{n}\rceil = 2p+2 & \text{if } p^2 + \dfrac{3p+1}{2} \leq n \leq p^2 + 2p. \end{cases}$$

Proof. Since $p^2 + p < (p + \frac{1}{2})^2 < p^2 + p + 1$, we can readily verify the following identities.

$$\lceil 2\sqrt{n}\rceil = \begin{cases} 2p & \text{if } n = p^2 \\ 2p+1 & \text{if } p^2 < n \leq p^2 + p \\ 2p+2 & \text{if } p^2 + p + 1 \leq n \leq p^2 + 2p \end{cases}$$

$$\frac{n}{\lfloor\sqrt{n}\rfloor} + \frac{n}{\lfloor\sqrt{n+\sqrt{n}}\rfloor} = \begin{cases} \dfrac{n}{p} + \dfrac{n}{p} & \text{if } p^2 \leq n \leq p^2 + p \\ \dfrac{n}{p} + \dfrac{n}{p+1} & \text{if } p^2 + p + 1 \leq n \leq p^2 + 2p. \end{cases}$$

If $n \leq p^2 + \frac{3p}{2}$, then $n < p^2 + \frac{3p}{2} + \frac{p}{4p+2} = \frac{(4p^3+2p^2)+3p(2p+1)+p}{4p+2} = \frac{2p^3+4p^2+2p}{2p+1} = \frac{(2p+2)(p^2+p)}{2p+1}$, which implies that $\frac{n}{p} + \frac{n}{p+1} = \frac{n(2p+1)}{p^2+p} < 2p+2$. Similarly, if $n \geq p^2 + \frac{3p+1}{2}$, then $n > p^2 + \frac{3p}{2} + \frac{p}{4p+2} = \frac{(2p+2)(p^2+p)}{2p+1}$, which implies that $\frac{n}{p} + \frac{n}{p+1} = \frac{n(2p+1)}{p^2+p} > 2p+2$. We will use these inequalities in our case analysis below.

- If $n = p^2$, then $\lceil 2\sqrt{n}\rceil = \frac{n}{\lfloor\sqrt{n}\rfloor} + \frac{n}{\lfloor\sqrt{n+\sqrt{n}}\rfloor}$, and so $t(n) = 2p = \frac{2n}{p}$.
- If $p^2 < n < p^2 + \frac{p}{2}$, then $2p+1 > \frac{2n}{p}$, and so $t(n) = \frac{n}{\lfloor\sqrt{n}\rfloor} + \frac{n}{\lfloor\sqrt{n+\sqrt{n}}\rfloor} = \frac{2n}{p}$.
- If $p^2 + \frac{p}{2} \leq n \leq p^2 + p$, then $2p+1 \leq \frac{2n}{p}$, and so $t(n) = \lceil 2\sqrt{n}\rceil = 2p+1$.
- If $p^2 + p + 1 \leq n \leq p^2 + \frac{3p}{2}$, then $2p+2 > \frac{n}{p} + \frac{n}{p+1}$, and so $t(n) = \frac{n}{\lfloor\sqrt{n}\rfloor} + \frac{n}{\lfloor\sqrt{n+\sqrt{n}}\rfloor} = \frac{n}{p} + \frac{n}{p+1} = \frac{n(2p+1)}{p^2+p}$.
- If $p^2 + \frac{3p+1}{2} \leq n \leq p^2 + 2p$, then $2p+2 < \frac{n}{p} + \frac{n}{p+1}$, and so $t(n) = \lceil 2\sqrt{n}\rceil = 2p+2$.

This completes the proof. □

By inspection, we can verify that $2\sqrt{n} \leq t(n) < 2\sqrt{n}+1$ in each of the above cases. Therefore, $\lceil 2\sqrt{n}\rceil$ and $\frac{n}{\lfloor\sqrt{n}\rfloor} + \frac{n}{\lfloor\sqrt{n+\sqrt{n}}\rfloor}$ are "close" in the sense that for any n, these two expressions differ by at most 1.

Lemma 5.3. *Let n be a positive integer. For each integer $1 \leq k \leq n$, define $f_1(k) = k + \lceil \frac{n}{k}\rceil$ and $f_2(k) = \frac{n}{k} + \frac{n}{\lfloor\frac{n}{k}\rfloor}$. Then, $\min\{f_1(k), f_2(k)\} \geq t(n)$, for all k. Moreover, this is the optimal lower bound, i.e., there exists at least one index $1 \leq k \leq n$ with $\min\{f_1(k), f_2(k)\} = t(n)$.*

Proof. Fix n. We first prove that $f_1(k) \geq \lceil 2\sqrt{n}\rceil$ for all $1 \leq k \leq n$, which implies by definition that $f_1(k) \geq t(n)$ for each k. Let $n = p^2 + q$, where $0 \leq q \leq 2p$.

If $q = 0$, then $f_1(k) \geq \lceil 2\sqrt{n}\rceil = 2p$ is equivalent to $k + \lceil\frac{n}{k}\rceil \geq 2p$, or $\lceil\frac{n}{k}\rceil \geq 2p - k$. Since $(p-k)^2 \geq 0$, $n = p^2 \geq 2pk - k^2 = k(2p-k)$. Therefore, $\lceil\frac{n}{k}\rceil \geq \frac{n}{k} \geq 2p - k$.

If $1 \leq q \leq p$, then $f_1(k) \geq \lceil 2\sqrt{n}\rceil = 2p+1$ is equivalent to $\lceil\frac{n}{k}\rceil \geq 2p-k+1$. Since $(p-k)^2 \geq 0$, $n > p^2 \geq 2pk-k^2 = k(2p-k)$, and so $\frac{n}{k} > 2p-k$. It follows that $\lceil\frac{n}{k}\rceil \geq 2p-k+1$.

If $p+1 \leq q \leq 2p$, then $f_1(k) \geq \lceil 2\sqrt{n}\rceil = 2p+2$ is equivalent to $\lceil\frac{n}{k}\rceil \geq 2p-k+2$. Since p and k are integers, $(p-k)(p-k+1) \geq 0$, and so $p^2 + p \geq 2pk - k^2 + k = k(2p-k+1)$, from which we get $\frac{n}{k} > \frac{p^2+p}{k} \geq 2p-k+1$. It follows that $\lceil\frac{n}{k}\rceil \geq 2p-k+2$.

Note that in all three cases, equality occurs if $k = p = \lfloor\sqrt{n}\rfloor$. Therefore, we have shown that $f_1(k) \geq \lceil 2\sqrt{n}\rceil$ for all $1 \leq k \leq n$, with at least one value of k for which equality occurs.

Now let us prove that $f_2(k) \geq t(n)$ for each k. This will conclude the proof of the lemma. We split our analysis into the four cases described in Lemma 5.2, which conveniently allows us to apply the formula for $t(n)$.

Case 1: $p^2 \leq n < p^2 + \frac{p}{2}$.

The desired inequality $f_2(k) \geq \frac{2n}{p}$ is equivalent to $\frac{n}{k} + \frac{n}{\lfloor\frac{n}{k}\rfloor} \geq \frac{2n}{p}$, which simplifies to $\lfloor\frac{n}{k}\rfloor(\frac{2k}{p} - 1) \leq k$. If $2k - p \leq 0$, the result is trivial, so assume otherwise. We divide both sides by $2k - p > 0$, and so it suffices to prove that $\lfloor\frac{n}{k}\rfloor \leq \frac{kp}{2k-p}$. We consider two further subcases: when $p^2 + \frac{p}{2} \geq (2p+1)k - k^2$, and when $p^2 + \frac{p}{2} < (2p+1)k - k^2$. In fact, for each of our four cases, we will separate our analysis into two subcases.

First suppose that $p^2 + \frac{p}{2} \geq (2p+1)k - k^2$. Then

$$(2p+1)k - \left(p^2 + \frac{p}{2}\right) \leq k^2$$
$$\left(p + \frac{1}{2}\right)(2k - p) \leq k^2$$
$$\left(p^2 + \frac{p}{2}\right)(2k - p) \leq k^2 p$$
$$\frac{p^2 + \frac{p}{2}}{k} \leq \frac{kp}{2k - p}.$$

Therefore, $\left\lfloor \frac{n}{k} \right\rfloor \leq \frac{n}{k} < \frac{p^2+\frac{p}{2}}{k} \leq \frac{kp}{2k-p}$.

Now suppose that $p^2+\frac{p}{2} < (2p+1)k-k^2$. Then $\lfloor \frac{n}{k} \rfloor \leq \lfloor \frac{p^2+\frac{p}{2}}{k} \rfloor \leq 2p-k$. Since $2(k-p)^2 \geq 0$, we have $4pk-2k^2-2p^2+kp \leq kp$, which is equivalent to $2p - k \leq \frac{kp}{2k-p}$. Therefore, we have $\lfloor \frac{n}{k} \rfloor \leq 2p - k \leq \frac{kp}{2k-p}$, with equality iff $k = p = \lfloor \sqrt{n} \rfloor$.

Case 2: $p^2 + \frac{p}{2} \leq n \leq p^2 + p$.

The desired inequality $f_2(k) \geq 2p + 1$ is equivalent to $\lfloor \frac{n}{k} \rfloor (\frac{(2p+1)k}{n} - 1) \leq k$. If $(2p + 1)k \leq n$, the inequality is trivial, so assume that $\frac{(2p+1)k}{n} - 1 > 0$. Then, it suffices to prove that $\lfloor \frac{n}{k} \rfloor \leq \frac{kn}{(2p+1)k-n}$.

First suppose that $n \geq (2p + 1)k - k^2$. Then

$$(2p+1)k - n \leq k^2$$
$$n((2p+1)k - n) \leq k^2 n$$
$$\frac{n}{k} \leq \frac{kn}{(2p+1)k - n}.$$

Therefore, $\left\lfloor \frac{n}{k} \right\rfloor \leq \frac{n}{k} \leq \frac{kn}{(2p+1)k-n}$.

Now suppose that $n < (2p + 1)k - k^2$. Then $\lfloor \frac{n}{k} \rfloor \leq 2p - k$. From $n \geq p^2 + \frac{p}{2}$ and $p^2 \geq k(2p - k)$, we have

$$n \geq p^2 + \frac{p}{2}$$
$$n \geq \frac{p^2(2p+1)}{2p}$$
$$n \geq \frac{k(2p-k)(2p+1)}{2p}$$
$$2pn \geq k(2p-k)(2p+1)$$
$$(2p-k)(2p+1)k - 2pn + kn \leq kn$$
$$(2p-k)((2p+1)k - n) \leq kn$$
$$2p - k \leq \frac{kn}{(2p+1)k - n}.$$

Therefore, $\left\lfloor \frac{n}{k} \right\rfloor \leq 2p - k \leq \frac{kn}{(2p+1)k-n}$.

Case 3: $p^2 + p < n \leq p^2 + \frac{3p}{2}$.

The desired inequality $f_2(k) \geq \frac{n(2p+1)}{p^2+p}$ is equivalent to $\lfloor \frac{n}{k} \rfloor (\frac{(2p+1)k}{p^2+p} - 1) \leq k$.

If $(2p + 1)k \leq p^2 + p$, the inequality is trivial, so assume otherwise. Then, it suffices to prove that $\lfloor \frac{n}{k} \rfloor \leq \frac{k(p^2+p)}{(2p+1)k-(p^2+p)}$.

Since $(2p + 3)(2p + 1) = (2p + 2)^2 - 1$, we have $\frac{(2p+3)(2p+1)}{2p+2} < 2p + 2$, or $2p + 2 > \frac{(p+\frac{3}{2})(2p+1)}{p+1}$.

First suppose that $p^2 + \frac{3p}{2} \geq (2p + 2)k - k^2$. Then

$$p^2 + \frac{3p}{2} \geq (2p+2)k - k^2$$
$$p^2 + \frac{3p}{2} > \frac{\left(p + \frac{3}{2}\right)(2p+1)}{p+1}k - k^2$$
$$\left(p^2 + \frac{3p}{2}\right)(p+1) > \left(p + \frac{3}{2}\right)(2p+1)k - k^2(p+1)$$

$$\left(p+\frac{3}{2}\right)((2p+1)k-(p^2+p)) < k^2(p+1)$$
$$\left(p^2+\frac{3p}{2}\right)((2p+1)k-(p^2+p)) < k^2(p^2+p)$$
$$\frac{p^2+\frac{3p}{2}}{k} < \frac{k(p^2+p)}{(2p+1)k-(p^2+p)}.$$

Therefore, $\left\lfloor \frac{n}{k} \right\rfloor \le \frac{n}{k} \le \frac{p^2+\frac{3p}{2}}{k} < \frac{k(p^2+p)}{(2p+1)k-(p^2+p)}$.

Now suppose that $p^2+\frac{3p}{2} < (2p+2)k-k^2$. Then $\lfloor \frac{n}{k} \rfloor \le \lfloor \frac{p^2+\frac{3p}{2}}{k} \rfloor \le 2p-k+1$.

Since k and p are both integers, $(k-p)(k-p-1) \ge 0$, with equality iff $k=p=\lfloor\sqrt{n}\rfloor$ or when $k=p+1=\lfloor\sqrt{n}\rfloor+1$. Thus, we have

$$(k-p)(k-p-1) \ge 0$$
$$(k-p)^2-(k-p) \ge 0$$
$$2pk+k-p^2-p \le k^2$$
$$(2p+1)k-(p^2+p) \le k^2$$
$$(2p+1)((2p+1)k-(p^2+p)) \le k^2(2p+1)$$
$$(2p+1-k)((2p+1)k-(p^2+p)) \le k(p^2+p)$$
$$2p-k+1 \le \frac{k(p^2+p)}{(2p+1)k-(p^2+p)}.$$

Therefore, $\left\lfloor \frac{n}{k} \right\rfloor \le 2p-k+1 \le \frac{k(p^2+p)}{(2p+1)k-(p^2+p)}$, with equality holding iff $k=\lfloor\sqrt{n}\rfloor$ or $k=\lfloor\sqrt{n}\rfloor+1$.

Case 4: $p^2+\frac{3p+1}{2} \le n \le p^2+2p$.

The desired inequality $f_2(k) \ge 2p+2$ is equivalent to $\lfloor \frac{n}{k} \rfloor (\frac{(2p+2)k}{n}-1) \le k$. If $(2p+2)k \le n$, the inequality is trivial, so assume otherwise. Then, it suffices to prove that $\lfloor \frac{n}{k} \rfloor \le \frac{kn}{(2p+2)k-n}$.

First suppose that $n \ge (2p+2)k-k^2$. Then

$$(2p+2)k-k^2 \le n$$
$$(2p+2)kn-k^2n \le n^2$$
$$(2p+2)kn-n^2 \le k^2n$$
$$\frac{n}{k} \le \frac{kn}{(2p+2)k-n}.$$

Therefore, $\left\lfloor \frac{n}{k} \right\rfloor \le \frac{n}{k} \le \frac{kn}{(2p+2)k-n}$.

Now suppose that $n < (2p+2)k-k^2$. Then $\lfloor \frac{n}{k} \rfloor \le 2p-k+1$. Since $(k-p)(k-p-1) \ge 0$, we have $k^2-(2p+1)k+p(p+1) \ge 0$, or $p^2+p \ge (2p+1)k-k^2$. Also, $\frac{(2p+2)(p^2+p)}{2p+1} = p^2+\frac{3p}{2}+\frac{p}{4p+2} < p^2+\frac{3p+1}{2}$. Thus, we have

$$n \ge p^2+\frac{3p+1}{2}$$
$$n > \frac{(2p+2)(p^2+p)}{2p+1}$$
$$(2p+1)n > (2p+2)(p^2+p)$$
$$(2p+1)n \ge (2p+2)((2p+1)k-k^2)$$
$$(2p+1)(2p+2)k-(2p+1)n \le k^2(2p+2)$$
$$(2p+1)((2p+2)k-n) \le k^2(2p+2)$$
$$(2p+1)((2p+2)k-n)-k^2(2p+2)+kn \le kn$$
$$(2p+1-k)((2p+2)k-n) \le kn$$
$$2p-k+1 \le \frac{kn}{(2p+2)k-n}.$$

Therefore, we have $\left\lfloor \frac{n}{k} \right\rfloor \le 2p-k+1 \le \frac{kn}{(2p+2)k-n}$.

This clears all the cases, and so we have shown that $f_2(k) \ge t(n)$ for each $1 \le k \le n$. Earlier we showed that $f_1(k) \ge \lceil 2\sqrt{n}\rceil \ge t(n)$. Therefore, we conclude that $\min\{f_1(k), f_2(k)\} \ge t(n)$, for all $1 \le k \le n$. Furthermore, we showed that in Cases 1 and 3, $f_2(\lfloor\sqrt{n}\rfloor) = t(n)$ and in Cases 2 and 4, $f_1(\lfloor\sqrt{n}\rfloor) = \lceil 2\sqrt{n}\rceil = t(n)$. Therefore, for any integer n, there is at least one index $1 \le k \le n$ for which $\min\{f_1(k), f_2(k)\} = t(n)$, which implies that our lower bound is indeed optimal. □

Lemma 5.4. *Let $n \geq 2$ be a fixed integer. We say that a pair (x, y) of real numbers with $x, y > 1$ is **n**-amicable if $(\lceil y \rceil - 1)x > n$ and $(\lceil x \rceil - 1)y > n$. Then $\inf\{x + y : (x, y) \text{ is } n\text{-amicable}\} = t(n)$.*

Proof. Let $f(n) = \inf\{x + y : (x, y) \text{ is } n\text{-amicable}\}$. We will prove that $f(n) = t(n)$.

Given any fixed $y > 1$, (x, y) is n-amicable if x satisfies $x > \frac{n}{\lceil y \rceil - 1}$ and $\lceil x \rceil > \frac{n}{y} + 1$. The latter inequality is equivalent to $x > \frac{n}{y} + 1$ if y divides n, and to $x > \lfloor \frac{n}{y} \rfloor + 1$ otherwise, so we see that in either case, the second inequality is equivalent to $x > \lfloor \frac{n}{y} \rfloor + 1$. Hence the two inequalities are equivalent to

$$x > \frac{n}{\lceil y \rceil - 1} \quad \text{and} \quad x > \left\lfloor \frac{n}{y} \right\rfloor + 1.$$

Let $X = X(y)$ denote the set of real numbers x satisfying both inequalities for a fixed value of $y > 1$. Then

$$\inf X = \max\left\{ \frac{n}{\lceil y \rceil - 1}, \left\lfloor \frac{n}{y} \right\rfloor + 1 \right\},$$

with the infimum taking place over all numbers greater than this lower bound. Since $y > 1$, it follows that $\inf X \leq n$. Furthermore, the second inequality insists that $\inf X \geq 1$. Therefore, $\inf X = \frac{n}{k}$ or $\inf X = k$, for some positive integer $1 \leq k \leq n$. This integer k can be determined as a function of y: if $\inf X = \frac{n}{k}$, then $k = \lceil y \rceil - 1$, and if $\inf X = k$, then $k = \lfloor \frac{n}{y} \rfloor + 1$.

So for any fixed $y > 1$, the infimum of the set of all $x + y$ such that (x, y) is n-amicable is the same as the infimum of the subset of all such (x, y) where $x = k + \varepsilon$ or $x = \frac{n}{k} + \varepsilon$ for some integer k and all arbitrarily small $\varepsilon > 0$. We extend this key insight to determine the value of $f(n)$, namely the infimum of the set of $x + y$ over *all* n-amicable pairs (x, y). To do this, we minimize the value of $x + y$ over all n-amicable pairs (x, y) with $x = k + \varepsilon$. We then do the same for the case $x = \frac{n}{k} + \varepsilon$, and compare the results. In essence, we are now fixing x (i.e., fixing k), and determining the smallest possible value of y as a function of k.

For each $1 \leq k \leq n$, define $g(k)$ to be the infimum of the set of $x + y$ where (x, y) is n-amicable and $x = k + \varepsilon$ over all arbitrarily small $\varepsilon > 0$ (say $\varepsilon < 1$). Similarly, define $h(k)$ to be the infimum of the set of all $x + y$ where (x, y) is n-amicable and $x = \frac{n}{k} + \varepsilon$ over all arbitrarily small $\varepsilon > 0$.

By definition, $f(n)$ is the minimum of $\min\{g(k), h(k)\}$ as k ranges from 1 to n. We will prove that $g(k) = f_1(k) = k + \lceil \frac{n}{k} \rceil$ and $h(k) = f_2(k) = \frac{n}{k} + \frac{n}{\lfloor \frac{n}{k} \rfloor}$, as defined in Lemma 5.3. By this lemma, $\min\{f_1(k), f_2(k)\} \geq t(n)$ for all $1 \leq k \leq n$, with equality occurring when $k = \lfloor \sqrt{n} \rfloor$. Thus, this will prove the desired result that $f(n) = t(n)$.

Hence, it suffices to establish that $g(k) = f_1(k)$ and $h(k) = f_2(k)$. Note that for any fixed x, (x, y) is amicable iff $\lceil y \rceil > \frac{n}{x} + 1$ and $y > \frac{n}{\lceil x \rceil - 1}$.

If $x = k + \varepsilon$, then we require $\lceil y \rceil > \frac{n}{k+\varepsilon} + 1$ and $y > \frac{n}{k}$. Then, (x, y) is not n-amicable when $y = \lceil \frac{n}{k} \rceil$, but is n-amicable when $y = \lceil \frac{n}{k} \rceil + \varepsilon'$, for any $\varepsilon' > 0$. Hence, in this case, $g(k) = k + \lceil \frac{n}{k} \rceil = f_1(k)$.

If $x = \frac{n}{k} + \varepsilon$, then we require $\lceil y \rceil > \frac{n}{\frac{n}{k}+\varepsilon} + 1$ and $y > \frac{n}{\lceil \frac{n}{k}+\varepsilon \rceil - 1} = \frac{n}{\lfloor \frac{n}{k} \rfloor}$. The latter inequality does not hold if $y = \frac{n}{\lfloor \frac{n}{k} \rfloor}$, but does if $y = \frac{n}{\lfloor \frac{n}{k} \rfloor} + \varepsilon'$, for any $\varepsilon' > 0$. We check that this value of y also satisfies the former inequality: if $y = \frac{n}{\lfloor \frac{n}{k} \rfloor} + \varepsilon' \geq \frac{n}{\frac{n}{k}} + \varepsilon' = k + \varepsilon'$, then $\lceil y \rceil \geq k + 1$, implying that $\lceil y \rceil \geq k + 1 = \frac{n}{\frac{n}{k}} + 1 > \frac{n}{\frac{n}{k}+\varepsilon} + 1$. Thus, (x, y) is n-amicable when $y = \frac{n}{\lfloor \frac{n}{k} \rfloor} + \varepsilon'$, for any $\varepsilon' > 0$. Hence, in this case, $h(k) = \frac{n}{k} + \frac{n}{\lfloor \frac{n}{k} \rfloor} = f_2(k)$.

We have proven that $g(k) = f_1(k)$ and $h(k) = f_2(k)$, and so we conclude that $f(n) = \min_{1 \leq k \leq n}\{\min\{g(k), h(k)\}\} = \min_{1 \leq k \leq n}\{\min\{f_1(k), f_2(k)\}\} = t(n)$. □

We are finally ready to prove Theorem 3.1. In addition to our lemmas, we will repeatedly apply Theorem 2.5, which states that for any graph G on n vertices,

$$\max\left\{ \omega(G), \frac{n}{\alpha(G)} \right\} \leq \chi_f(G) \leq \chi_c(G) \leq \chi(G).$$

Proof of Theorem 3.1. By Theorems 2.5 and 1.1, $\chi_f(G) + \chi_f(\overline{G}) \leq \chi_c(G) + \chi_c(\overline{G}) \leq \chi(G) + \chi(\overline{G}) \leq n + 1$. Similarly, $\chi_f(G)\chi_f(\overline{G}) \leq \chi_c(G)\chi_c(\overline{G}) \leq \chi(G)\chi(\overline{G}) \leq \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor$. From three applications of Theorem 2.5,

$$\chi_c(G)\chi_c(\overline{G}) \geq \chi_f(G)\chi_f(\overline{G}) \geq \frac{n\chi_f(G)}{\alpha(\overline{G})} = \frac{n\chi_f(G)}{\omega(G)} \geq n.$$

Thus, we have justified three of the four bounds. To complete the proof, we prove that $t(n) \leq \chi_f(G) + \chi_f(\overline{G})$. By Corollary 4.4, the optimal lower bound of $\chi_f(G) + \chi_f(\overline{G})$ is the infimum of the set of possible sums $a_1 + a_2$ such that $r_{\chi_f}(a_1 + \varepsilon, a_2 + \varepsilon) > n$, for any $\varepsilon > 0$. Set $x = a_1 + \varepsilon$ and $y = a_2 + \varepsilon$. By Theorem 4.3, we require x and y to be chosen so that $\lceil (\lceil x \rceil - 1)y \rceil > n$ and $\lceil (\lceil y \rceil - 1)x \rceil > n$. This is equivalent to the inequalities $(\lceil x \rceil - 1)y > n$ and $(\lceil y \rceil - 1)x > n$, since n is an integer.

In other words, we seek to find the n-amicable pair (x, y) so that its sum $x + y$ is as small as possible. By Lemma 5.4, the infimum of the set of all possible sums $x + y$ equals $t(n)$, implying that $\inf\{a_1 + a_2 : r_{\chi_f}(a_1 + \varepsilon, a_2 + \varepsilon) > n\} = t(n)$. Therefore, we have proven that $\chi_f(G) + \chi_f(\overline{G}) \geq t(n)$. By Theorem 2.5, we also have $\chi_c(G) + \chi_c(\overline{G}) \geq t(n)$. This completes the proof of Theorem 3.1. □

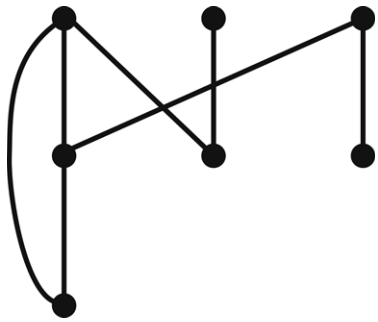

Fig. 1. A Graph in $T(7, 3, 3)$.

To verify Theorem 3.2, we only need to establish for each $n \geq 1$ the existence of *one* extremal graph for each of our four bounds.

We require the following definition and lemma.

Definition 5.5 ([9]). For each ordered triplet (n, x, y) with $x+y-1 \leq n \leq xy$, the set $\boldsymbol{T(n, x, y)}$ of graphs is defined as follows: consider a rectangular array M with x rows and y columns, where we place at most one dot in each of the xy entries of M. We place a dot in each entry of the first row and first column of M, which accounts for $x + y - 1$ dots. Now place $n - (x + y - 1)$ dots in any of the remaining entries of M. Then a graph G in the family $T(n, x, y)$ is formed by taking the n dots of M as the vertices of G, and defining adjacency as follows:

(a) Any two dots in the same column are adjacent.
(b) No two dots in the same row are adjacent.
(c) Any two dots which belong to distinct rows and columns may or may not be adjacent.

Fig. 1 illustrates a graph in $T(7, 3, 3)$.

Note that for any $G \in T(n, x, y)$, we have $\chi(G) = \omega(G) = x$ and $\chi(\overline{G}) = \omega(\overline{G}) = y$. By Theorem 2.5, this implies that $\chi_c(G) = \chi_f(G) = x$ and $\chi_c(\overline{G}) = \chi_f(\overline{G}) = y$. The following result classifies *all* extremal graphs for the original Nordhaus–Gaddum inequality $\chi(G) + \chi(\overline{G}) \geq \lceil 2\sqrt{n} \rceil$ (see Theorem 1.1).

Theorem 5.6 ([9]). *Let G be a graph on n vertices. Then, $\chi(G) + \chi(\overline{G}) = \lceil 2\sqrt{n} \rceil$ iff $G \in T(n, x, y)$, where $x + y = \lceil 2\sqrt{n} \rceil$.*

To complete the proof of our main result, we prove Theorem 3.2, which enables us to conclude that the Nordhaus–Gaddum inequalities found in Theorem 3.1 are indeed optimal. For the most difficult case among our bounds, our extremal graph will be a star extremal *circulant*, i.e., a symmetric vertex-transitive graph of the form $G = C_{n,S}$, where G has vertex set $\mathbb{Z}_n$, and two vertices u and v in G are adjacent iff their (circular) distance appears in the generating set $S \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, that is, $|u - v| \in S$.

Proof of Theorem 3.2. For each of our four bounds, it suffices to find one graph on n vertices for which equality is attained.

Let $G = K_n$. Then, $\omega(G) = \chi(G) = n$, which implies that $\chi_f(G) = \chi_c(G) = n$, by Theorem 2.5. By the same argument, $\chi_f(\overline{G}) = \chi_c(\overline{G}) = 1$. Hence, for this graph G, $\chi_f(G) + \chi_f(\overline{G}) = \chi_c(G) + \chi_c(\overline{G}) = n + 1$, and $\chi_f(G)\chi_f(\overline{G}) = \chi_c(G)\chi_c(\overline{G}) = n$.

Let $G = K_m \cup \overline{K_{n-m}}$ be the disjoint union of K_m and $n - m$ isolated vertices, where $m = \lfloor \frac{n+1}{2} \rfloor$. Then, $\omega(G) = \chi(G) = m$, implying that $\chi_f(G) = \chi_c(G) = m$. Also, $\overline{G} = K_{n-m} + \overline{K_m}$, where $H_1 + H_2$ is formed from the disjoint union of H_1 and H_2 by adding in all edges between a vertex of H_1 and a vertex of H_2. Thus, $\omega(\overline{G}) = \chi(\overline{G}) = n-m+1$, implying that $\chi_f(\overline{G}) = \chi_c(\overline{G}) = n-m+1$. Since $m = \lfloor \frac{n+1}{2} \rfloor$, we have

$$\begin{aligned}\chi_f(G) \cdot \chi_f(\overline{G}) &= \chi_c(G) \cdot \chi_c(\overline{G}) \\ &= \left\lfloor \frac{n+1}{2} \right\rfloor \cdot \left(n + 1 - \left\lfloor \frac{n+1}{2} \right\rfloor \right) \\ &= \left\lfloor \left(\frac{n+1}{2} \right)^2 \right\rfloor.\end{aligned}$$

The last line follows from a simple case analysis (n even and n odd).

Finally, we verify the existence of a graph G for which $\chi_f(G) + \chi_f(\overline{G}) = \chi_c(G) + \chi_c(\overline{G}) = t(n)$. Since $t(n)$ is defined to be the minimum of two functions, we consider both possibilities separately.

Case 1: $t(n) = \lceil 2\sqrt{n} \rceil$.

By Theorem 5.6, $\chi(G) + \chi(\overline{G}) = \lceil 2\sqrt{n} \rceil = t(n)$ iff $G \in T(n, x, y)$. For any such graph G, $\chi(G) = \omega(G)$ and $\chi(\overline{G}) = \omega(\overline{G})$. By Theorem 2.5, we must have $\chi_f(G) + \chi_f(\overline{G}) = \chi_c(G) + \chi_c(\overline{G}) = \chi(G) + \chi(\overline{G}) = t(n)$. Any such graph $G \in T(n, x, y)$ is an extremal graph.

Case 2: $t(n) = \frac{n}{\lfloor\sqrt{n}\rfloor} + \frac{n}{\lfloor\sqrt{n+\sqrt{n}}\rfloor}$.

By Lemma 5.2, this case only occurs when $p^2 \le n < p^2 + \frac{p}{2}$ or $p^2 + p + 1 \le n \le p^2 + \frac{3p}{2}$ (recall that $p = \lfloor\sqrt{n}\rfloor$). For the values of n for which $\lceil 2\sqrt{n}\rceil = \frac{n}{\lfloor\sqrt{n}\rfloor} + \frac{n}{\lfloor\sqrt{n+\sqrt{n}}\rfloor}$, an extremal graph must exist from the analysis in the previous paragraph (Case 1).

Thus, we may assume that $t(n) = \frac{n}{\lfloor\sqrt{n}\rfloor} + \frac{n}{\lfloor\sqrt{n+\sqrt{n}}\rfloor} > \lceil 2\sqrt{n}\rceil$. It is easy to check that this requires $n \ge 7$.

Define $d = \lfloor\sqrt{n}\rfloor - 1$, and let $G = C_{n,\{1,2,\ldots,d\}}$ be the circulant on $\mathbb{Z}_n$, where two distinct vertices u and v are adjacent iff $|u - v| \le d$. In other words, G is the dth power of the cycle C_n. Note that $\overline{C_{n,\{1,2,\ldots,d\}}} = C_{n,\{d+1,d+2,\ldots,\lfloor\frac{n}{2}\rfloor\}}$, which is also a circulant.

To prove that G is star extremal, we cite the theorem by Gao and Zhu [11] which states that $C_{n,\{1,2,\ldots,d\}}$ is star extremal for any $n \ge 2d$. By this theorem, G is star extremal for each $n \ge 7$ since $n \ge 2d = 2\lfloor\sqrt{n}\rfloor - 2$. To prove that $\overline{G}$ is star extremal, we cite the theorem by Lih et al. [17] which states that $C_{n,\{a,a+1,\ldots,b\}}$ is star extremal for any $n \ge 2b$ and $b \ge \frac{5a}{4}$. By this theorem, $\overline{G}$ is star extremal for each $n \ge 7$, since $n \ge 2 \cdot \lfloor\frac{n}{2}\rfloor$ and $\lfloor\frac{n}{2}\rfloor \ge \frac{5(d+1)}{4} = \frac{5\lfloor\sqrt{n}\rfloor}{4}$.

Therefore, we conclude that both G and $\overline{G}$ are star extremal. It can be verified (see [13]) that $\alpha(G) = \lfloor\frac{n}{d+1}\rfloor$ and $\alpha(\overline{G}) = d+1$. By this result and also Theorem 2.2, we have

$$\chi_f(G) = \chi_c(G) = \frac{n}{\alpha(G)} = \frac{n}{\lfloor\frac{n}{d+1}\rfloor} = \frac{n}{\lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor},$$

$$\chi_f(\overline{G}) = \chi_c(\overline{G}) = \frac{n}{\alpha(\overline{G})} = \frac{n}{d+1} = \frac{n}{\lfloor\sqrt{n}\rfloor}.$$

Recall that $p^2 \le n < p^2 + \frac{p}{2}$ or $p^2 + p + 1 \le n \le p^2 + \frac{3p}{2}$. In both these cases, it is straightforward to show that

$$\left\lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\right\rfloor = \left\lfloor\sqrt{n+\sqrt{n}}\right\rfloor.$$

Therefore, $\chi_f(G) + \chi_f(\overline{G}) = \chi_c(G) + \chi_c(\overline{G}) = \frac{n}{\lfloor\sqrt{n}\rfloor} + \frac{n}{\lfloor\sqrt{n+\sqrt{n}}\rfloor} = t(n)$, as required.

Thus, for all four bounds, we have determined the existence of an extremal graph. This completes the proof of Theorem 3.2, and hence our Nordhaus–Gaddum inequalities are indeed optimal. □

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