On Lie solvable restricted enveloping algebras

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Abstract
In this note we study the Lie derived lengths of a restricted enveloping algebra $u(L)$, for a non-abelian restricted Lie algebra $L$ over a field of positive characteristic $p$. For $p > 2$ we show that if the Lie derived length of $u(L)$ is minimal then $u(L)$ is Lie nilpotent. Moreover, we investigate the case when the strong Lie derived length of $u(L)$ is minimal. For odd $p$ we establish a classification of Lie centrally metabelian restricted enveloping algebras.

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1. Introduction and statement of the main results

Let $R$ be a unital associative algebra over a field $F$. Recall that $R$ can be viewed as a Lie algebra with Lie multiplication defined by $[x, y] = xy - yx$, for all $x, y \in R$. For subspaces $A, B \subseteq R$, we denote by $[A, B]$ the linear span of all elements $[a, b]$, with $a \in A$ and $b \in B$. The Lie derived series of $R$ is defined inductively by $\delta^{[0]}(R) = R$ and $\delta^{[n+1]}(R) = [\delta^{[n]}(R), \delta^{[n]}(R)]$. Moreover, following [9], we consider the series of associative ideals of $R$ defined by $\delta^{(0)}(R) = R$ and $\delta^{(n+1)}(R) = [\delta^{(n)}(R), \delta^{(n)}(R)]R$. We say that $R$ is Lie solvable (respectively strongly Lie solvable) if $\delta^{[n]}(R) = 0$ ($\delta^{(n)}(R) = 0$) for some $n$. In this case, the minimal $n$ with such a property is called the Lie derived length (respectively strong Lie derived length) of $R$ and denoted by $d_{\text{Lie}}(R)$ ($d_{\text{Lie}}^{(n)}(R)$). Clearly, strong Lie solvability implies Lie solvability of $R$ (and $d_{\text{Lie}}^{(n)}(R) \leq d_{\text{Lie}}(R)$).
Let $L$ be a restricted Lie algebra over a field $F$ of characteristic $p > 0$ and denote by $u(L)$ the restricted (universal) enveloping algebra of $L$. An element $x$ of $L$ is $p$-nilpotent if there exists a non-negative integer $m$ such that $x[p]^m = 0$; a subset $S$ of $L$ is $p$-nilpotent if there is an $m$ such that $S[p]^m = \{x[p]^m \mid x \in S\} = 0$. In the main theorem of [5], D. Riley and A. Shalev characterized the Lie solvable restricted enveloping algebras for odd $p$. On the other hand, in [13] the author proved (without restrictions on the characteristic) that $u(L)$ is strongly Lie solvable if and only if the commutator subalgebra $L'$ of $L$ is finite-dimensional and $p$-nilpotent. It turns out that for $p > 2$ the Lie solvability of $u(L)$ is equivalent to the strongly Lie solvability (this is no longer true for $p = 2$: see Example 1 in [13]). Moreover, if $u(L)$ is strongly Lie solvable then it is possible to have $dl_{Lie}(u(L)) < dl_{Lie}(u(L))$ (see Example 2 of [13]). While it is possible to compute the Lie nilpotency classes of $u(L)$ by fairly satisfactory methods (see [6,15]), the computation of the Lie derived lengths represents a harder task. For more results on this topic the reader is referred to [7,13,14].

The purpose of this paper is to present some further contributions to this problem. In [13], the minimal value for the Lie derived length of non-commutative restricted enveloping algebras was determined. Indeed, it was proved that if $L$ is not abelian then

$$dl_{Lie}(u(L)) \geq \lceil \log_2(p + 1) \rceil$$

and such lower bound is actually attained for every $p$. In Section 2 we investigate the structure of $L$ when the Lie derived length or strong Lie derived length of $u(L)$ coincides with this lower bound. Our first main result is the following.

**Theorem 1.** Let $L$ be a restricted Lie algebra over a field $F$ of characteristic $p > 2$. If $dl_{Lie}(u(L)) = \lceil \log_2(p + 1) \rceil$ then $u(L)$ is Lie nilpotent.

It should be noted that Theorem 1 does not hold in characteristic 2 (cf. [14]). In the next theorem we characterize the non-commutative restricted enveloping algebras with minimal strong Lie derived length.

**Theorem 2.** Let $L$ be a restricted Lie algebra over a field $F$ of characteristic $p > 0$. Then $dl_{Lie}(u(L)) = \lceil \log_2(p + 1) \rceil$ if and only if one of the following conditions is satisfied:

(i) $p = 2$, $dim_F L' \leq 2$, $L'$ is central, and $L'[p] = 0$;
(ii) $p = 2$, $dim_F L' = 1$, and $L'[p] = 0$;
(iii) $p > 2$, $dim_F L' = 1$, $L'$ is central, and $L'[p] = 0$.

We recall that an associative algebra $R$ is called *Lie metabelian* if $\delta^{[2]}(R) = 0$, and *Lie centrally metabelian* if $[\delta^{[2]}(R), R] = 0$. The Lie metabelian restricted enveloping algebras were characterized in [14]. Moreover, for $p > 3$ D. Riley and V. Tasić showed that $u(L)$ is Lie centrally metabelian if and only if $L$ is abelian (see [7]). In Section 3, we complete the characterization in odd characteristic by solving the (more difficult) case $p = 3$. Actually, we obtain the following result.
Theorem 3. Let \( L \) be a restricted Lie algebra over a field \( F \) of characteristic \( p > 2 \). Then \( u(L) \) is Lie centrally metabelian if and only if either \( L \) is abelian or all of the following conditions hold: \( p = 3 \), \( \dim_F L' = 1 \), \( L' \) is central, and \( L'[^p] = 0 \).

As a consequence of such a result, for \( p > 2 \) a Lie centrally metabelian restricted enveloping algebra is in fact Lie metabelian. This is no longer true for \( p = 2 \).

Finally, we mention that all the questions considered in the present paper arise also in the theory of group algebras and have been investigated by several authors (see [3,8–12,16]). It is interesting that the natural theoretic analogues of Theorems 1–3 do not hold for this class of algebras: we refer the reader to [3,11,12,16].

2. Minimal Lie derived lengths

The notations used throughout this paper are essentially standard. Let \( L \) be a restricted Lie algebra over a field \( F \) of positive characteristic \( p \). We adopt the left-normed convention for longer commutators. For \( x, y \in L \), we write \([x, n y] \) to mean \([x, y, \ldots, y]\), where \( y \) appears in the latter expression \( n \) times. For a subset \( S \) of \( L \) we denote by \( S_p \) the restricted subalgebra generated by \( S \). Note that, if \( I \) is an ideal then \( I_p \) is a restricted ideal of \( L \). If an element \( x \) of \( L \) is \( p \)-nilpotent, the minimal non-negative integer \( n \) such that \( x[^p]^n = 0 \) is denoted by \( e(x) \). We write \( \omega(L) \) for the augmentation ideal of \( u(L) \), namely, the associative ideal generated by \( L \) in \( u(L) \).

It is well known that \( \omega(L) \) is nilpotent if and only if \( L \) is finite-dimensional and \( p \)-nilpotent (see [5, Lemma 2.4]): in this case the minimal \( m \) such that \( \omega(L)^m = 0 \) is denoted by \( t(L) \).

Proof of Theorem 1. Without loss of generality, we may assume \( F \) to be algebraically closed. In view of Theorem 1.3 of [5], \( L' \) is finite-dimensional and \( p \)-nilpotent. Hence, by Theorem 1.1 of [5], it is enough to prove that \( L \) is nilpotent. Assume, if possible, that this is not true. Since \( \dim_F L' < \infty \), Lemma 2.3 of [4] implies that \( L/\zeta_2(L) \) is finite-dimensional as well, where \( \zeta_2(L) \) is the second term of the ascending central series of \( L \). By Engel’s Theorem, it follows that \( L \) cannot satisfy any Engel condition. Consider \( a, b \in L \) such that \([a, n b] \neq 0 \) for every \( n \in \mathbb{N} \) and denote by \( H \) the Lie subalgebra generated by \( a \) and \( b \). As \( \dim_F L' < \infty \), we have also \( \dim_F H < \infty \). Since \( F \) is algebraically closed and \( H \) is not nilpotent, it follows that \( H \) contains a 2-dimensional non-abelian subalgebra. As a consequence we can choose \( x \) and \( y \) in \( H \) in such a way that \([x, y] = x \). By standard calculations (see relation (6) in the proof of Theorem 1 in [13]), for all non-negative integers \( r_1, r_2, s_1, s_2 \) we have

\[
[x^{r_1}y^{s_1}, x^{r_2}y^{s_2}] = x^{r_1+r_2}((y-r_2)^{s_1}y^{s_2}-(y-r_1)^{s_2}y^{s_1}).
\]

(1)

We claim that, for all non-negative integers \( h \) and \( k \) satisfying \( k < p-h \), the element \( x^{2^h}y^k \) is contained in \( \delta^{[h+1]}(u(L)) \). We proceed by induction on \( h \). Suppose first \( h = 0 \). For every \( 0 \leq k \leq p - 1 \), we have

\[
[x y^k, y] = x y^k
\]

and then \( x y^k \in \delta^{[1]}(u(L)) \). Assume now \( h \geq 1 \). The inductive hypothesis implies that \( \delta^{[h]}(u(L)) \) contains all elements \( x^{2^h-1}y^v \) with \( 0 \leq v \leq p - h \). Relationship (1) yields

\[
[x^{2^{h-1}}y, x^{2^{h-1}}] = -2^{h-1}x^{2^h},
\]
so that $x^{2h} \in \delta^{[h+1]}(u(L))$, since $p \neq 2$. By (1) one has

$$[x^{2h-1}y, x^{2h-1}] = x^{2h}(-2^{h+1}y + 2^{2(h-1)})$$

and thus, as $x^{2h} \in \delta^{[h+1]}(u(L))$ and $p \neq 2$, it follows that $x^{2h}y \in \delta^{[h+1]}(u(L))$. Suppose we have already shown that $\delta^{[h+1]}(u(L))$ contains all elements $x^{2h}, x^{2h}y, \ldots, x^{2h}y^{k-1}$. By (1), one has

$$[x^{2h-1}y^{k+1}, x^{2h-1}] = x^{2h}((y - 2^{h-1})^{k+1} - y^{k+1})$$

$$= x^{2h}\left(\sum_{j=0}^{k}(-1)^{k+1-j}\binom{k+1}{j}2^{(h-1)(k+1-j)}y^{j}\right).$$

Since $x^{2h}y^{r} \in \delta^{[h+1]}(u(L))$ for every $0 \leq r < k$, also the element $2^{h-1}(k+1)x^{2h}y^{k}$ is contained in $\delta^{[h+1]}(u(L))$. Since $p \neq 2$ and $p$ does not divide $k+1$, we can conclude that $x^{2h}y^{k} \in \delta^{[h+1]}(u(L))$, completing the inductive step.

Put $n = \lceil \log_2(\frac{p+1}{2}) \rceil$, so that $\delta^{[n+1]}(u(L)) = 0$. Notice that $0 < p - 2^{n} < p - n$. Hence the previous part of the proof implies that the element $x^{2n}$ is contained in $\delta^{[n+1]}(u(L))$. Thus $x^{2n} = 0$ and so, as $2^{n} < p$, this yields a contradiction to the PBW Theorem for restricted Lie algebras (see, e.g., Theorem 5.6 in Chapter 2 of [17]), and completes the proof. □

**Remark 1.** If $L$ is a non-nilpotent restricted Lie algebra over a field $F$ of characteristic $p > 2$, then the previous result implies that $\text{dL}_{\text{Lie}}(u(L)) \geq \lceil \log_2(2(p+1)) \rceil$. Actually, a lower bound is the best possible. It is attained, for instance, if $L$ is the non-abelian 2-dimensional restricted Lie algebra. More generally, if $\dim_F L' = 1$ and $L'|[p] = 0$, then one has $\delta(L'_p) = p$. According to Lemma 1 of [13], we get (recall that $p \neq 2$):

$$\text{dL}_{\text{Lie}}(u(L)) \leq \lceil \log_2 2\delta(L'_p) \rceil = \lceil \log_2 2(p+1) \rceil.$$

Thus $\text{dL}_{\text{Lie}}(u(L)) = \lceil \log_2 2(p+1) \rceil$.

**Proof of Theorem 2.** Suppose that $\text{dL}_{\text{Lie}}(u(L)) = \lceil \log_2 (p+1) \rceil$. If $p = 2$, then, by [14], $L$ satisfies one of the conditions (i) or (ii) of the statement.

Assume that $p > 2$. An easy verification by induction (cf. [1, Proposition 3.8]) shows that, for every positive integer $n$, one has

$$\omega(L'_p)^{2^{n-1}} \subseteq \delta^{(n)}(u(L)).$$

Hence

$$\lceil \log_2 (\delta(L'_p) + 1) \rceil \leq \text{dL}_{\text{Lie}} u(L).$$

(2)

From Theorem 1.3 of [5] we already know that $L'$ is $p$-nilpotent. We claim that the $p$-map $[p]$ acts trivially on $L'$. Suppose, if possible, that this is not true. Then there exists $x \in L'$ such that $e(x) > 1$. By Theorem 3.4 of [6] and Lemma 1 of [15], one has

$$p^2 \leq p^{e(x)} = t([x]_p) \leq t(L'_p).$$
From elementary arithmetical considerations and from (2) it follows that
\[
d_{\text{Lie}}^{\text{u}(L)} > \lceil \log_2 (p^2 + 1) \rceil > \lceil \log_2 (p + 1) \rceil,
\]
a contradiction. Therefore \( L'[p] = 0 \) and, in particular, \( L' \) is a restricted subalgebra of \( L \). Furthermore, as \( L' \) is finite-dimensional (by Theorem 1.3 of [5]), by Engel’s Theorem \( L' \) is nilpotent. Suppose that \( \dim_F L' > 1 \). Thus, since \( L' \) is nilpotent it contains a 2-dimensional abelian restricted subalgebra \( H \). Hence
\[
t(L') \geq t(H) = 2p - 1
\]
and then, since \( p > 2 \), by (2) one has \( d_{\text{Lie}}^{\text{u}(L)} > \lceil \log_2 (p + 1) \rceil \), a contradiction. Finally, by Theorem 1, \( L \) is nilpotent and thus \([L, L'] = 0\), completing the first part of the proof.

Conversely, if \( L \) verifies one of the conditions of the statement, the claim follows from [14] for even \( p \), and by Proposition 3 of [13] for odd \( p \).

A unital associative algebra \( A \) is said to be strongly Lie nilpotent if \( A^{(m)} = 0 \), where \( A^{(1)} = A \) and \( A^{(m+1)} = [A^{(m)}, A]A \). The smallest \( m \) such that \( A^{(m+1)} = 0 \) is denoted by \( \text{cl}_{\text{Lie}}^{A} \). For arbitrary associative algebras, the strong Lie nilpotency is a stronger condition than Lie nilpotency (see [2]). Nevertheless, for restricted enveloping algebras these properties turn out to be equivalent (see [5]). Furthermore, if \( \text{cl}_{\text{Lie}}^{\text{u}(L)} \) denotes the ordinary nilpotency class of \( \text{u}(L) \) regarded as a Lie algebra, then \( \text{cl}_{\text{Lie}}^{\text{u}(L)} \leq \text{cl}_{\text{Lie}}^{\text{u}(L)} \) and equality holds provided \( p > 3 \) (see [6]).

As a consequence of Theorem 2 we shall see that, for odd \( p \), the strong Lie derived length of a non-commutative restricted enveloping algebra is minimal if and only if its strong Lie nilpotency class is minimal, namely \( \text{cl}_{\text{Lie}}^{\text{u}(L)} = p \) (cf. [15]). In fact, the following holds

**Corollary 1.** Let \( L \) be a non-abelian restricted Lie algebra over a field \( F \) of characteristic \( p > 2 \). Then \( d_{\text{Lie}}^{\text{u}(L)} = \lceil \log_2 (p + 1) \rceil \) if and only if \( \text{cl}_{\text{Lie}}^{\text{u}(L)} = p \).

**Proof.** Consider the chain of restricted ideals of \( L \) defined inductively by

\[
D^{(1)}(L) = L, \quad D^{(2)}(L) = L'_p, \\
D^{(m+1)}(L) = \left( D^{(\lceil \frac{m+1}{p} \rceil)}(L)^{[p]} \right)_p + [D^{(m)}(L), L] \quad (m \geq 2).
\]

According to [6], if \( \text{u}(L) \) is strongly Lie nilpotent one has

\[
\text{cl}_{\text{Lie}}^{\text{u}(L)} = 1 + (p - 1) \sum_{m \geq 1} md_{(m+1)}
\]

where \( d_{(m)} = \dim_F (D^{(m)}(L)/D^{(m+1)}(L)) \).

Suppose, first, \( d_{\text{Lie}}^{\text{u}(L)} = \lceil \log_2 (p + 1) \rceil \). By Proposition 6.2 of [5] and Theorem 1, \( \text{u}(L) \) is strongly Lie nilpotent. Moreover, Theorem 2 implies that \( D^{(m)}(L) = 0 \) for every \( m > 2 \). Thus, by (3), we conclude that \( \text{cl}_{\text{Lie}}^{\text{u}(L)} = p \). Conversely, if \( \text{cl}_{\text{Lie}}^{\text{u}(L)} = p \) then by (3) we have necessarily \( d_{(2)} = 1 \) and \( d_{(m)} = 0 \) for every \( m > 2 \). This forces \( \dim_F L'_p = 1 \), \([L'_p, L] = 0\) and \( L'_p^{[p]} = 0 \). Hence Theorem 2 yields the claim. \( \square \)
3. Lie centrally metabelian restricted enveloping algebras

For an associative algebra \( A \), we denote by \( \zeta(A) \) the center of \( A \) and by \( \gamma_r(A) \) the terms of the Lie descending central series of \( A \) (defined inductively by \( \gamma_1(A) = A \) and \( \gamma_{r+1}(A) = [\gamma_r(A), A] \)). In the proof of Theorem 3, we shall make use of the following result stated in [7]:

**Lemma 1.** Let \( L \) be a restricted Lie algebra over a field of characteristic \( p > 2 \). If \( [[\gamma_n(u(L)), \gamma_r(u(L))], u(L)] = 0 \) for some \( n, r \geq 1 \), then \( L \) is nilpotent.

**Proof of Theorem 3.** In view of Theorem 2, the conditions expressed in the statement are clearly sufficient.

Conversely, suppose that \( u(L) \) is Lie centrally metabelian and \( L \) is not abelian. By [7] it is enough to consider the case \( p = 3 \). Because of Lemma 1, \( L \) is nilpotent; therefore there exist two non-commuting elements \( a \) and \( b \) of \( L \) such that \( z = [a, b] \) is central in \( L \). It is easy to see that \( a, b \) and \( z \) are \( F \)-linearly independent. We claim that \( z^3 = 0 \). In fact, we have \( a^2z = [a, a^2b] \in \delta^{[1]}(u(L)) \) and \( bz = [ba, b] \in \delta^{[1]}(u(L)) \), so that \( 2az^3 = [a^2z, bz] \in \delta^{[2]}(u(L)) \). Since \( u(L) \) is Lie centrally metabelian, it follows that \( 0 = [az^3, b] = z^4 \), hence the PBW Theorem forces \( z^3 = 0 \), as claimed.

In order to complete the proof, it will suffice to show that \( L' = Fz \). Suppose, if possible, that \( L' \neq Fz \); then consider three cases.

**Case 1:** there exists \( c \in L \) such that \( [a, c] \notin Fz \). Clearly, the elements \( a, b \) and \( c \) are \( F \)-linearly independent. Put \( t = [a, c] \) and \( v = [b, c] \). Since \( u(L) \) is Lie centrally metabelian, we have

\[
[b, v]z = [bz, v] = [(ba, b), [b, c]] \in \zeta(u(L))
\]

and

\[
[a, v]z = [az, v] = [[a, ab], [b, c]] \in \zeta(u(L)).
\]

Moreover, as \( [a, ab^2] = 2abz \), we have

\[
[a, v]bz + a[b, v]z = [abz, v] \in \zeta(u(L)). \tag{4}
\]

One concludes that

\[
0 = [a, [a, v]bz + a[b, v]z] = [a, v]z^2
\]

and, analogously, \( [b, v]z^2 = 0 \). By the PBW Theorem, it follows that \( [a, v] = \alpha z \) and \( [b, v] = \beta z \) for some \( \alpha, \beta \in F \). As \( z \in \zeta(u(L)) \), the Jacobi identity yields

\[
[b, t] = [b, [a, c]] = [a, [b, c]] = [a, v] = \alpha z. \tag{5}
\]

We claim that \( b \) and \( t \) commute. Assume the contrary, so that \( \alpha \neq 0 \). Relation (4) yields

\[
0 = [[a, v]bz + a[b, v]z, c] = (\alpha v + \beta t)z^2.
\]
Therefore, the PBW Theorem forces $\alpha v + \beta t = k z$ for some $k \in F$. As $\alpha \neq 0$, $v$ is an $F$-linear combination of the elements $t$ and $z$. In particular, $v$ and $t$ commute and then, by (5) and the fact that $u(L)$ Lie centrally metabelian:

$$\alpha z v = [b, t] v = [bv, t] = [[b, bc], [a, c]] \in \zeta(u(L)).$$

As a consequence,

$$0 = [a, \alpha z v] = \alpha z [a, v] = \alpha^2 z^2.$$

Since $\alpha \neq 0$, this latter relation violates the PBW Theorem; hence $[b, t] = 0$. If $[a, t] = 0$ one has

$$2at^2 z = [a^2 z, ct] = [[a, a^2 b], [ca, c]] \in \zeta(u(L))$$

and, as $[b, t] = 0$, this forces

$$0 = [at^2 z, b] = z^2 t^2. \quad (6)$$

Since $t$ and $z$ are $F$-linearly independent, relation (6) yields a contradiction to the PBW Theorem. On the other hand, if $[a, t] \neq 0$ notice that

$$[a, t] t = [at, t] = [[a, ac], [a, c]] \in \zeta(u(L)); \quad (7)$$

thus

$$0 = [a, t] t, t = [a, t], t t.$$

By the PBW Theorem, it follows that

$$[[a, t], t] = 0. \quad (8)$$

If $[a, t] = \lambda z$ for some $\lambda \in F$, then by (7) one has

$$0 = [a, [a, t] t] = \lambda^2 z^2.$$

Hence, the PBW Theorem forces $\lambda = 0$. Therefore $[a, t] = 0$, a contradiction. Therefore, $[a, t]$ and $z$ are $F$-linearly independent. One has

$$a[a, t] z = [az, at] = [[a, ab], [a, ac]] \in \zeta(u(L)).$$

As a consequence, by (8) one obtains

$$0 = [a[a, t] z, t] = [a, t]^2 z.$$

By the linear independence of $[a, t]$ and $z$, the last relation contradicts the PBW Theorem, and the proof for the first case is complete.

Case 2: there exists $c \in L$ such that $[b, c] \notin Fz$. The proof is analogous to the previous case.

Case 3: for every $c \in L$, $[a, c]$ and $[b, c]$ belong to $Fz$. Let $x, y \in L$ such that $[x, y]$ and $z$ are $F$-linearly independent. Put $w = [x, y]$. By assumption, the elements $[a, x], [a, y], [b, x]$ and
are in $Fz$. If $[a, x] = \mu z$ for some $\mu \in F \setminus \{0\}$, an argument similar to the one used in case 1 (replacing $b$ by $x$) yields a contradiction. Using a similar argument also for the other elements, we may assume that $[a, x] = [a, y] = [b, x] = [b, y] = 0$. As a consequence, the Jacobi identity yields

$$[a, w] = [a, [x, y]] = 0$$

and, analogously, $[b, w] = 0$.

Finally, one concludes

$$azw^2 = [aw, abw] = [[ax, y], [abx, y]] \in \xi(u(L))$$

and hence

$$0 = [azw^2, b] = z^2 w^2.$$  \hspace{1cm} (9)

Since $z$ and $w$ are $F$-linearly independent, the relation (9) contradicts the PBW Theorem, and the proof is complete. \hfill $\Box$

As a consequence of Theorem 3 and of [14] we have the following result.

**Corollary 2.** Let $L$ be a restricted Lie algebra over a field of characteristic $p > 2$. Then $u(L)$ is Lie centrally metabelian if and only if it is Lie metabelian.

The following example shows that Theorem 3 fails in characteristic 2.

**Example 1.** Let $H$ be the restricted Lie algebra over a field $F$ of characteristic 2 with a basis $\{x, y, z\}$ such that $[x, y] = z$, $[x, z] = [y, z] = 0$, $x^{[p]} = y^{[p]} = z^{[p]} = z$. It is straightforward to check that $u(L)$ is Lie centrally metabelian, but $L^{'[p]} \neq 0$. Note also that in this case $u(L)$ is not Lie metabelian (see [14]).

**References**