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An algebraic proof of a result by Gonzaga and Lara

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Abstract

The equality of two condition numbers of a rectangular matrix recently established by C.C. Gonzaga and H.J. Lara [Linear Algebra Appl. 261 (1997) 269–273] is now proved by purely algebraic means. © 1999 Published by Elsevier Science Inc. All rights reserved.

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1. Introduction

Let A be a real $m \times n$ matrix of rank m $(m < n)$. Denote by $\mathscr{R}(A)$ and $\mathscr{N}(A)$, respectively, the range space and null space of A, and let Z be a matrix whose rows form a basis of $\mathscr{N}(A)$. The symbol $\mathscr{D}_n$ stands for the set of all diagonal $n \times n$ matrices with a positive main diagonal. Throughout the paper, the matrix norm is the spectral norm.

The two numbers below were introduced in [5]:

$$\chi(A) = \sup\{\|A^{\mathrm{T}}(ADA^{\mathrm{T}})^{-1}AD\|: D \in \mathscr{D}_n\}, \tag{1}$$

$$\chi(Z) = \sup\{\|Z^{\mathrm{T}}(ZDZ^{\mathrm{T}})^{-1}ZD\|: D \in \mathscr{D}_n\}. \tag{2}$$

These numbers are the suprema of the norms of oblique projectors with the range space $\mathscr{R}(A^{\mathrm{T}})$ and $\mathscr{N}(A)$, respectively, and can be interpreted as the condition num-

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bers associated with the matrix A. According to Gonzaga and Lara [3], the numbers (1) and (2) are "intensively used in new results on complexity of linear programming algorithms".

It was proved in [3] that, in fact,

$$\chi(A) = \chi(Z). \tag{3}$$

The proof uses simple optimization theory considerations and some geometry.

The purpose of this note is to give another proof of relation (3). It is not simpler than the proof in [3] but it exploits only purely algebraic means. A certain optimization machinery is hidden here as well through the use of the principal angles between subspaces. However, an optimization of this kind has long been integrated into linear algebra, and does not contradict the claim above on the algebraic nature of our proof.

In Section 2, we state the facts that we need for the proof. The most important of them is a canonical form for oblique projectors introduced in [1]. We feel that this remarkable result deserves to be known better than it seems to be at present.

The proof in Section 3 is a very easy implication of the facts in Section 2.

In what follows, we use the symbol $M_n(\mathbb{C})$ for the set of complex $n \times n$ matrices and the symbol $\mathbb{R}^{m \times n}$ ($\mathbb{C}^{m \times n}$) for the set of real (complex) matrices of size $m \times n$. The symbol I_n stands for the identity matrix of order n.

2. The necessary facts

Fact 1 (The canonical form for projectors under unitary similarity). *Let $P \in M_n(\mathbb{C})$ be a projector. Then there is a unitary similarity that reduces P to a block diagonal form*

$$\begin{pmatrix} 1 & \sigma_1 \\ 0 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 & \sigma_k \\ 0 & 0 \end{pmatrix} \oplus I_m \oplus 0_s. \tag{4}$$

Here, $\sigma_1 \geqslant \cdots \geqslant \sigma_k > 0$; I_m and 0_s are the identity and zero matrices of the corresponding order; and the numbers $\sigma_1, \ldots, \sigma_k, k, m,$ and s are uniquely defined by the projector P.

This assertion was proved in [1]. A geometric interpretation of the numbers $\sigma_1 \geqslant \cdots \geqslant \sigma_k$ was given in [4].

Fact 2. *The numbers $\sigma_1, \ldots, \sigma_k$ in (4) are the tangents of the principal angles between the range spaces of P and the adjoint operator P^*.*

For the definition of the principal angles between subspaces, we refer the reader to [2, p. 484].

For the proof of (3), the following equivalent formulation of Fact 2 will be more convenient.

Fact 2′. *The numbers $\sigma_1, \ldots, \sigma_k$ are the cotangents of the principal angles between the range space and the null space of the projector P.*

We call the angle θ_1 defined by the relation

$$\cot\ \theta_1 = \sigma_1$$

the *first* principal angle between the range space and the null space of P. The assertion below is immediate from (4) and Fact 2′.

Fact 3. *The norm of P is equal to the cosecant of the first principal angle between $\mathscr{R}(P)$ and $\mathscr{N}(P)$.*

Fact 4 is a simple consequence of the CS decomposition (see [2, p. 75]).

Fact 4. *Let $\mathscr{L}$ and $\mathscr{M}$ be subspaces of $\mathbb{C}^n$. Then the nonzero principal angles between $\mathscr{L}$ and $\mathscr{M}$ are equal to the nonzero principal angles between their orthogonal complements $\mathscr{L}^{\perp}$ and $\mathscr{M}^{\perp}$.*

We now turn to formulas (1) and (2). Recall that the matrices A and Z in these formulas are real. We set

$$\mu(D) = \| A^{\mathrm{T}}(ADA^{\mathrm{T}})^{-1}AD \|, \tag{5}$$

$$\nu(D) = \| Z^{\mathrm{T}}(ZDZ^{\mathrm{T}})^{-1}ZD \| . \tag{6}$$

Then

$$\chi(A) = \sup_{D\in\mathscr{D}_n} \mu(D),$$

$$\chi(Z) = \sup_{D\in\mathscr{D}_n} \nu(D).$$

Let the columns of $Q_1 \in \mathbb{R}^{n\times r}$ and $Q_2 \in \mathbb{R}^{n\times(n-r)}$ define orthonormal bases for $\mathscr{R}(A^{\mathrm{T}})$ and $\mathscr{N}(A)$, respectively. The following two facts are very easy to check. Thus, we give only their formulations.

Fact 5. *The matrix*

$$\pi_1 = A^{\mathrm{T}}(ADA^{\mathrm{T}})^{-1}AD \tag{7}$$

is a projector with

$$\mathscr{R}(\pi_1) = \mathscr{R}(A^{\mathrm{T}}) = \mathscr{R}(Q_1) \tag{8}$$

and

$$\mathscr{N}(\pi_1) = \mathscr{R}(D^{-1}Q_2). \tag{9}$$

Fact 6. *The matrix*

$$\pi_2 = Z^{\mathrm{T}}(ZDZ^{\mathrm{T}})^{-1}ZD \tag{10}$$

is a projector with

$$\mathscr{R}(\pi_2) = \mathscr{R}(Z^{\mathrm{T}}) = \mathscr{R}(Q_2) \tag{11}$$

and

$$\mathscr{N}(\pi_2) = \mathscr{R}(D^{-1}Q_1). \tag{12}$$

3. The proof

Equality (3), i.e.,

$$\sup_{D\in\mathscr{D}_n} \mu(D) = \sup_{D\in\mathscr{D}_n} \nu(D)$$

is an immediate implication of the relation

$$\mu(D) = \nu(D^{-1}).$$

To prove the latter, observe that $\mu(D)$ is the cosecant of the first principal angle between the subspaces (8) and (9), where π_1 is the projector (7), and $\nu(D^{-1})$ is the cosecant of the first principal angle between $\mathscr{R}(Q_2)$ and $\mathscr{R}(DQ_1)$ (see (11) and (12)). These two subspaces are the orthogonal complements of $\mathscr{R}(\pi_1)$ and $\mathscr{N}(\pi_1)$, respectively. It remains to apply Fact 4.

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