A Network Model Approach for the Degree Correlation Mixing Pattern

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Abstract

Many social, biological or technological systems are recognized as complex networks with assortative or disassortative mixing pattern, which is lacking in typical theoretical network models, such as, Watts-Strogatz small-world model or Barabási-Albert scale-free network. In this paper, we propose a mechanism which models the emergence of the degree correlation property of complex networks. Numerical simulations indicate that this correlation-adjustable network model can exhibit fundamentally different degree correlation mixing patterns with a scale-free degree sequence. Moreover, the structural properties of networks, such as the average path length and clustering property are also investigated.

Keywords: Degree correlation; complex network; assortative; power-law

1. Introduction

Complex networks are the abstract representations of the real complex systems, in which the nodes represent typical system units and the connections are the interactions between pairs of units[1, 2]. The research of complex networks has interested a wide variety of fields, such as sociology, biology, economics, and technology[3]. Based on the empirical studies of networked systems, two distinct characters are discovered: the small-world property[3], which describes an average path length of a network is increased slowly with the
network size, and a scale-free degree distribution \[4\], which means the probability of a node in the network with \( k \) connections follows a power-law behavior \( P(k) \sim k^{-\gamma} \), where \( \gamma \) is the degree exponent.

Apart from the above two typical network properties, researchers also found that there are several different sorts of nodes in such complex networks, and the connections between nodes are often related with the sorts. The nodes are selectively linked to the nodes in the same sorts, such as, for instance, in many social networks people tend to associate preferentially with people who are similar to themselves in some way, which is called assortative mixing \[5, 6\]. On the other hands, in some information networks or technological networks, the dynamical nodes show a tendency to connect with different nodes, which is defined as disassortative mixing \[6\]. The dynamical processes, such as epidemic spreading, have been widely investigated in correlated scale-free complex networks \[7–10\].

The Pearson coefficient \( r \) is widely accepted as an index to measure the degree assortative \[5\]:

\[
r = \frac{\sum_{i=1}^{M} k_i j_i - \sum_{i=1}^{M} k_i \sum_{i=1}^{M} j_i}{\sqrt{\sum_{i=1}^{M} k_i^2 - \left(\frac{\sum_{i=1}^{M} k_i}{M}\right)^2} \sqrt{\sum_{i=1}^{M} j_i^2 - \left(\frac{\sum_{i=1}^{M} j_i}{M}\right)^2}}
\]

(1)

where \( k_i \) and \( j_i \) are the degrees of the end points of connection \( i \), respectively, and \( M \) is the number of the connections in the network. Positive values of \( r \) indicate assortative mixing, which means two nodes with similar degrees are tended to be connected together; while negative values refer to disassortative networks in which different nodes tend to be linked. If \( r \) is near zero, it implies that the network appears to be uncorrelated. It is interestingly observed that essentially all social networks measured appear to be assortative, but other networks, such as, information networks, technological networks, or biological networks, appear to be disassortative \[6\].

However, the typical theoretical network models, such as, Barabási-Albert scale-free network, do not work in revealing such degree correlation property. Accordingly, constructing a complex network with designated degree-degree correlation pattern is the first important step to study and understand how the correlation impacts on dynamics of a complex system. At this respect, Newman \[5\] obtained a positive correlation network model, based on generating functions and Monte Carlo algorithm. Another algorithm of rewiring the edges in BA network to change correlation parameter was also proposed \[11\]. The problem is, when the network size is large, the time complexity of this algorithm is \( \sim O(M^2) \) and the amount of computation is enormous. In this paper we propose a new network model algorithm to generate a complex network with adjustable degree correlation coefficient, in which the positive or negative degree correlation coefficient could be both obtained. Based on the scale-free degree sequence in the network nodes, we generate the assortative and disassortative mixing patterns with different node degree power law exponents, and investigate the corresponding network structure properties. We find that the models with extremely disassortative mixing pattern easily generate isolated subgraphs. The network average path lengths are monotonically decreasing as the degree correlation varying from disassortative to assortative. Moreover, the cluster coefficient of the novel correlation-adjustable network model does well describe the clustering property of most real-world networks.

This paper is organized as follows. In the section 2, the generating algorithm of a correlation-adjustable model is proposed. The novel network models with different scale-free degree distributions are constructed and compared in section 3. In section 4, the average path length and the cluster coefficient are investigated. Finally, section 5 concludes the whole paper.
2. Correlation-Adjustable Network Modeling

The correlation-adjustable network model is generated as follows. First of all, according to the generating function\[12\], we build the degree sequence with arbitrary distribution \(D = \{d_1, d_2, \ldots, d_N\}\), which could be a power law or any random distribution. For simplicity, the indexes of the nodes are sorted such that the degrees have a descending order, i.e., \(d_1 > d_2 > \ldots > d_N\), and \(d_i\) is endowed with the \(i\)th node as its degree. Secondly, from the first node, as each node is attached to another node, the probability to the ‘biggest’ (i.e., with the maximum degree) nodes is \(\alpha (0 \leq \alpha \leq 1)\), and to the ‘smallest’ (with the minimum degree) node is \(1-\alpha\), till the degree value denoted to this node has been all considered. Note that, here, we consider that the network models are undirected and without self-loop, the multiple connections between pairs of nodes or self-loops are not allowed. At last, repeat the above second step procedure until all \(N\) nodes complete their connections. We obtain a correlation-adjustable network with assigned degree distribution, which the correlation is determined through altering the value of \(\alpha\).

According to Eq. (1), the value of \(r\) is depended on the value of \(\sum kj\) when the degrees of nodes in the network are fixed. From the sorting principle\[13\], the result of two vectors’ product in well-ordering is larger than the value of in negative order, and that of in randomly-order is between the above two cases. Therefore, the larger \(\sum kj\) is, the larger value of assortative coefficient \(r\) is. So when \(\alpha\) is large, we will get a network model with positive assortative mixing; on the contrary, when \(\alpha\) is close to 0, \(r\) tends to be negative, we can get a disassortative mixing pattern.

It should be noted that we start from the ‘biggest’ node (\(d_1\) is the largest value in the degree sequence) which contribute to appearing isolated subgraphs with extremely low probability, except when \(\alpha = 0\). We will explain this special case in section 4.

3. Simulation Results

We generate the network degrees with the power-law distribution from the generating function\[12\] as its exponential varying. In order to compare the heterogeneous network with homogeneous one, we pick up 3 and 5, respectively. In all numerical computations, the correlation coefficients \(r\) are obtained by averaging 100 groups of models. With varied network scale \(N\), we have the correlation coefficient \(r\) of the network models with different power-law degree sequences, as shown in Fig 1. \(r\) shows a monotonically increase trend with the parameters \(\alpha\) increasing.

When the network degree sequence is heterogeneously distributed (\(\gamma = 3\)), the minimum value of correlation coefficient \(r\) is started around -0.3 to -0.4 (varies as the different network sizes \(N\)) when \(\alpha = 0\) (as shown in Fig 1-a). As the increasing of \(\alpha\), the \(r\) rapidly rises to zero at \(0 < \alpha < 0.05\). When \(\alpha > 0.05\), \(r\) converge to 0.6~0.7. For homogeneous networks \(\gamma = 5\), we get the similar curves as Fig 1-b, when \(0 \leq \alpha \leq 0.1\), \(r\) is rapidly growing from -0.6~0.7 to 0.37~0.65. When \(\alpha > 0.1\), \(r\) slowly rises to near 1.

From Fig 1 we conclude that for the different network scale \(N\), the upward trend of \(r\) shows a good consistency. It is worth noting that, for the network whose degree distribution follows a power-law distribution, the value of \(\alpha\) is far less than 0.5 when \(r\) is equal to 0. This is mainly for the reason that in the power-law distribution, most of the nodes degrees are similarly small and only a very small number of nodes degrees are large. When a link connects with two nodes, the chosen nodes have a high probability of both small degrees, which means the connected nodes have the similar node property. Only when the nodes are
Fig. 1. The degree correlation coefficients $r$ of the correlation-adjust network models with different power-law degree distributions. (a) heterogeneous networks $\gamma = 3$; (b) homogeneous networks $\gamma = 5$.

Intentionally chosen by quite different properties ($\alpha \to 0$), we may obtain the network with disassortative mixing pattern.

For a fixed network scale $N=1000$, $r$ of the networks with different power-law coefficients are visualized in Fig 2. When $\alpha$ increases to a small value ($\ll 0.5$), the curves increase dramatically. As $\alpha > 0.5$, the trend of increment slows down. In particularly, for extremely heterogeneous networks ($\gamma = 2.1$), we even could not obtain assortative mixing. Table 1 concludes the ranges of the $r$ in different scale-free networks. We find that the variation range of $r$ is reduced as $\gamma$ decreases.

Fig. 2. The degree correlation coefficients $r$ of the scale-free networks with exponent $\gamma = 2.1, 2.2, 2.3, 3$ and $5$. $N = 1000$. 
Table 1. The minimum and maximum values for $r$ when $\gamma=5, 3, 2.3, 2.2$ and $2.1$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Min($r$)</th>
<th>Max($r$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>-0.66</td>
<td>0.94</td>
</tr>
<tr>
<td>3</td>
<td>-0.29</td>
<td>0.71</td>
</tr>
<tr>
<td>2.3</td>
<td>-0.73</td>
<td>0.32</td>
</tr>
<tr>
<td>2.2</td>
<td>-0.72</td>
<td>0.22</td>
</tr>
<tr>
<td>2.1</td>
<td>-0.67</td>
<td>-0.12</td>
</tr>
</tbody>
</table>

4. Network Topological Properties

We take $\gamma=2.3$, and $N=1000$ in the construction of a correlation-adjustable model as an example to consider the network structural properties of networks with different degree correlations. The average path length and the clustering coefficients are revealed in Fig 3. According to our modeling mechanism, when $\alpha = 0$, the big nodes are only connected with the small ones, which would cause such situation: In a scale-free network, each big node is surrounded by many small nodes, but these few hubs are isolated from with each other. The whole network is separated to several ‘quasi-star’ sub graphs. When $\alpha = 1$, the hubs are almost fully connected with each other, those small nodes are around this ‘core’. In this case, the network is compact, so the distance between any two nodes is shortened. Therefore, as $\alpha$ changing from 0 to 1, the network degree correlation is altered from disassortative to assortative, correspondingly, the average path length shows an obviously decreasing tendency. The values of clustering coefficient $C$ are mainly focused on 0.01 to 0.3, which is well fitted with the clustering property of most of real networks. The similar results could be found as $\gamma=3$ and $5$.

![Fig 3. The network structural properties, the clustering coefficient $C^{[14]}$ and the average path length $d$, of the correlation-adjustable networks, as $\gamma = 2.3$, $N=1000$.](image_url)
5. Conclusion

In this paper, we have demonstrated an approach to obtain a new network model with an alterable degree correlation. By adjusting a parameter $\alpha$, a scale-free network with assortative or disassortative mixing pattern is easily generated. In the light of the power-law exponent of degree distribution, the correlation coefficient $r$ of proposed correlation-adjustable mode has different variation ranges. More heterogeneous degree distribution, less variation region of $r$. Through analyzing its structural properties, we found assortative mixing networks are more compact than disassortative ones. The former has much shorter average path length than the latter. Furthermore, the clustering coefficient $C$ of resulting model shows that it is a better realistical model for many real networks. And, the proposed modeling algorithm is simple and easy to achieve, with small of time complexity. Further work about the dynamics of such network models is necessary in this direction.

Acknowledgements

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References