Characteristic-Free Representation Theory of
the General Linear Group
II. Homological Considerations

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INTRODUCTION

In our first paper of this series [1], we indicated how we were led to consider resolutions of Schur and Weyl modules of a particular form. In order to prove the existence of these resolutions, we were forced to enlarge the family of skew shapes to a class $J$ containing new shapes whose corresponding Schur and Weyl modules had heretofore not been studied. With these shapes in hand, we proved in [1] the existence of some fundamental exact sequences and described how, from these exact sequences, we could use a mapping cone construction to build up the resolutions we were seeking. For this mapping cone construction, we needed maps, and to provide maps we needed projectivity of tensor products of divided powers. This led to the study of the Schur algebra and its decomposition into orthogonal idempotents. In Sections 1 and 2 we review the information about the Schur algebra that we need to carry out our program. Fortunately there is a very clear and detailed exposition of this subject in the notes of J. A. Green [5] from which we borrowed very heavily.¹ In fact, the main function of the first two sections is to condense and translate into our notation and terminology the relevant sections of Green's notes.

In Sections 3 and 4 we define the family of shapes, $J$, that we will study,

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¹ See the Appendix of this paper for a guide to converting some of our notation into that of [5].
and prove the existence of resolutions of Weyl modules in terms of sums of tensor products of divided powers. Section 5 is a very brief review of the Universal Coefficient Theorem in the form that we use repeatedly in subsequent sections. In Section 6 we move from Weyl to Schur modules and from divided powers to exterior powers. In fact we define a functor $\Omega$ from the category of sums of tensor products of divided powers to the corresponding category of exterior powers and show, in particular, that $\Omega$ induces an isomorphism from $\text{Hom}_A(D_\lambda, D_\mu)$ to $\text{Hom}_A(A_\lambda, A_\mu)$ where $A$ is the Schur algebra. This enables us to prove the existence of resolutions of Schur modules in terms of exterior powers. Section 7 continues the study of the functor $\Omega$ from the category of Weyl modules to the category of Schur modules, and extends the result about the isomorphism between $\text{Hom}_A(D_\lambda, D_\mu)$ and $\text{Hom}_A(A_\lambda, A_\mu)$ to the statement that $\Omega$ induces an isomorphism $\text{Ext}_A^i(K(\alpha), K(\beta)) \rightarrow \text{Ext}_A^i(L(\alpha), L(\beta))$ for all $i \geq 0$, where $\alpha, \beta$ are shapes in $I$, $K(\cdot)$ is the Weyl module and $L(\cdot)$ is the Schur module of designated shape.

In Section 8 we prove that Schur algebras over a field and over $\mathbb{Z}$ have finite global dimension. Section 9 contains a computation of a special $\text{Ext}^1$ and indicates how this provides some information about intertwining numbers. Finally, in Section 10 we include a sketch of some results that we hope will lead to more explicit information about the resolutions we have introduced.

We conclude this introduction by tying in the content of Sections 6 and 7 with the introduction to the first paper [1] of this series where various connections with symmetric polynomials were discussed. There is an important involutory ring automorphism $\omega$ on the ring of symmetric functions in a countably infinite set $\{x_1, x_2, \ldots\}$ of variables (see [8, I.2]). The involution $\omega$ takes the elementary symmetric function

$$e_i(x) = \sum_{i_1 < \cdots < i_i} x_{i_1} \cdots x_{i_i}$$

to the complete symmetric function

$$h_i(x) = \sum_{i_1 \leq \cdots \leq i_i} x_{i_1} \cdots x_{i_i}$$

and the Schur function $s_{\lambda/\mu}$ to $s_{\lambda/\mu}$ where $\lambda^\prime$ denotes the transpose of a partition $\lambda$ in the sense of Young diagrams (see Sect. 2 of [2]). Moreover, the involution $\omega$ preserves the classical scalar product on symmetric functions.

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This result has been independently obtained by S. Donkin. A proof is contained in the paper “On Schur Algebras and Related Algebras” to appear in the Journal of Algebra in two parts. In fact he proves that “generalized Schur algebras” over a p.i.d have finite global dimension for which he gives an explicit bound.
The homogeneous component in degree $r$ of the ring of symmetric functions is the formal character group of the Schur algebra $A_R(n, r)$, the universal algebra for the homogeneous polynomial representations of degree $r$ of the algebraic group scheme $\text{GL}_n(R)$ over a principal ideal domain $R$, provided that the inequality $n \geq r$ is satisfied. In the classical case where $R$ is a field of characteristic zero, the Schur algebra $A = A_R(n, r)$ is semisimple and the inner product of two symmetric functions is just the intertwining number of the corresponding $A$-modules. In fact, it can be realized as an exact involutory functor $\Omega$ on the finite dimensional modules over $A$. The situation over arbitrary fields or rings is quite a bit more complicated. However, as observed earlier, we do construct a functor $\Omega$ from Weyl modules to Schur modules which preserves extension groups over $A$. Combining this with the contravariant duality discussed at the end of Section 2, we get natural isomorphisms

$$\text{Ext}^i_A(K(\alpha), K(\beta)) \cong \text{Ext}^i_A(K(\bar{\beta}), K(\bar{\alpha}))$$

for all $i \geq 0$, for any Schur algebra $A = A_R(n, r)$ with $n \geq r$, and for any pair of skew shapes $\alpha, \beta$ of weight $r$, thus obtaining a somewhat surprising extension of the useful classical reciprocity

$$\langle s_\alpha^*, s_\beta^* \rangle = \langle s_\beta, s_\alpha^* \rangle$$

on "intertwining numbers" of skew Schur functions.

1. Schur Algebras

Let $F$ be a free module of rank $n$ over a commutative ring $R$. The symmetric group $\Gamma = \Gamma(r)$ on the set $\{1, \ldots, r\}$ acts on the $r$th tensor power

$$F^{\otimes r} = \bigotimes_{\sigma \in \Gamma} F$$

of the module $F$, on the right, by permutation of tensor factors, i.e.,

$$(x_1 \otimes \cdots \otimes x_r) \sigma = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)}, \quad \sigma \in \Gamma$$

where $\sigma \in \Gamma$ and $x_i \in F$.

**Definition 1.1.** The Schur algebra $A = A_R(n, r)$ is the algebra $\text{End}_\Gamma(F^{\otimes r})$ of endomorphisms of the $\Gamma$-module $F^{\otimes r}$.

For any $R$-module $M$ we have the action of $\Gamma$ on $M^{\otimes r}$, by permutation
of tensor factors as in (2), and a natural map \( \phi: \text{End}_R(M)^{\otimes r} \to \text{End}_R(M^{\otimes r}) \)
of algebras given by

\[
(\alpha_1 \otimes \cdots \otimes \alpha_r)(m_1 \otimes \cdots \otimes m_r) = \alpha_1(m_1) \otimes \cdots \otimes \alpha_r(m_r).
\]

It is easy to see that

\[
[\phi(\sigma)(m)](\sigma) = (\sigma)(\sigma(m))
\]

for all \( \sigma \in \Gamma, \quad \alpha \in \text{End}_R(M)^{\otimes r}, \quad \text{and} \quad m \in M^{\otimes r} \). It therefore follows that if \( \alpha \in \text{End}_R(M)^{\otimes r} \) is an invariant of the group \( \Gamma \), then \( \phi(\alpha) \in \text{End}_R(M^{\otimes r}) \).

Moreover, if \( \phi \) is an isomorphism, it follows from (3) that \( \phi \) induces an isomorphism from the \( \Gamma \)-invariants of \( \text{End}_R(M)^{\otimes r} \) to \( \text{End}_R(M^{\otimes r}) \). In particular, when \( M \) is our free module \( F \) above, the morphism \( \phi \) is an isomorphism so that our Schur algebra \( A = \text{End}_R(F^{\otimes r}) \) is isomorphic to the submodule of \( \Gamma \)-invariants of the free \( R \)-module \( E^{\otimes r} \). Since \( (E^{\otimes r})^\Gamma \) is the submodule of \( r \)-fold symmetric tensors, and since this submodule is naturally isomorphic to the \( r \)th divided power of \( E \), \( D_r(E) \), we can identify the Schur algebra \( A \) with the \( r \)th divided power \( D_r(F) = D_r(\text{End}_R(F)) \).

The Schur algebra \( A = D_r(E) \) possesses a universal property which we now describe. If \( R \) is an infinite field, then a representation \( f: \text{GL}(n, R) \to \text{GL}(m, R) \) is said to be a polynomial representation (respectively homogeneous of degree \( r \)) if the matrix entries of \( f(X) \) are polynomials (respectively homogeneous polynomials of degree \( r \)) in the matrix entries \( x_{ij} \) of the matrix \( X \). Every polynomial representation of \( \text{GL}(n, R) \) decomposes into a direct sum of homogeneous polynomial representations. Identifying \( \text{GL}(n, R) \) with \( \text{GL}(F) \), we have that every homogeneous polynomial representation of \( \text{GL}(F) \)

\[
f: \text{GL}(F) \to \text{GL}(V)
\]
of degree \( r \) can be factored uniquely through the Schur algebra \( A = D_r(\text{End}_R(F)) \) as

\[
\begin{array}{ccc}
\text{GL}(F) & \longrightarrow & \text{GL}(V) \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & \text{End}_R(V)
\end{array}
\]

where \( \alpha(g) = g^{\otimes r} \). Conversely, every finite dimensional representation \( f' \) of the Schur algebra \( A \) gives rise to a unique homogeneous representation \( f \) of \( \text{GL}(F) \) of degree \( r \). Consequently the theory of homogeneous polynomial representations of \( \text{GL}(F) \) of degree \( r \) is exactly the study of finite dimen-
sional representations of the Schur algebra $A$. It should be observed that
the above discussion holds over any commutative ring $R$ once one for-
mulates a suitable definition of a polynomial representation (of degree $r$) of
the algebraic group scheme $GL(F)$.

All of the foregoing material is found in [5, 2] and [7, 4], although the
organization of the material in [5] is slightly different. We will briefly run
through the alternative description of the Schur algebra as given in [5].
(The reader will notice that our Definition 1.1 is Theorem 2.6c in [5].)

Let $F$ and $E$ be as in the preceding paragraphs. Then $E = \text{End}_R(F)$ is
naturally isomorphic to the module $F \otimes F^*$ where $F^*$ denotes the linear
dual $\text{Hom}_R(F, R)$ of the module $F$. We let $c_F$ denote the element of $F \otimes F^*$
which corresponds to the identity map in $F$. The map $\mu: F \otimes F \to F$, given
by composition has as its linear dual $\delta: E^* \to E^* \otimes E^*$ the map $F^* \otimes F \to
F^* \otimes F \otimes F^* \otimes F$ given by $x \otimes y \to x \otimes c_F \otimes y$, where we are using the
natural isomorphism $E^* \cong F^* \otimes F$. This map $\delta$ induces a map $S_r(E^*) \to
S_r(E^* \otimes E^*)$, where $S_r$ denotes the $r$th symmetric power. Furthermore, we
have the map $S_r(E^* \otimes E^*) \to S_r(E^*) \otimes S_r(E^*)$ given by

$$(x_1 \otimes y_1) \cdots (x_r \otimes y_r) \to x_1 \cdots x_r \otimes y_1 \cdots y_r.$$ 

(This map is actually the map $S_r(G \otimes H) \to S_r(G) \otimes S_r(H)$ given generally by
the Cauchy filtration of the symmetric algebra $S(G \otimes H)$ of the tensor
product of any two free $R$-modules [2, III.1].) The composition

$$\Delta: S_r(E^*) \to S_r(E^* \otimes E^*) \to S_r(E^*) \otimes S_r(E^*)$$

makes $S_r(E^*)$ into a coalgebra over $R$. When $R$ is an infinite field $K$, the
$r$th symmetric power $S_r(E^*)$ is naturally isomorphic to the homogeneous
polynomial functions of degree $r$ on the vector space $E = \text{End}_R(F)$ (or
equivalently on $GL(F)$), and the coalgebra $S_r(E^*)$ is precisely the coalgebra
$A_k(n, r)$ described in [5.2.1]. As the dual $D_r(E)^* \text{ of } D_r(E)$ is naturally
isomorphic to $S_r(E^*)$, the dual of the map $\Delta: S_r(E^*) \to S_r(E^*) \otimes S_r(E^*)$
converts the module $D_r(E)$ into an algebra. This algebra structure on $D_r(E)$
coincides with the one obtained by viewing $D_r(E) = (E^*)^r$ as a subalgebra
of $E^{\otimes r}$ which we saw earlier in this section to be naturally isomorphic to
the Schur algebra $\text{End}_F(F^{\otimes r})$.

2. SOME PROPERTIES OF $D_r(F)$

As in the preceding section, we let $F$ be a free $R$-module of rank $n$, and
$E = \text{End}(F) = \text{Hom}_R(F, F)$. The group $GL(F)$ acts on the left and right of $E$
by composing an automorphism $\alpha \in GL(F)$ with an endomorphism $\phi \in E$
on the left and right respectively (i.e., $\alpha \circ \phi$ and $\phi \circ \alpha$). When we identify $E$ with $F \otimes F^*$, these actions of $GL(F)$ on $F \otimes F^*$ translate to the following:

$\alpha \cdot (x \otimes y) = \alpha(x) \otimes y$

$(x \otimes y) \cdot \alpha = x \otimes \alpha^*(y)$.

As a left $GL(F)$ module, $E$ is the direct sum $F \oplus \cdots \oplus F$ of $n$ copies of $F$. It follows that the Schur algebra, $D_\lambda(E)$, decomposes as a left $GL(F)$-module into the direct sum

$$D_\lambda(E) = \bigoplus_\mu D_{\mu}(F) \oplus \cdots \oplus D_{\mu_n}(F),$$

where the summation is taken over all sequences of non-negative integers $\mu = (\mu_1, \ldots, \mu_n)$ of weight $r$. Since the Schur algebra $A = D_\lambda(E)$ is the universal algebra for the homogeneous polynomial representations of $GL(F)$ of degree $r$, (1) is a direct sum decomposition of the Schur algebra $A$ into left ideals (see [5, 3.21]). Consequently we have established the following proposition.

**Proposition 2.1.** Let $F$ be a free $R$-module of rank $n$, and let $A = D_\lambda(E)$ be the Schur algebra corresponding to polynomial representations of $GL(F)$ of degree $r$, where $E = \operatorname{End}_R(F)$. Then for all sequences of non-negative integers $\lambda = (\lambda_1, \ldots, \lambda_n)$ of weight $r$, the module $D_\lambda(F) = D_{\lambda_1}(F) \otimes \cdots \otimes D_{\lambda_n}(F)$ is $A$-projective.

We will now analyze the decomposition (1) more carefully. Choose a basis $f_1, \ldots, f_n$ for $F$, and the dual basis $\phi_1, \ldots, \phi_n$ for $F^*$. For each $i = 1, \ldots, n$, let $F_i = F \otimes \phi_i$ be the submodule of $F \otimes F^*$ consisting of all tensors of the form $f \otimes \phi_i$ with $f \in F$. Then we can write $E$ as an internal direct sum

$$E = F_1 \oplus \cdots \oplus F_n$$

of left $GL(F)$ submodules. Thus (1) can be rewritten as

$$D_\lambda(E) = \bigoplus_\mu D_{\mu}(F_1) \otimes \cdots \otimes D_{\mu_n}(F_n).$$

The element $c_e \in F \otimes F^*$ which corresponds to the identity element $1$ of $E$ under the natural isomorphism $E \simeq F \otimes F^*$ is the sum $\sum f_i \otimes \phi_i$. It is easy to see that $1^{(r)}$ is the identity element of the Schur algebra $D_\lambda(E) = A$ and that

$$1^{(r)} = \sum (f_1 \otimes \phi_1)^{(i_1)} \otimes \cdots \otimes (f_n \otimes \phi_n)^{(i_n)}$$

(4)
under the decomposition (3). Consequently $D_\lambda(F)$ is generated as a left $A$-module by the element

$$f_1^{(\lambda_1)} \otimes \cdots \otimes f_n^{(\lambda_n)},$$

which we will denote by $f_\lambda$.

The choice of basis $\{f_1, \ldots, f_n\}$ for $F$ determines an isomorphism of $GL(F)$ with $GL(n)$, and we let $T$ denote the maximal torus of $GL(F)$ corresponding to the subgroup of diagonal matrices of $GL(n)$. The decomposition $E = F_1 \oplus \cdots \oplus F_n$ in (2) is a direct sum decomposition of $E$ as a right $T$-module. It follows that (3) is also a right $T$-module decomposition. Using this fact, we will show that the modules $D_\lambda(F)$ are induced from $T$ in a way that we shall now make explicit.

Let $H$ be the $R$-linear span in $E = F \otimes F^*$ of the tenors $h_i = f_i \otimes \phi_i$, for $i = 1, \ldots, n$. Since $H$ is a subalgebra of $E$, $D_\lambda(H)$ is a subalgebra of $D_\lambda(E)$. The algebra $D_\lambda(H)$ plays the same role for the algebraic group $T$ that the Schur algebra $D_\lambda(E)$ plays for $GL(F)$, i.e., it is the universal algebra for the homogeneous polynomial representations of $T$ of degree $r$.

We let $R_\lambda$ denote the $R$-span of the idempotent $h_\lambda$ in $H$, and write $H = R_1 \times \cdots \times R_n$ as a direct product of algebras. Similarly, given a sequence of non-negative integers $\lambda = (\lambda_1, \ldots, \lambda_n)$ of weight $r$, we let $R(\lambda)$ denote the $R$-submodule $D_\lambda(R_1) \otimes \cdots \otimes D_\lambda(R_n)$ of $D_\lambda(H)$ generated by the element

$$h_\lambda = h_1^{(\lambda_1)} \otimes \cdots h_n^{(\lambda_n)},$$

which is an idempotent in $D_\lambda(H)$. It is easy to see that

$$D_\lambda(H) = \sum_\lambda R(\lambda)$$

is a decomposition of $D_\lambda(H)$ as a direct product of the algebras $R(\lambda)$. We can view the free $R$-module $R(\lambda)$ of rank one as a two-sided $T$-module because it is a two-sided $D_\lambda(H)$-module. With this point of view we can rewrite (3) as the direct sum

$$D_\lambda(E) = \sum_\lambda D_\lambda(F) \otimes R(\lambda)$$

of $(GL(F), T)$-bimodules.

For convenience, we let $A_T$ denote the subalgebra $D_\lambda(H)$ of the Schur algebra $A = D_\lambda(E)$. It follows immediately from (7) and (8) that $D_\lambda(F)$ is, as we suggested earlier, an induced module, i.e.,

$$D_\lambda(F) = A \otimes_{A_T} R(\lambda)$$

as a left $A$-module.
Now, if $M$ is any left $A$-module, we have

$$\text{Hom}_A(D_\lambda(F), M) = \text{Hom}_{A_T}(R(\lambda), M). \quad (10)$$

From the universal properties of the algebras $A$ and $A_T$ we can conclude that if $M$ is a homogeneous polynomial representation of $GL(F)$ of degree $r$, then

$$\text{Hom}_{GL(F)}(D_\lambda(F), M) = \text{Hom}_{A_T}(R(\lambda), M). \quad (11)$$

It is customary to identify $\text{Hom}_R(R(\lambda), M)$ as a $T$-submodule $M_\lambda$ of $M$, called the weight submodule of $M$ corresponding to the weight $\lambda$, and to refer to non-zero elements of $M_\lambda$ as weight vectors of $M$, at least when $R$ is a field.

Finally, we want to briefly discuss contravariant duality and some consequences. We begin by observing that the composite

$$\text{End}_R(F) \xrightarrow{\cong} F \otimes F^* \xrightarrow{\cong} F^* \otimes F \xrightarrow{\cong} \text{End}_R(F^*)$$

of natural isomorphisms is an algebra anti-isomorphism. It then follows from the discussion in the beginning of Section 1 that the natural isomorphism

$$D_\lambda(\text{End}_R(F)) \xrightarrow{\cong} D_\lambda(\text{End}_R(F^*))$$

is an anti-isomorphism of Schur algebras. If $M$ is any left module over $A = \text{End}_R(F)$, then its linear dual $M^*$ is naturally a right module over $A$, and hence can be made naturally into a left module over $D_\lambda(\text{End}_R(F^*))$. As an example, if we take $M = D_\lambda(F)$, then the natural $R$-module isomorphism $D_\lambda(F)^* \cong S_\lambda(F^*)$ tells us that $M^*$ can be identified with the left module $S_\lambda(F^*)$ over $D_\lambda(\text{End}_R(F^*))$. Now if $M, N$ are left $A$-modules which are free $R$-modules of finite rank, then there is a natural isomorphism between $\text{Hom}_A(M, N)$ and $\text{Hom}_A(N^*, M^*)$. It follows that when $R$ is a field $K$, the linear dual provides a contravariant natural equivalence between finitely generated modules over $D_\lambda(\text{End}(F))$ and $D_\lambda(\text{End}(F^*))$. Consequently, the dual $D_\lambda(F^*) = S_\lambda(F^*)$ of the projective $A$-module $D_\lambda(F)$ in Proposition 2.1 is an injective left module over the Schur algebra $D_\lambda(\text{End}_R(F^*))$, which establishes the following proposition.

**Proposition 2.2.** Let $F$, $R$, $n$, $A$, $r$, and $\lambda$ be as in the statement of Proposition 2.1. If $R$ is a field then the module $S_\lambda(F) = S_{\lambda_1}(F) \otimes \cdots \otimes S_{\lambda_n}(F)$ is $A$-injective.

It should be pointed out that a choice of basis of $F$ determines an isomorphism $F \cong F^*$ and hence an isomorphism $D_\lambda(\text{End}(F)) \cong$
$D_r(\text{End}(F^*))$ of Schur algebras. Using this isomorphism, the dual $M^*$ of a left module over $A = D_r(\text{End}_r(F))$ can be given the structure of a left $A$-module, which is denoted by $M^0$ in [5]. In this notation, we have $D_\lambda(F)^0 \cong S_\lambda(F)$ as $A$-modules for any sequence $\lambda$ of non-negative integers.\(^3\)

More generally, we would like to record here the fact that, for any pair $\lambda, \mu$ of partitions, the $A$-modules $K_{i/\mu}(F)^0$ and $L_{\lambda/\mu}(F)$ are isomorphic, as a consequence of the natural isomorphism $K_{i/\mu}(F)^* \cong L_{\lambda/\mu}(F^*)$ (see [2, II.4.1]).

3. THE FAMILY OF SHAPES $J$

Let $J_n$ denote the set of all relative sequences $\gamma = (\lambda_1, ..., \lambda_{n+1})/(\mu_1, ..., \mu_{n+1})$ where $(\lambda_1, ..., \lambda_{n+1})$ and $(\mu_1, ..., \mu_n)$ are partitions, and there exists a non-negative integer $i < n + 1$ such that the following conditions are satisfied:

\[
\begin{align*}
\mu_1 &\geq \mu_2 \geq \cdots \geq \mu_i \geq \mu_{n+1} > \mu_{i+1} \geq \cdots \geq \mu_n, \\
\lambda_n - \lambda_{n+1} &\geq n - i.
\end{align*}
\]

Notice that $\gamma$ is a skew partition if and only if $i = n$. More generally, given a shape $\gamma$ as above, if we let $\alpha$ denote the skew partition

\[
\alpha = (\lambda_1, ..., \lambda_n, \lambda_{n+1} + n - i)/(\mu_1, ..., \mu_i, \mu_{n+1}, \mu_{i+1} + 1, ..., \mu_n + 1),
\]

then $\gamma = \alpha(t; 0)$ where $t = n - i$ (see [1, 6] for notation). Conversely, given a skew partition $\alpha = (\lambda_1, ..., \lambda_{n+1})/(\mu_1, ..., \mu_{n+1})$ with $n + 1$ rows, it is easy to see that the shape $\alpha(t; 0)$

\[
(\lambda_1, ..., \lambda_n, \lambda_{n+1} - t)/(\mu_1, ..., \mu_i, \mu_{i+2} - 1, ..., \mu_{n+1} - 1, \mu_{i+1}),
\]

where $t = n - i$, belongs to $J_n$. So $J_n$ is exactly the family of all shapes of the form $\alpha(t; 0)$ where $\alpha$ is a skew partition with $n + 1$ rows, and $0 \leq t \leq n$. The integer $i$ in (1) is uniquely determined by $\gamma$ and will occasionally be denoted by $i(\gamma)$.

It will be important for us to know that shapes of the form $\alpha(t; 1)$ (see [1] again for notation) are also in $J_n$. In the notation of (4), the shape $\alpha(t; 1)$ can be written as

\[
(\lambda_1, ..., \lambda_n, \lambda_{n+1} - t - 1)/
(\mu_1, ..., \mu_i, \mu_{i+2} - 1, ..., \mu_{n+1} - 1, \mu_{i+1} - 1).
\]

\(^3\) Using the notation of [5], this is equivalent to the fact that $S_\lambda(F)$ is isomorphic to the right $\lambda$-weight subspace $^4A_\lambda(n, r)$, defined in [5, 4.5b], from which Proposition 2.2 also follows.
We let \( k \leq n \) be the largest integer such that \( \mu_{i+1} = \mu_{k+1} \), and we take \( \beta \) to be the skew partition
\[
\beta = (\lambda_1, \ldots, \lambda_n, \lambda_{n+1} - (k - i) - 1) / \\
(\mu_1, \ldots, \mu_i, \mu_{i+1} - 1, \ldots, \mu_{k+1} - 1, \mu_{k+2}, \ldots, \mu_{n+1}).
\]

(6)

It is then easy to see that \( \alpha(t; 1) = \beta(s; 0) \) where \( s = n - k \leq n - i = t \), and we know \( \beta(s; 0) \) is in \( J_n \).

We let \( J \) denote the union of the families \( J_n \) for all integers \( n \geq 0 \). For any shape \( \gamma \) in \( J \) we define \( j(\gamma) \) to be the sum of the number of overlaps in each pair of rows of \( \gamma \). More precisely, if \( \gamma \in J_n \) and we take \( 1 \leq i_1 < i_2 \leq n + 1 \), then the number of overlaps between the \( i_1 \)th and \( i_2 \)th rows of \( \gamma \) is
\[
\begin{align*}
[\lambda_{i_2} - \mu_{i_1}] & \quad \text{if } i_2 < n + 1 \text{ or if } i_1 \leq i(\gamma) \\
\lambda_{n+1} - \mu_{n+1} & \quad \text{if } i_2 = n + 1 \text{ and } i_1 > i(\gamma),
\end{align*}
\]

(7)

where \( [\lambda_{i_2} - \mu_{i_1}] = \max(0, \lambda_{i_2} - \mu_{i_1}) \).

Notice that if \( j(\gamma) = 0 \), then \( \lambda_{n+1} = \mu_{n+1} \) and \( \gamma \) is equivalent to the skew partition \( \beta = (\lambda_1, \ldots, \lambda_n)/(\mu_1, \ldots, \mu_n) \) with the property that \( \lambda_{i_2} \leq \mu_i \) for all \( 1 \leq i_1 < i_2 \leq n \). In other words, the path components of the shape \( \beta \) are exactly the rows of \( \beta \), and so we have \( K_\gamma(F) = D_\gamma(F) \) and \( L_\gamma(F) = A_\gamma(F) \) when \( j(\gamma) = 0 \).

We need one more observation about the function \( j(\gamma) \). Let \( \gamma = \alpha(t; 0) \) as in (4), and let \( h \) be the number of entries from among \( \mu_1, \ldots, \mu_i \) (recall that \( \gamma \in J_n \) and \( t = n - i \)) which are less than \( \lambda_{n+1} - t \). When \( h > 0 \), it is not hard to see that \( j(\alpha(t + 1; 0)) = j(\alpha(t; 1)) = j(\alpha(t; 0)) - h \). Recall that if \( \alpha = (\lambda_1, \ldots, \lambda_{n+1})/(\mu_1, \ldots, \mu_{n+1}) \), then
\[
\begin{align*}
\alpha(t; 0) = (\lambda_1, \ldots, \lambda_n, \lambda_{n+1} - (n - i))/ \\
(\mu_1, \ldots, \mu_i, \mu_{i+2} - 1, \ldots, \mu_{n+1} - 1, \mu_{i+1}); \\
\alpha(t; 1) = (\lambda_1, \ldots, \lambda_n, \lambda_{n+1} - (n - i) - 1)/ \\
(\mu_1, \ldots, \mu_i, \mu_{i+2} - 1, \ldots, \mu_{n+1} - 1, \mu_{i+1} - 1); \\
\alpha(t + 1; 0) = (\lambda_1, \ldots, \lambda_n, \lambda_{n+1} - (n - i) - 1)/ \\
(\mu_1, \ldots, \mu_i, \mu_{i+1} - 1, \ldots, \mu_{n+1} - 1, \mu_i).
\end{align*}
\]

Thus, to compare \( j(\alpha(t; 0)) \) and \( j(\alpha(t; 1)) \), one simply has to compare overlaps with the last row, and the condition that \( \lambda_{n+1} - t > \mu_{i-h} \geq \cdots \geq \mu_i \) shows that in these \( h \) rows the number of overlaps decreases by exactly one. The comparison of overlaps in \( \alpha(t; 1) \) and \( \alpha(t + 1; 0) \) is effected by
observing that the overlaps lost by shortening the last row in going from \(\alpha(t;1)\) to \(\alpha(t+1;0)\) are gained by lengthening the \(j\)th row of \(\alpha(t+1;0)\) by the same amount.

4. The Resolutions \(D(\gamma, F)\)

Let \(F\) be a free \(R\)-module of finite rank, and let \(A\) denote the Schur algebra \(D(\text{End}_R(F))\) of degree \(r\). Given a shape \(\gamma\) in \(J_n\) of weight \(r\), we will construct a resolution \(D(\gamma, F))\) of the Weyl module \(K_\gamma(F)\) over \(A\). The terms of the complex \(D(\gamma, F)\) will consist of direct sums of modules of the form

\[ D_\lambda(F) = D_{\lambda_1} F \otimes \cdots \otimes D_{\lambda_{n+1}} F, \]

where \(\lambda\) is a sequence \(\lambda = (\lambda_1, \ldots, \lambda_{n+1})\) of non-negative integers of weight \(r\) and the length of \(D(\gamma, F)\) will be at most \(j(\gamma)\).

The constructions will proceed by induction on \(j(\gamma)\) and the number \(n\). When \(j(\gamma) = 0\), we have \(D_\lambda(F) = K_\lambda(F)\), so that we can take \(D(\gamma, F)\) to be the complex \(0 \rightarrow D_\lambda(F) \rightarrow 0\) of length zero. Observing that \(j(\gamma) = 0\) when \(n = 0\), let us assume that \(j(\gamma) > 0\) and \(n > 0\).

We will first take care of the case when \(\text{rank} \ F > n\) and later derive from this the general case. The reason for this is that we want to guarantee that the modules (1) be projective \(A\)-modules. At the end of this section we will show that modules of type (1) need not be projective if \(\text{rank} \ F \leq n\).

Since the shape \(\gamma\) is in \(J_n\), we can represent \(\gamma\) in the form \(\alpha(t;0)\) where \(\alpha = (\lambda_1, \ldots, \lambda_{n+1})/(\mu_1, \ldots, \mu_{n+1})\) is a skew partition. Therefore we have

\[ \gamma = (\lambda_1, \ldots, \lambda_n, \lambda_{n+1} - t)/(\mu_1, \ldots, \mu_2, \mu_{i+2}, \mu_{i+3}, \ldots, \mu_{n+1} - 1, \mu_{i+1}), \]

where \(t = n - i\).

When \(\lambda_{n+1} \leq \mu_n\), all the shapes \(\alpha(t;0)\) are empty except for \(\alpha(0;0) = \alpha\), and it is easy to see that \(K_\gamma(F) = K_\beta(F) \otimes D_{\alpha_{n+1}}(F)\) where \(\beta\) is the skew partition \((\lambda_1, \ldots, \lambda_n)/(\mu_1, \ldots, \mu_n)\) and \(\alpha_{n+1} = \lambda_{n+1} - \mu_{n+1}\). By induction on \(n\) we know that \(K_\beta(F)\) has a resolution \(D(\beta, F)\) of length at most \(j(\beta) = j(\alpha)\), and so we can take \(D(\alpha, F)\) to be complex \(D(\beta, F) \otimes D_{\alpha_{n+1}}(F)\). Thus, we can assume that \(\gamma = \alpha(t;0)\), and that \(\lambda_{n+1} > \mu_n\).

Our next step is to exhibit a short exact sequence

\[ 0 \rightarrow K_\sigma(F) \rightarrow K_\tau(F) \rightarrow K_\rho(F) \rightarrow 0, \]

where \(\sigma\) and \(\tau\) are shapes in \(J_n\) of weight \(r\) with \(j(\sigma)\) and \(j(\tau)\) less than \(j(\gamma)\).

For convenience we shall treat the case \(t = 0\) separately, i.e., \(\gamma = \alpha(0;0)\).

Let \(k\) denote the number of terms of \((\mu_1, \ldots, \mu_n)\) such that \(\mu_k < \lambda_{n+1}\). The
assumption that \( \mu_1 < \lambda_{n+1} \) guarantees that \( h > 0 \). We can therefore take \( \tau = \alpha(0; 1) \), \( \sigma = \alpha(1; 0) \), and obtain from \([1, 7.5]\) the desired short exact sequence (3). The fact that \( j(\tau) \) and \( j(\iota) \) are less than \( j(\gamma) \) has already been noted at the end of Section 3.

We are tempted to repeat, almost verbatim, the argument we have just given when \( \gamma = \alpha(t; 0) \) with \( t = n - i > 0 \). We can certainly assume, still using the notation of (2), that \( \mu_{i+1} < \lambda_{n+1} - t \), for equality would imply that \( K_\gamma(F) = K_\beta(F) \) where \( \beta \) is the skew partition

\[
(\lambda_1, \ldots, \lambda_n) / (\mu_1, \ldots, \mu_i, \mu_{i+2} - 1, \ldots, \mu_{n+1} - 1)
\]

in \( J_{n-1} \), and then we would take \( D(\gamma, F) \) to be \( D(\beta, F) \). If we let \( h \) denote the number of terms of \( (\mu_1, \ldots, \mu_i) \) such that \( \mu_k > \lambda_{n+1} - t \), and if \( h > 0 \) as it was in the case of \( t = 0 \), then again we can take \( \tau = \alpha(t; 1) \), \( \sigma = \alpha(t + 1; 0) \) and, invoking \([1, 7.5]\) together with our remark at the end of Section 3, we obtain the exact sequence (3) that we are looking for. The only snag in the argument is that we might have \( h = 0 \) (as is always the case, for instance, if \( t = n \)). If this happens, there is a standard way to "adjust" \( \gamma \) to obtain an equivalent shape for which the corresponding number \( h \) is positive. This modification proceeds as follows.

We are assuming that \( \mu_1 \geq \cdots \geq \mu_i \geq \lambda_{n+1} - t \), and that \( \mu_{i+1} < \lambda_{n+1} - t \). Let \( p \) be the largest integer between \( i \) and \( n-1 \) such that \( \mu_{i+2} = \mu_{p+2} \), and let \( q = \mu_{i+1} - \mu_{i+2} + 1 = \cdots = \mu_{i+1} - \mu_{p+2} + 1 \). If we set

\[
\gamma' = (\lambda_1, \ldots, \lambda_n, \lambda_{n+1} - t - q) / (\mu_1, \ldots, \mu_2, \mu_{i+2} - 1, \ldots, \mu_{n+1} - 1, \mu_{i+1} - q),
\]

it is clear that \( \gamma' \) and \( \gamma \) are equivalent, i.e., \( K_\gamma(F) \cong K_\iota(F) \) and \( j(\gamma') = j(\gamma) \). Furthermore, if we take \( \beta \) to be the skew partition

\[
(\lambda_1, \ldots, \lambda_n, \lambda_{n+1} - q - (p - i + 1)) / (\mu_1, \ldots, \mu_i, \mu_{i+1} - q, \mu_{i+2} - 1, \ldots, \mu_{p+2} - 1, \mu_{p+3}, \ldots, \mu_{n+2})
\]

then \( \gamma' = \beta(s; 0) \) where \( s = n - (p + 1) \).

If \( p = i \), the number of terms of \( (\mu_1, \ldots, \mu_i, \mu_{i+1} - q) \) which are less than \( \lambda_{n+1} - t - q \) is precisely \( p - i + 1 = 1 > 0 \). If \( p > 1 \), the number of terms of \( (\mu_1, \ldots, \mu_i, \mu_{i+1} - q, \mu_{i+2} - 1, \ldots, \mu_{p+1} - 1) \) which are less than \( \lambda_{n+1} - t - q \) is precisely \( p - i + 1 > 0 \). Thus we may now choose \( \tau = \beta(s; 1) \) and \( \sigma = \beta(s + 1; 0) \) to obtain the desired exact sequence (3).

Having established the existence of the sequence (3), we can now construct the resolution \( D(\gamma, F) \). By induction on \( j(\gamma) \), there exist resolutions \( D(\sigma, F) \) and \( D(\tau, F) \) of \( K_\sigma(F) \) and \( K_\iota(F) \). Since \( D(\sigma, F) \) is a projective resolution over \( A \), the injection \( K_\sigma(F) \rightarrow K_\iota(F) \) can be lifted (by com-
parison theorem) to a map of complexes $D(\sigma, F) \to D(\tau, F)$ which is unique up to chain homotopy. We take $D(\gamma, F)$ to be the mapping cone of any such lifting and obtain a resolution of $K_\gamma(F)$ of length at most $j(\gamma)$. Since any two liftings are homotopic, the complex $D(\gamma, F)$ is determined up to isomorphism. This completes the construction of $D(\gamma, F)$ in the case $\text{rank } F \geq n + 1$.

Now let us remove the restriction that $\text{rank } F \geq n + 1$. Let $F'$ be a free $R$-module of rank $\geq n + 1$ such that $F$ is a summand of $F'$, and let $A'$ denote the Schur algebra $D_\gamma(\text{End}_R(F'))$. Then, using the arguments of [5, pp. 83, 102 ff], we have a "restriction" functor $C_{A'} \to C_A$ from the category of $A'$-modules to the category of $A$-modules, which is exact. We therefore may take $D(\gamma, F)$ to be the restriction of the resolution $D(\gamma, F)'$ over $A'$. As the discussion in [5] shows, the result is independent of the choice of $F'$ so that we can view $D(\gamma, F)$ as a resolution in the category of polynomial functors [8].

We conclude this section with the promised example of a module of the form $D_{a_1}F \otimes \cdots \otimes D_{a_{n+1}}F$ which is not $A$-projective when $\text{rank } F \leq n$. The simplest such case occurs when $\text{rank } F = 2$ and the module in question is $F \otimes F \otimes F$. Now it can easily be shown that for any free module $F$, we have the exact sequence

$$0 \to D_3F \xrightarrow{\alpha} D_2F \otimes F \otimes F \otimes D_2F \xrightarrow{\nu} F \otimes F \otimes F \xrightarrow{u} A^3F \to 0,$$

where $F \otimes F \otimes F \to A^3F$ is multiplication and the other maps are appropriate diagonalizations. If $\text{rank } F = 2$, then $A^3F = 0$ so that we have the short exact sequence

$$0 \to D_3F \xrightarrow{\alpha} D_2F \otimes F \otimes F \otimes D_2F \xrightarrow{\nu} F \otimes F \otimes F \otimes F \to 0.$$

Suppose that $F \otimes F \otimes F$ were $A$-projective. Then the above sequence would split and we would have maps $\alpha: D_2F \otimes F \to D_3F$, $\beta: F \otimes D_2F \to D_3F$ such that the composition

$$D_3F \xrightarrow{\alpha} D_2F \otimes F \otimes F \otimes D_2F \xrightarrow{\alpha + \beta} D_3F$$

is the identity. Let us compute $\text{Hom}_A(D_2F \otimes F, D_3F)$. By the discussion of Section 2, this is the weight submodule of $D_3F$ corresponding to the weight $(2, 1)$. Since $D_3F$ has basis $f_1(2), f_2(2), f_1f_2(2), f_2^2, f_1(3), f_2(3), f_1f_2(3)$, where $\{f_1, f_2\}$ is a basis of $F$, the $(2, 1)$ eight submodule of $D_3F$ is generated by $f_1f_2(2)$. The $A$-map corresponding to this generator is the multiplication map $D_2F \otimes F \to A^3D_3F$. Thus any map from $D_2F \otimes F \to D_3F$ must be a multiple of $\alpha_0$ by an element of the ground ring $R$. A similar argument leads to the same results for $\text{Hom}_A(F \otimes D_2F, D_3F)$. 
Since the compositions
\[ D_3 F \rightarrow D_2 F \otimes F \rightarrow D_1 F \]
\[ D_3 F \rightarrow F \otimes D_2 F \rightarrow D_1 F \]
are each 3 times the identity map, we see that if 3 is not invertible in \( R \), we can find no maps \( \alpha \) and \( \beta \) such that the composition (*) is the identity. Thus \( F \otimes F \otimes F \) is not \( A \)-projective if \( \text{rank } F = 2 \).

5. Change of Rings and Ext

In this section we recall two well-known elementary facts from homological algebra which will be used in subsequent sections.

Let \( R \rightarrow \bar{R} \) be the homomorphism of commutative rings, and let \( A \) be an \( R \)-algebra. We let \( \bar{A} \) denote the \( \bar{R} \)-algebra \( \bar{R} \otimes_R A \) and, more generally, we let \( \bar{M} \) denote \( \bar{R} \otimes_R M \) for any left \( A \)-module \( M \). Then clearly \( \bar{M} \) is an \( \bar{A} \)-module in a natural way.

For any pair of left \( A \)-modules \( M \) and \( N \), there is a natural homomorphism
\[
\alpha: \bar{R} \otimes_R \text{Hom}_A(M, N) \rightarrow \text{Hom}_A(\bar{M}, \bar{N})
\]
defined by
\[
\alpha(\bar{r} \otimes g)(\bar{s} \otimes m) = \bar{r} \otimes g(m).
\]

**Proposition 5.1.** The map \( \alpha \) is an isomorphism when \( M \) is a finitely generated projective left \( A \)-module. Moreover, if \( \bar{R} \) is a flat \( R \)-module, then \( \alpha \) is an isomorphism for any finitely presented \( A \)-module \( M \).

The proof of the above is elementary, as is that of the following form of the universal coefficient theorem.

**Lemma 5.2.** Let \( R \) be a hereditary commutative ring, \( \mathbb{X} \) an \( R \)-projective complex, and \( R \rightarrow \bar{R} \) a ring homomorphism. Then there is a short exact sequence
\[
0 \rightarrow \bar{R} \otimes_R H^i(\mathbb{X}) \rightarrow H^i(\bar{R} \otimes_R \mathbb{X}) \rightarrow \text{Tor}^R_i(\bar{R}, H^{i+1}(\mathbb{X})) \rightarrow 0
\]
for each \( i \).

With these facts behind us, we can easily prove the following theorem.

**Theorem 5.3.** Let \( R \) be a commutative hereditary ring, \( R \rightarrow \bar{R} \) a
homomorphism of commutative rings, and $A$ an $R$-algebra. Let $M$ and $N$ be left $A$-modules which are free $R$-modules. Furthermore, assume that $M$ has a resolution $P$ over $A$ by finitely generated projective $A$-modules. Then there is a short exact sequence of $R$-modules

$$0 \rightarrow \mathbb{R} \otimes R \text{Ext}_i^j(M, N) \rightarrow \text{Ext}_i^j(\mathbb{M}, \mathbb{N}) \rightarrow \text{Tor}_i^R(\mathbb{R}, \text{Ext}_{i-1}^j(M, N)) \rightarrow 0$$

(2)

for each $i \geq 0$ where, as in 5.1, we have set $\mathbb{A} = \mathbb{R} \otimes A$, etc.

Proof. Let $\mathbb{X}$ be the complex $\text{Hom}_A(P, N)$. Then $H^i(\mathbb{X}) = \text{Ext}_i^j(M, N)$. Since $M$ is $R$-free, the resolution $P \rightarrow M$ is $R$-split, so that $\mathbb{R} \otimes_R P$ is an $\mathbb{A}$-projective resolution of $\mathbb{M} = \mathbb{R} \otimes_R M$. Thus

$$\text{Ext}_i^j(\mathbb{M}, \mathbb{N}) = H^i(\text{Hom}_A(\mathbb{R} \otimes_R P, \mathbb{N})).$$

By 5.1, we know that $\text{Hom}_A(\mathbb{R} \otimes_R P, \mathbb{N}) \cong \mathbb{R} \otimes_R \text{Hom}_A(P, N) = \mathbb{R} \otimes_R \mathbb{X}$, since we are assuming the terms of $P$ to be finitely generated projective $A$-modules.

To apply 5.2 to our situation, we need only show that $\mathbb{X}$ is an $R$-projective complex. But any finitely generated projective $A$-module $P$ is a summand of a direct sum of a finite number of copies of $R$, so that the $R$-module $\text{Hom}_A(P, N)$ is a summand of a finite number of copies of the free $R$-module $N$, and hence is $R$-projective. Thus $\mathbb{X} = \text{Hom}_A(P, N)$ is an $R$-projective complex and 5.2 applies to yield the exact sequence (2).

Remark. From the notation used in the above theorem, the reader will have guessed that the applications we have in mind (see Sect. 7) occur when $A = D_r(E)$ and $M = K_r(F)$ for some shape $\gamma$ in $J$. In that case (at least when rank $F \geq r$), we have seen that $D(\gamma, F)$ is a finitely generated (even finite) projective resolution of $K_r(F)$. Hence, when $R = \mathbb{Z}$, $M = K_r(F)$, and $N$ is any finitely generated $A$-module which is $R$-free, Theorem 5.3 is applicable. It is also worth noting here that if $(\lambda_1, \ldots, \lambda_p)$ is a sequence of non-negative integers of weight $r$, then $A_{\lambda}(F) = K_{\gamma}(F)$ where $\gamma$ is the skew shape $\sigma/\mu$ in $J$ given by

$$\sigma = (p, \ldots, p, p - 1, \ldots, p - 1, \ldots, 1, \ldots, 1)$$

$$\mu = (p - 1, \ldots, p - 1, p - 2, \ldots, p - 2, \ldots, 0, \ldots, 0).$$
6. The Resolutions $\Lambda(y, F)$

Let $F$ be a free $R$-module of finite rank, and let $A$ denote the Schur algebra $D_r(\text{End}_R(F))$. Given a shape $y \in J$ of weight $r$, we will describe a resolution $\Lambda(y, F)$ of the Schur module $L_y(F)$ over $A$. The terms of the complex $\Lambda(y, F)$ consist of direct sums of tensor products

$$A_i(F) = A^{\lambda_1} F \otimes \cdots \otimes A^{\lambda_k} F, \quad \sum \lambda_i = r \quad (1)$$

of exterior powers of $F$.

The resolutions $\Lambda(y, F)$ can be constructed in exactly the same manner as $\mathbb{D}(y, F)$. It is, however, necessary to use a more involved argument in place of the comparison theorem to lift maps because the resolutions $\Lambda(y, F)$ are not projective over $A$. Before describing the lifting argument, we need to make an explicit connection between $\Lambda(y, F)$ and $\mathbb{D}(y, F)$. As in Section 4, we will first assume that the rank of $F$ is at least as large as the degree $r$ of the Schur algebra. This will be sufficient for our purposes because, as in Section 4, we can replace $F$ by a free $R$-module $F'$ of sufficiently high rank which contains $F$ as a summand, and then restrict the resolution $\Lambda(y, F')$ to obtain $\Lambda(y, F)$.

Let $B$ denote the endomorphism algebra $\text{End}_A(F^{\otimes r})$ of the projective left $A$-module $F^{\otimes r} = F \otimes_R \cdots \otimes_R F$. Since rank $F \geq r$, $B$ is naturally isomorphic to the group algebra of the symmetric group $\Gamma(r)$ on the set $\{1, \ldots, r\}$, where the elements of $\Gamma(r)$ act on $F^{\otimes r}$ from the right by permuting tensor factors as described in the beginning of Section 1. This is a well-known fact which is an immediate consequence, for example, of the formula (10) of Section 2 in the special case $\lambda = (1, \ldots, 1, 0, \ldots, 0)$ and $M = F^{\otimes r}$,

$$\text{Hom}_A(F^{\otimes r}, F^{\otimes r}) \cong (F^{\otimes r})_{\lambda},$$

where each permutation $\sigma \in \Gamma(r)$ corresponds to the element $f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(r)}$ of weight $\lambda$ in the notation in Section 2.

The starting point for the connection between $\Lambda(y, F)$ and $\mathbb{D}(y, F)$ is the sign involution $\omega : B \to B$ on the group algebra $B = R[\Gamma(r)]$ which sends each permutation $\sigma$ in $\Gamma(r)$ to $\text{sgn}(\sigma) \sigma$. Before discussing the general case we will briefly review the classical situation where $R$ is a field of characteristic zero, and for convenience we will denote by $M$ the (left $A$, right $B$)-module $F^{\otimes r}$. Over a field of characteristic zero, the group algebra $B$, and hence its centralizer, the Schur algebra $A = \text{End}_R(M)$, are semisimple. Consequently, there is a natural equivalence between the representations of $B$ and $A$, called the Schur functor in [5], which gives a correspondence
"I → MI" between the left ideals of B and the left A-submodules of M. On the other hand, by a change of rings argument, treating B as a B-module via the automorphism ω: B → B, one sees that there is a natural equivalence from the category of left B-modules to itself which sends a left ideal I of B to the left ideal I' = ω(I). Combining this with the Schur functor, one obtains an involutory natural equivalence Ω from the category of representations of A to itself whose effect on A-submodules of M is to send MI to MI'. In particular, Ω exchanges A(F) with D,F = S,F(F). In order to see explicitly the effect of Ω on Hom A(MI, MJ) one chooses idempotents e, f in the algebra B for which I = Be, J = Bf and then embeds the R-module Hom A(Me, MF) in Hom A(M, M) = B using the injection Hom(πe, i', j), where πe is the projection M → Me and i' is the inclusion MF → M. The image of this embedding is the R-submodule eBf of B, and the involution ω clearly maps eBf isomorphically onto ω(e)Bo(f), the embedded image of Hom A(Mω(e), Mω(f)).

We now return to the general case where R is an arbitrary commutative ring. We let C A denote the full subcategory of C A (the category of left A-modules) whose objects are finite direct sums of modules of the type shown in (1). Similarly we let C B denote the full subcategory of C B whose objects are finite direct sums of tensor products of divided powers of F (also of weight r). We define a functor Ω: C A → C B by sending the object D,A(F) in C B to A,F in C A and extending to direct sums in the obvious way. In order to define Ω on the maps, we have to define


To do this, keeping in mind the classical case as a guide, we make use of the sign involution ω on the group algebra B = R[Γ(r)].

If we let aμ: A,F → F⊗r denote the tensor product of the injections AμF → F⊗μμi, and let mμ: F⊗r → A,F denote the tensor product of the multiplication maps F⊗μ → A,F, then we can embed Hom A(A,F, A,F) in B = End A(F⊗r) by the map Hom A(mμ, aμ): Hom A(A,F, A,F) → Hom A(F⊗r, F⊗r). Similarly we can map the R-module Hom A(D,A(F), D,F) into B by the map Hom A(nμ, bμ) where bμ: D,F → F⊗r and nμ: F⊗r → D,A(F) are the appropriate analogues of aμ and mμ.

Using these maps, we can define the map (2) to be the composite map,

\[ \text{Hom}_A(D,A(F), D,F) \xrightarrow{\text{Hom}(nμ, bμ)} B \xrightarrow{\omega} B, \]  

provided that we show that the image of (3) is contained in the image of Hom A(mμ, aμ). However, we know from 5.1 that Hom A(D,A(F), D,F) commutes with change of the ground ring R, so we may assume that R = Z. Since aμ is a Z-split injection, and mμ is a Z-split surjection, it is clear that Hom A(mμ, aμ) is a Z-split injection. Therefore the case R = Z follows from
the classical case $R = \mathbb{Q}$ discussed earlier. In order to illuminate the description of the classical $\Omega$ sketched earlier, we will examine this example in a little more detail.

For any sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of weight $\sum \lambda_i$ equal to $r$, we let $\Gamma(\lambda)$ denote the Young subgroup corresponding to $\lambda$ of the symmetric group $\Gamma(r)$. As an abstract group $\Gamma(\lambda)$ is the product $\Gamma(\lambda_1) \times \cdots \times \Gamma(\lambda_k)$ of symmetric groups and it is viewed as a subgroup of $\Gamma(r)$ by letting $(\sigma_1, \ldots, \sigma_k)$ in $\Gamma(\lambda)$ act on $\{1, \ldots, r\}$ in the usual manner whereby $\sigma_i \in \Gamma(\lambda_i)$ permutes the subset

$$\{\lambda_1 + \cdots + \lambda_{i-1} + 1, \ldots, \lambda_1 + \cdots + \lambda_i\}$$

of $\{1, \ldots, r\}$. One can then define a symmetrizer $e_{\lambda}$ and an antisymmetrizer $e'_{\lambda}$ in the group algebra $B = \mathbb{Q}[\Gamma(r)]$ by setting

$$e_{\lambda} = \sum_{\sigma \in \Gamma(r)} \sigma, \quad e'_{\lambda} = \sum_{\sigma \in \Gamma(r)} \text{sgn}(\sigma) \sigma.$$

Since we are over $\mathbb{Q}$, the $A$-modules $D(\lambda, F)$ and $S(\lambda, F)$ can both be identified with the submodule $(F^{\otimes r}) e_{\lambda}$ of $F^{\otimes r}$. Similarly, $A(\lambda, F)$ can be identified with $(F^{\otimes r}) e'_{\lambda}$. It is worth pointing out that $e_{\lambda} = n_{\lambda} \circ b_{\lambda}$ and $e'_{\lambda} = m_{\lambda} \circ a_{\lambda}$.

Now that the notation is set up, we can proceed with the examination of $\Omega$. An element $T$ of $B = \text{End}_A(F^{\otimes r})$ factors through $S(\lambda, F)$, i.e., is in $\text{Hom}_A(S(\lambda, F), F^{\otimes r})$, if and only if $T$ is invariant under left multiplication by the idempotent $e_{\lambda}/r!$ in $B$, i.e., $T$ is in the right ideal $e_{\lambda}B$. (Keep in mind that $B$ acts on $F^{\otimes r}$ from the right.) Similarly, $T \in B$ is in $\text{Hom}_A(F^{\otimes r}, S(\lambda, F))$ if and only if $T$ is in the left ideal $Be_{\lambda}$. Combining these two observations, we get $\text{Hom}_A(S(\lambda, F), S(\lambda, F)) = e_{\lambda}Be_{\lambda}$. Similar observations lead to the equality $\text{Hom}_A(A(\lambda, F), A(\lambda, F)) = e'_{\lambda}Be'_{\lambda}$, and it is clear that $\omega$ maps $e_{\lambda}Be_{\lambda}$ isomorphically onto $e'_{\lambda}Be'_{\lambda}$ as expected.

Returning to the general case where $R$ is an arbitrary commutative ring, in order to see that the $\Omega$ we have defined is really a functor, one has to check that $\Omega$ preserves identity maps and commutes with composition. Since $(\Omega)$ commutes with change of the ground ring $R$, it is sufficient to check that $\Omega$ is a functor when $R = \mathbb{Z}$, which again follows immediately from the classical case $R = \mathbb{Q}$ where we know $\Omega$ to be a functor.

It is clear from construction that the functor $\Omega$ is additive. Therefore given any shape $\gamma$ in $J$ of weight $r$, we can define $\Lambda(\gamma, F)$ to be the complex $\Omega(D(\gamma, F))$ obtained by applying the functor $\Omega$ to the resolution $D(\gamma, F)$ of $K(\gamma, F)$.

**Theorem 6.1.** The complex $\Lambda(\gamma, F)$ is a resolution of $L(\gamma, F)$.  


Proof. We will prove the theorem by induction on \( j(\gamma) \), and the number, \( n + 1 \), of rows of \( \gamma \), proceeding in a manner parallel to the construction of \( D(\gamma, F) \) in Section 4.

When \( j(\gamma) = 0 \), we have \( A_\gamma(F) = L_\gamma(F) \), and \( \Lambda(\gamma, F) = \Omega(D(\gamma, F)) \) is the complex \( 0 \to A_\gamma(F) \to 0 \) which is trivially a resolution of \( L_\gamma(F) \). Recalling that \( j(\gamma) \) equals zero when \( n = 0 \), let us assume that \( j(\gamma) > 0 \) and \( n > 0 \). As in Section 4 we can represent \( \gamma \) in the form \( \alpha(t; 0) \) where \( \alpha = (\lambda_1, \ldots, \lambda_{n+1})/(\mu_1, \ldots, \mu_{n+1}) \) is a skew partition. When \( \lambda_{n+1} \leq \mu_n \), we have \( L_\alpha(F) = L_{\rho}(F) \otimes A^{\mu_{n+1}}(F) \), and it is clear that \( A(\gamma, F) = \Lambda(\beta, F) \otimes A^{\mu_{n+1}}(F) \) is a resolution of \( L_\alpha(F) \) because \( \Lambda(\beta, F) \) is a resolution of \( L_\alpha(F) \). In the more interesting case where \( \lambda_{n+1} > \mu_n \), we have from [1, 6.16], a short exact sequence

\[
0 \to L_\sigma(F) \to L_\tau(F) \to L_\gamma(F) \to 0,
\]

where the shapes \( \sigma, \tau \), are those of \( (3) \) in Section 4. By induction on \( j(\gamma) \) we known that \( \Lambda(\sigma, F) \) and \( \Lambda(\tau, F) \) are resolutions of \( L_\sigma(F) \) and \( L_\tau(F) \). It is clear that \( \Lambda(\gamma, F) \) is the mapping cone of the map

\[
\Lambda(\sigma, F) \to \Lambda(\tau, F)
\]

obtained by applying the functor \( \Omega \) to the map

\[
D(\sigma, F) \to D(\tau, F)
\]

of complexes chosen to lift the injection \( K_\sigma(F) \to K_\tau(F) \). We recall from [1] that the injection \( L_\sigma(F) \to L_\tau(F) \) is induced, on the generator level, by a map

\[
A_\sigma(F) \to A_\tau(F).
\]

It is easy to check from the definition of \( \Omega \) and the recipe given in the second paragraph of [1, Sect. 7] that the map

\[
D_\sigma(F) \to D_\tau(F)
\]

which induces the injection \( K_\sigma(F) \to K_\tau(F) \) of \( (3) \) in Section 4, is sent by \( \Omega \) to the map \( (7) \). But the map \( (8) \) is just the component in dimension zero of the map \( (6) \) of resolutions. Consequently, the map \( (5) \) is a lifting of the map \( L_\sigma(F) \to L_\tau(F) \) and it follows immediately that the mapping cone \( \Lambda(\gamma, F) \) of \( (5) \) is a resolution of \( L_\gamma(F) \). This concludes the proof of Theorem 6.1.

For future reference we will show that the map \( (2) \) is an isomorphism when the ground ring \( R \) is \( \mathbb{Z} \). This will later be used in the proof in Section 7 that the same is true over any ground ring \( R \).
Since $\text{Hom}_A(m_{\lambda}, a_{\mu})$ is a $\mathbb{Z}$-split injection, it is clearly sufficient to show that $\text{Hom}_A(n_{\lambda}, b_{\mu})$ is also a $\mathbb{Z}$-split injection, because we know that (2) is an isomorphism when $R = \mathbb{Q}$ (and $\mathbb{Q}$ is flat over $\mathbb{Z}$). Now the map $\text{Hom}_A(n_{\lambda}, b_{\mu})$ can be factored as the composite of

$$\text{Hom}_A(D_{\lambda}F, D_{\lambda}F) \xrightarrow{\text{Hom}(1, b_{\mu})} \text{Hom}_A(D_{\lambda}F, F^{\otimes r})$$

and

$$\text{Hom}_A(D_{\lambda}F, F^{\otimes r}) \xrightarrow{\text{Hom}(n_{\lambda}, 1)} \text{Hom}_A(F^{\otimes r}, F^{\otimes r}).$$

The map (9) is a $\mathbb{Z}$-split injection because the injection $b_{\mu} : D_{\mu}F \to F^{\otimes r}$ is $\mathbb{Z}$-split. Therefore we need only show that the map in (10) is a $\mathbb{Z}$-split injection.

For $\lambda = (\lambda_1, ..., \lambda_k)$ we let $\Gamma(\lambda)$ denote the Young subgroup of $\Gamma(r)$ corresponding to $\lambda$ as described earlier. With our identifcations, $D_{\lambda}(F)$ is exactly the submodule of $F^{\otimes r}$ which is invariant under the action of the subgroup $\Gamma(\lambda)$ of the group $\Gamma(r)$ acting on $F^{\otimes r}$ on the right by permuting tensor factors. Since the $\mathbb{Z}$-module $\text{Hom}_A(D_{\lambda}F, F^{\otimes r})$ is isomorphic to the weight submodule $(F^{\otimes r})_{\lambda}$ of $F^{\otimes r}$ (see Sect. 2), it is clear that $\text{Hom}_A(D_{\lambda}F, F^{\otimes r})$ has a $\mathbb{Z}$-basis consisting of the elements $\{\tau \circ b_{\lambda} | \tau \in X\}$ where $X$ denotes a set of representatives of the right cosets of $\Gamma(\lambda)$ in $\Gamma(r)$ and $\tau \in \Gamma(\lambda)$ is viewed as an $A$-endomorphism of $F^{\otimes r}$ under the canonical identification of the group ring $B = \mathbb{Z}[\Gamma(r)]$ with $\text{End}_A(F^{\otimes r})$. It is easy to see that the map $\text{Hom}_A(n_{\lambda}, 1)$ of (10) takes $\tau \circ b_{\lambda}$ to the summation

$$\sum_{\sigma \in \Gamma(\lambda)} \sigma \tau$$

in the group ring. But this is just the sum of all permutations in the right coset of $\Gamma(\lambda)$ represented by $\tau$. So $\text{Hom}_A(n_{\lambda}, 1)$ is a $\mathbb{Z}$-split injection as claimed.

### 7. The Functor $\Omega : C^K_A \to C^L_A$

Let $A = D_0(\text{End}_F(F))$ be a Schur algebra of degree $r$ where $F$ is a free $R$-module of rank at least $r$. We let $C^K_A$ denote the full subcategory of $C_A$ whose objects are finite direct sums of $K_\gamma(F)$ where $\gamma$ is allowed to range over all shapes in $J$ of weight $r$. Similarly we denote by $C^L_A$ the full subcategory of $C_A$ whose objects are finite direct sums of $L_\gamma(F)$. We will extend $\Omega : C^K_A \to C^L_A$ to a functor $C^K_A \to C^L_A$, also denoted by $\Omega$, by sending $K_\gamma(F)$ to $L_\gamma(F)$. In order to describe $\Omega$ on maps, we consider $\psi \in \text{Hom}_A(K_\gamma(F), K_\delta(F))$. From the comparison theorem, we know that $\psi$
can be lifted to a map of resolutions $\Psi: D(\gamma, F) \to D(\delta, F)$, and that this lifting is unique up to chain homotopy. We then have a map $\Omega(\Psi): \Lambda(\gamma, F) \to \Lambda(\delta, F)$ which is determined up to homotopy by $\psi$, and so we can take $\Omega(\psi): L_\gamma(F) \to L_\delta(F)$ to be the unique map induced on homology by $\Omega(\Psi)$. It is clear that $\Omega$ is $R$-linear. This section is devoted to the investigation of the map $\Omega$.

The simplest case to consider is that in which we have two sequences $\lambda$ and $\mu$ of weight $r$. Then $\Omega$ induces a map from $\text{Hom}_A(\Lambda_\lambda F, \Lambda_\mu F)$ to $\text{Hom}_A(\Lambda_\lambda F, \Lambda_\mu F)$. We want to show that this map is an isomorphism. Now by 5.1, since $D_\lambda F$ is $A$-projective of finite type, we know that $\text{Hom}_A(D_\lambda F, \Lambda_\mu F)$ commutes with change of ground ring. If we can show the same is true for $\text{Hom}_A(\Lambda_\lambda F, \Lambda_\mu F)$, then it will be sufficient to consider the case when the ground ring is $\mathbb{Z}$.

To simplify notation, let us denote modules and algebras over $\mathbb{Z}$, by the subscript zero, and delete the zero when we extend these modules and algebras to other rings $R$. Thus $F_0$ denotes a free abelian group while $F = R \otimes_\mathbb{Z} F_0$, and $A_0$ denotes $D_\lambda(E_0) = D_\lambda(\text{End}(F_0))$ with $A$ denoting $D_\lambda(E) = R \otimes_\mathbb{Z} D_\lambda(E_0)$.

**Lemma 7.1.** Let $M_0$ be an $A_0$-module which is free of finite rank over $\mathbb{Z}$ and let $\mu$ be a sequence of weight $r$. Then $\text{Ext}^i_{A_0}(M_0, S_\mu(F_0)) = 0$ for $i > 0$.

**Proof.** Since $A_0$ is noetherian, $M_0$ has an $A_0$-projective resolution by finitely generated projective $A_0$-modules. Thus we may apply 5.2 (since $\mathbb{Z}$ is hereditary) for any ring $R$ and, in particular, obtain $R \otimes \text{Ext}^i_{A_0}(M_0, S_\mu(F_0)) \cong \text{Ext}^i_R(M, S_\mu(F_0))$. If we let $R = \mathbb{Z}/(p)$ or $\mathbb{Q}$ we now by 2.2 that $\text{Ext}^i_R(M, S_\mu(F_0))$ is a finitely generated abelian group. Thus we have $\text{Ext}^i_{A_0}(M_0, S_\mu(F_0)) = 0$ since $R \otimes \text{Ext}^i_{A_0}(M_0, S_\mu(F_0)) = 0$ for $R = \mathbb{Q}$ or $R = \mathbb{Z}/(p)$ for all primes $p$.

**Lemma 7.2.** With the hypotheses of 7.1, we have $R \otimes \text{Hom}_{A_0}(M_0, S_\mu(F_0)) \cong \text{Hom}_A(M, S_\mu(F))$ for any commutative ring $R$.

**Proof.** Again we may apply 5.2 and obtain the short exact sequence

$$0 \to R \otimes \text{Hom}_{A_0}(M_0, S_\mu F_0) \to \text{Hom}_A(M, S_\mu F) \to T \to 0,$$

where $T$ denotes the module $\text{Tor}^2_R(R, \text{Ext}^1_{A_0}(M_0, S_\mu F_0))$. By 7.1 we know that $\text{Ext}^1_{A_0}(M_0, S_\mu(F_0)) = 0$, so we are done.

**Proposition 7.3.** The map

$$\Omega: \text{Hom}_A(D_\lambda F, A_\mu F) \to \text{Hom}_A(A_\lambda F, S_\mu F) \tag{1}$$

is an isomorphism of $R$-modules.
Proof. The $R$-modules in (1) are finitely generated and commute with the change of the ground ring $R$, so they are free $R$-modules of finite rank. The map $\Omega$ itself commutes with change of $R$ because the resolutions used to construct $\Omega$ do, and therefore it is sufficient to prove the proposition when $R = \mathbb{Z}$. Since the $\mathbb{Z}$-modules in (1) are free of finite rank, one can show that $\Omega_0$ is an isomorphism by showing that it is an isomorphism when extended to $\mathbb{Q}$ and to $\mathbb{Z}/(p)$ for every rational prime $p$. But we know from the classical theory discussed in Section 6 that the map in (1) is an isomorphism when $R = \mathbb{Q}$, and so the free $\mathbb{Z}$-modules in (1) have the same rank. Consequently, we need only show that the map in (1) is a surjection when the ground ring $R$ is a field.

Let $b_\alpha: D_\alpha(F) \to F^{\otimes r}$ and $m_\mu: F^{\otimes r} \to A_\mu(F)$ be as in Section 6. We observe that $\Omega(b_\alpha)$ is the injection $a_\alpha: A_\alpha(F) \to F^{\otimes r}$ and $\Omega(m_\mu)$ is the surjection $F^{\otimes r} \to S_\mu(F)$ which is the tensor product of appropriate multiplication maps. Therefore the map $\text{Hom}_A(\Omega(b_\alpha), \Omega(m_\mu))$ can be factored as the composite of surjective maps

$$\text{Hom}_A(F^{\otimes r}, F^{\otimes r}) \xrightarrow{\alpha} \text{Hom}_A(F^{\otimes r}, S_\mu F) \xrightarrow{\beta} \text{Hom}_A(A_\alpha F, S_\mu F),$$

where $\alpha$ is surjective because $F^{\otimes r}$ is $A$-projective and $\beta$ is surjective because $S_\mu(F)$ is an injective $A$-module. (Do not forget that we are now over a field.) Therefore we have a commutative diagram

$$\begin{align*}
\text{Hom}_A(F^{\otimes r}, F^{\otimes r}) & \xrightarrow{\alpha} \text{Hom}_A(D_\alpha F, A_\alpha F) \\
\downarrow & \quad \downarrow \\
\text{Hom}_A(F^{\otimes r}, F^{\otimes r}) & \xrightarrow{\beta \alpha} \text{Hom}_A(A_\alpha F, S_\mu F),
\end{align*}$$

where $\beta \alpha$ is surjective and the left vertical map is an isomorphism. It follows that the right vertical map is also a surjection as desired.

**Proposition 7.4.** $\text{Ext}_A^i(K_\alpha(F), A_\mu(F)) = 0$ for $i > 0$ and any shape $\alpha \in J$.

Proof. By the application of 5.2, we reduce to the case of proving 7.4 over $\mathbb{Z}$. But in this case we have $\text{Ext}_A^i(K_\alpha(F_0), A_\mu(F_0))$ is a finitely generated abelian group, so that it suffices to show that tensoring with $\mathbb{Q}$ and $\mathbb{Z}/(p)$ for every prime $p$ gives zero. Again by applying 5.2, we see that it is enough to prove 7.4 when the ground ring $R$ is $\mathbb{Q}$ or $\mathbb{Z}/(p)$ or more generally, when $R$ is a field. We assume, then, for the remainder of this proof, that we are working over a field.

Since $\mathbb{D}(\alpha, F)$ is a projective resolution of $K_\alpha(F)$ over $A$, we have

$$\text{Ext}_A^i(K_\alpha(F), A_\mu(F)) = H^i(\text{Hom}_A(\mathbb{D}(\alpha, F), A_\mu(F))).$$

(3)
By 7.3, the complex $\text{Hom}_A(D(\alpha, F), A_\mu(F))$ is isomorphic to $\text{Hom}_A(\Lambda(\alpha, F), S_\mu(F))$. Since $S_\mu(F)$ is an injective $A$-module when $R$ is a field, the augmented cochain complex

$$0 \rightarrow \text{Hom}_A(L_\alpha(F), S_\mu(F)) \rightarrow \text{Hom}_A(\Lambda(\alpha, F), S_\mu(F))$$

is exact because the augmented complex

$$\Lambda(\alpha, F) \rightarrow L_\alpha(F) \rightarrow 0$$

is exact. It therefore follows that (3) vanishes for $i > 0$.

**Corollary 7.5.** $\text{Ext}^i_A(A_\beta(F), A_\mu(F)) = 0$ for $i > 0$.

**Proof.** We saw in the remark at the end of Section 5 that $A_\beta(F) = K_\gamma(F)$ for a skew shape $\gamma$. Therefore we may apply 7.4.

**Proposition 7.6.** Let $\beta$ be a shape in $J$ of weight $r$. Then

(I) $\text{Ext}^i_A(A_\beta(F), L_\beta(F)) = 0$ for $i > 0$;

(II) $\Omega: \text{Hom}_A(D_\beta(F), K_\beta(F)) \rightarrow \text{Hom}_A(A_\beta(F), L_\beta(F))$

is an isomorphism of $R$-modules.

**Proof.** From 7.5 we know that $\Lambda(\beta, F)$ is a finite resolution of $L_\beta(F)$ by modules $X_j$ with the property that $\text{Ext}^i_A(A_\beta(F), X_j) = 0$ for $i > 0$. It therefore follows trivially that $\text{Ext}^i(A_\beta(F), L_\beta(F))$ is zero for $i > 0$, while

$$H_0(\text{Hom}_A(A_\beta(F), \Lambda(\beta, F))) = \text{Hom}_A(A_\beta(F), L_\beta(F)).$$

The vanishing of $\text{Ext}^i_A(A_\beta(F), L_\beta(F))$ enables us, as before (by 5.2), to reduce the proof of (II) to the case $R = \mathbb{Z}$ (since both modules involved, and $\Omega$, commute with change of ground ring).

Now over $\mathbb{Z}$, the argument at the end of Section 6 established that $\Omega_0$ induces an isomorphism of complexes

$$\text{Hom}_{A_0}(D_\beta(F_0), \Lambda(\beta, F_0)) \rightarrow \text{Hom}_{A_0}(A_\beta(F_0), \Lambda(\beta, F_0)).$$

which in turn induces an isomorphism on the homology. By (6) we know that the zeroth homology of the right-hand term is $\text{Hom}_{A_0}(A_\beta(F_0), L_\beta(F_0))$. The projectivity of $D_\beta(F_0)$ tells us that the zeroth homology of the left-hand term of (7) is $\text{Hom}_{A_0}(D_\beta(F_0), K_\beta(F_0))$, since $H_0(\Lambda(\beta, F_0)) = K_\beta(F_0)$. This concludes the proof of (II).

Finally, let $\alpha, \beta$ be shapes in $J$ of weight $r$. From 7.6(II) we know that $\Omega$ induces an isomorphism of cochain complexes

$$\text{Hom}_A(D_\beta(F), K_\beta(F)) \rightarrow \text{Hom}_A(\Lambda(\alpha, F), L_\beta(F)).$$
Since $D(x, F)$ is a projective resolution of $K_x(F)$ over $A$, the cohomology of the left-hand complex in (8) is just $\text{Ext}^*(K_x(F), K_y(F))$. However, by 7.6(I) we know that $\text{Ext}^*(L_x(F), L_y(F))$ can be calculated as the cohomology of the right-hand complex of (8). We have therefore proved the following theorem:\footnote{If $a, b$ are partitions and the characteristic of the ring $R$ is different from 2, then the case $i = 0$ of this theorem can also be deduced from Theorem 3.7 of [3], using the result of G. D. James in [5, 6.3f].}

**THEOREM 7.7.** For shapes $\alpha, \beta$ in $J$ of weight $r$, there are isomorphisms of $R$-modules

$$\text{Ext}^i_x(K_x(F), K_y(F)) \rightarrow \text{Ext}^i_y(L_x(F), L_y(F))$$

induced by the functor $\Omega$ through the chain map in (8).

### 8. FINITE GLOBAL DIMENSION OF SCHUR ALGEBRAS

In this section we will prove that the global dimension of the Schur algebra over a field or over the ring, $\mathbb{Z}$, of integers is finite.

Let $F$ be a vector space of dimension $n$ over a field $K$, and let $A$ denote the Schur algebra $D_\lambda(\text{End}_K(F))$ over $K$. We will first consider the case where $K$ is an infinite field. In order to prove that $A$ has finite global dimension, it is sufficient to show that the simple left $A$-modules have finite homological dimension. Let $\pi(n, r)$ denote the set of all partitions $\lambda = (\lambda_1, ..., \lambda_n)$ of weight $|\lambda| = r$, and for each $\lambda$ in $\pi(n, r)$ let $F_\lambda$ denote the quotient of the Weyl module $V_\lambda = K_\lambda(F)$ by its radical. Over an infinite field, it is shown in [5, 3.5a, 5.4b] that the set of modules $\{F_\lambda \mid \lambda \in \pi(n, r)\}$ forms a complete collection of non-isomorphic simple $A$-modules.

Our next observation is that all our Weyl modules $V_\lambda$ for $\lambda \in \pi(n, r)$ have finite homological dimension (i.e., projective dimension) over $A$. This follows from the fact that the resolution $D(\lambda, F)$ of $K_\lambda(F) = V_\lambda$ constructed in Section 4 consists of terms which are direct sums of modules $D_\mu(F)$ where $\mu$ is a sequence (or weight) of length less than or equal to $n$. Since $n = \text{rank } F$, the modules $D_\mu(F)$ are $A$-projective, so that $D(\lambda, F)$ is a finite projective resolution of $V_\lambda$ over $A$. Using this, we will prove that the simple modules $F_\lambda$ also have finite homological dimension by an induction argument on the set $\pi(n, r)$ which is totally ordered under the lexicographic ordering $\leq$. The smallest element in $\pi(n, r)$ is partition

$$\lambda_0 = \underbrace{(t + 1, ..., t + 1)}_{s}, \underbrace{t, ..., t}_{n-s}$$

where $r = m + s$ with $0 \leq s < n$.\footnote{If $x, y$ are partitions and the characteristic of the ring $R$ is different from 2, then the case $i = 0$ of this theorem can also be deduced from Theorem 3.7 of [3], using the result of G. D. James in [5, 6.3f].}
It is easy to see (e.g., using the duality results in [2, II.4]) that $V_{\omega_0} = K_{\omega_0}(F)$ is isomorphic to

\[ A_{\omega_0}(F) = \bigotimes_{i} A_{\omega}F \]

which is clearly irreducible because it is generated by any of its weight vectors, as $A_{\omega}F$ itself is, and any non-zero submodule has a weight vector. Therefore $F_{\omega_0} = V_{\omega_0}$ and thus $F_{\omega_0}$ has finite homological dimension.

For the induction step, let $\lambda \in \pi(n, r)$ be any partition other than the smallest, and assume that $\text{hd}_{F_{\mu}} < \infty$ for all partitions $\mu < \lambda$ in $\pi(n, r)$. Since $F_{\lambda}$ is the quotient of $V_{\lambda}$ by its radical, and $V_{\lambda}$ has finite homological dimension, we need only show that the radical of $V_{\lambda}$ has finite homological dimension. It is a well-known fact that $A$ is lexicographically the largest weight occurring in the weight space decomposition of $V_{\lambda}$, and that it occurs with multiplicity one in each of $V_{\lambda}$ and $F_{\lambda}$ (see the proof of (5.4b) and the discussion in (3.5) in [S]). It follows that the only possible composition factors of the radical of $V_{\lambda}$ are those $F_{\mu}$ with $\mu < \lambda$. But by our induction assumption, all of these $F_{\mu}$ have finite homological dimension and therefore so does $F_{\lambda}$.

If $K$ is a finite field, let us take $\bar{K}$ to be the algebraic closure of $K$. Then $A = \bar{K} \otimes_k A$ is the Schur algebra $D_\lambda(\text{End}_k(\bar{K} \otimes F))$ over $K$. Since $A$ is a finite-dimensional algebra we can apply Theorem 5.3 to $K$, $\bar{K}$ and $A$ to conclude that for any finitely generated $A$-module $M$, there is an isomorphism

\[ \bar{K} \otimes_k \text{Ext}^i_A(M, N) \cong \text{Ext}^i_A(M, N), \]

where $N$ is an arbitrary $A$-module. It follows immediately that the global dimension of $A$ is less than or equal to the global dimension of $\bar{A}$; so it is finite. This gives us the following theorem.

**Theorem 8.1.** Let $K$ be a field, $F$ a finite dimensional vector space over $K$, and $A = D_\lambda(\text{End}_K(F))$ the Schur algebra over $K$. Then $A$ has finite global dimension.

We conclude this section with a proof of the following result.

**Theorem 8.2.** Let $F$ be a free $\mathbb{Z}$-module of finite rank and let $A$ be the Schur algebra $D_\lambda(\text{End}_{\mathbb{Z}}(F))$. Then $A$ has finite global dimension.

**Proof.** For each prime $p$ we let $d_p$ denote the global dimension of the Schur algebra $\mathbb{Z}_p \otimes A$ over the field $\mathbb{Z}_p = \mathbb{Z}/(p)$. Since $\mathbb{Z}_p \otimes A$ is the centralizer $\text{End}_{\Gamma(r)}(\mathbb{Z}_p \otimes F^{\otimes r})$ of the group algebra of $\Gamma(r)$, we know that $\mathbb{Z}_p \otimes A$ is semisimple whenever $p > r!$, so that $d_p = 0$ except for a finite
number of primes. We will take $d$ to be the maximum of the integers $d_p$ and show that $\text{gl.dim}(A) \leq d + 2$.

It is sufficient to show that the homological dimension of any finitely generated $A$-module is at most $d + 2$. Since $A$ is itself a finitely generated $\mathbb{Z}$-free module, any finitely generated $A$-module is the quotient of two finitely generated $\mathbb{Z}$-free $A$-modules. It therefore suffices to show that the homological dimension of any finitely generated $A$-module $M$ which is $\mathbb{Z}$-free is at most $d + 1$ or, what is the same, that $\text{Ext}^i_A(M, N) = 0$ for all $i > d + 1$ and any $A$-module $N$. Again, any $A$-module $N$ is the quotient of two $\mathbb{Z}$-free $A$-modules, so from the long exact sequence for $\text{Ext}_A(M, -)$, we need only show that $\text{Ext}^i_A(M, N) = 0$ for $i = d + 1$ and any $A$-module $N$ which is $\mathbb{Z}$-free.

For any field $K$ we have (from 5.3) the short exact sequence

$$0 \rightarrow K \otimes \text{Ext}^i_A(M, N) \rightarrow \text{Ext}^i_A(\overline{M}, \overline{N}) \rightarrow \text{Tor}_i^A(K, \text{Ext}^{i+1}_A(M, N)) \rightarrow 0,$$

where $\overline{A} = K \otimes A$, $\overline{M} = K \otimes M$, and $\overline{N} = K \otimes N$.

If we take $K = \mathbb{Q}$ the semi-simplicity of $A = \mathbb{Q} \otimes A$ forces $\mathbb{Q} \otimes \text{Ext}^i_A(M, N)$ to vanish for $i > 0$, so that $\text{Ext}^i_A(M, N)$ is torsion for all $i > 0$. On the other hand, taking $K = \mathbb{Z}/(p)$ in the above exact sequence, the middle term vanishes for $i > d$, since $d \geq d_p$. Therefore $\text{Ext}^{i+1}_A(M, N)$ has no $p$-torsion for $i + 1 > d + 1$. Since $p$ is arbitrary, it follows that the abelian group $\text{Ext}^{i+1}_A(M, N)$ is zero for $i + 1 > d + 1$ as desired.

9. ON EXTENSIONS OF WEYL MODULES

In this section we will discuss the groups $\text{Ext}^i_A(K(\lambda), K(\mu))$ for certain special pairs of partitions $\lambda$ and $\mu$. We will assume that the rank of $F$ is large enough so that $K_\lambda(F)$ and $K_\mu(F)$ are both non-zero. Also, for convenience we shall let $K(\gamma_1)$ and $D(\gamma_1)$ denote $K_{\gamma_1}(F)$ and $D_{\gamma_1}(F)$, respectively.

We will first consider the case $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\lambda_1 + 1, \lambda_2 - 1)$. We want to show that over the ring $\mathbb{Z}$ the group $\text{Ext}^1_A(K(\lambda), K(\mu))$ is cyclic of order $\lambda_1 - \lambda_2 + 2 = \mu_1 - \mu_2$, and is generated by the class of the extension

$$0 \rightarrow K(\mu) \rightarrow K(\beta) \rightarrow K(\lambda) \rightarrow 0,$$

where $\beta$ is the skew partition $(\lambda_1 + 1, \lambda_2)/(1, 0)$.

From [1] we have the projective resolution $D(\lambda)$

$$\cdots \rightarrow \sum_{t \geq 2} c_t D(\lambda_1 + t, \lambda_2 - t) \rightarrow \sum_{t \geq 1} D(\lambda_1 + t, \lambda_2 - t) \rightarrow D(\lambda) \rightarrow K(\lambda) \rightarrow 0,$$

where $\beta$ is the skew partition $(\lambda_1 + 1, \lambda_2)/(1, 0)$. From [1] we have the projective resolution $D(\lambda)$.
of the Weyl module $K(\lambda)$, where the integers $c_t$ are certain binomial coefficients and the remaining terms of the resolution are of the same type with increasing lower bounds on the index of summation $t$.

Now apply $\text{Hom}_A(-, K(\mu))$ to (2) and get

$$0 \to 0 \to \text{Hom}_A(D(\lambda), K(\mu)) \to \sum_{t \geq 1} \text{Hom}_A(D(\lambda_1 + t, \lambda_2 - t), K(\mu)) \to \cdots,$$

(3)

We will show that (3) is isomorphic to the complex

$$0 \to 0 \to \mathbb{Z} \xrightarrow{\lambda_1 - \lambda_2 + 2} \mathbb{Z} \to 0 \to 0 \to \cdots$$

(4)

which implies that $\text{Ext}^1_A(K(\lambda), K(\mu)) = \mathbb{Z}/(\lambda_1 - \lambda_2 + 2)$.

Now $\text{Hom}_A(D(\lambda), K(\mu)) = K(\mu)_\lambda$ is one-dimensional because there is only one “standard” tableau

\begin{align*}
1 & \cdots & 1 & 2 \\
2 & \cdots & 2
\end{align*}

(5)

of shape $\mu$ and content $\lambda$. Similarly $\text{Hom}_A(D(\lambda_1 + t, \lambda_2 - t), K(\mu))$ is one-dimensional when $t = 1$, spanned by the tableau

\begin{align*}
1 & \cdots & 1 \\
2 & \cdots & 2
\end{align*}

(6)

and is zero for $t > 1$.

The map $\text{Hom}(D(\lambda), K(\mu)) \to \text{Hom}(D(\mu), K(\mu))$, under the identification $K(\mu)_\lambda \to K(\mu)_\mu$, sends the tableau (5) to the sum

\begin{align*}
(\lambda_1 + 1) \begin{array}{c}
1 & \cdots & 1 & 1 \\
2 & \cdots & 2
\end{array} & + & \begin{array}{c}
1 & \cdots & 1 & 2 \\
1 & 2 & \cdots & 2
\end{array}
\end{align*}

(7)

where the coefficient $\lambda_1 + 1$ comes from multiplication in the divided power algebra. Similarly, if one straightens the second tableau in (7), one gets $-(\lambda_2 - 1)$ times the tableau in (6). This proves that (5) gets sent to $(\lambda_1 + 1 - \lambda_2 + 1)$ times the canonical tableau (6) and so we have shown that (3) is (4).

See [2, II.2.13]. The content of a tableau is also called the weight of a tableau [8, 1.1].
Now we have $\text{Ext}_d^i(K(\lambda), K(\mu))$ is generated by the cocycle in $\text{Hom}(D(\mu), K(\mu))$ corresponding to the tableau (6). In order to check that this cocycle is the class of the extension (1), it is sufficient to check that the following diagram commutes

$$
\begin{array}{cccccc}
D(\lambda_1 + 1, \lambda_2 - 1) & \longrightarrow & D(\lambda_1, \lambda_2) & \longrightarrow & K(\lambda) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K(\mu) & \longrightarrow & K(\beta) & \longrightarrow & K(\lambda) & \longrightarrow & 0
\end{array}
$$

(8)

where the vertical maps are the usual maps. But this we know: it is part of the diagram that leads to the construction of the resolution (2).

A similar computation shows that when $\lambda = (\lambda_1, \ldots, \lambda_n)$ is any partition and the partition $\mu = (\mu_1, \ldots, \mu_n)$ is of the form

$$
(\lambda_1, \ldots, \lambda_i, \lambda_{i+1} + 1, \lambda_{i+2} - 1, \lambda_{i+3} - 1, \ldots, \lambda_n)
$$

(9)

for some $1 \leq i \leq n - 1$, then $\text{Ext}_d^i(K(\lambda), K(\mu))$ is a cyclic group of order $\lambda_i - \lambda_{i+1} - 2$ generated by the class of the extension

$$
0 \rightarrow K(\mu) \rightarrow K(\lambda + \delta/\delta) \rightarrow K(\lambda) \rightarrow 0,
$$

(10)

where $\delta$ is the sequence $(0^{i-1}, 1, 0, 1^{n-i-1})$.

We next consider the case $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\mu = (\lambda_1 + 1, \lambda_2, \lambda_3 - 1)$. The first of several terms of the projective resolution $D(\lambda)$ of $K(\lambda)$ can be read explicitly from the description given in the next section. Using this it can be seen easily that the complex $\text{Hom}_A(D(\lambda), K(\mu))$ is zero in dimension two, and is free of rank two in each of the dimensions zero and one. In dimension zero, it is clear that $\text{Hom}_A(D_{\lambda_1} \otimes D_{\lambda_2} \otimes D_{\lambda_3}, K(\mu))$ is generated by the morphisms induced by

$$
\alpha_1 : D_{\lambda_1} \otimes D_{\lambda_2} \otimes D_{\lambda_3} \rightarrow D_{\lambda_1 + 1} \otimes D_{\lambda_2 - 1} \otimes D_{\lambda_3}
$$

$$
\alpha_2 : D_{\lambda_1} \otimes D_{\lambda_2} \otimes D_{\lambda_3} \rightarrow D_{\lambda_1 + 1} \otimes D_{\lambda_2 - 1} \otimes D_{\lambda_3}
$$

where $\alpha_1$ and $\alpha_2$ are the composition of diagonalization maps and multiplication maps indicated below:

$$
\begin{array}{c}
\alpha_1 : D_{(\lambda_1, \lambda_2, \lambda_3)} \xrightarrow{1} D_{(\lambda_1, 1, \lambda_2 - 1, 1, \lambda_3 - 1)} \xrightarrow{m} D_{(\lambda_1 + 1, \lambda_2, \lambda_3 - 1)} \\
\alpha_2 : D_{(\lambda_1, \lambda_2, \lambda_3)} \xrightarrow{1} D_{(\lambda_1, \lambda_2, 1, \lambda_3 - 1)} \xrightarrow{m} D_{(\lambda_1 + 1, \lambda_2, \lambda_3 - 1)}
\end{array}
$$

$$
\begin{array}{c}
D_{(\lambda_1, 1, \lambda_2, \lambda_3 - 1)} \xrightarrow{m} D_{(\lambda_1 + 1, \lambda_2, \lambda_3 - 1)} \\
D_{(\lambda_1, \lambda_2, 1, \lambda_3 - 1)} \xrightarrow{m} D_{(\lambda_1 + 1, \lambda_2, \lambda_3 - 1)}
\end{array}
$$
In dimension one we have the map induced by

$$\beta_1: D_{(\lambda_1, \lambda_2, \lambda_3 - 1)} \xrightarrow{1 \otimes 1 \otimes \varphi} D_{(\lambda_1 + 1, \lambda_2 - 1, \lambda_3)} \xrightarrow{1 \otimes m \otimes 1} D_{(\lambda_1, \lambda_2, \lambda_3 - 1)}$$

generating the weight space, $\text{Hom}(D_{\lambda_1 + 1} \otimes D_{\lambda_2 - 1} \otimes D_{\lambda_3}, K(\mu))$, and the map induced by

$$\beta_2: D_{(\lambda_1, \lambda_2 + 1, \lambda_3 - 1)} \xrightarrow{1 \otimes \varphi \otimes 1} D_{(\lambda_1, \lambda_2, \lambda_3 - 1)} \xrightarrow{m \otimes 1 \otimes 1} D_{(\lambda_1 + 1, \lambda_2, \lambda_3 - 1)}$$

generating the weight space, $\text{Hom}(D_{\lambda_1} \otimes D_{\lambda_2 + 1} \otimes D_{\lambda_3 - 1}, K(\mu))$.

With these explicit identifications of the weight modules, it is easy to see, using the straightening laws, that $\text{Hom}_A(\mathbb{D}(\lambda), K(\mu))$ is the following complex,

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{[\lambda_1 - \lambda_2 + 2 \lambda_2 - \lambda_3 + 1]} \mathbb{Z} \oplus \mathbb{Z} \to 0,$$

with $\text{Ext}^1(K(\lambda), K(\mu))$ being the cokernel of the above matrix. Clearly this cokernel is isomorphic to $\mathbb{Z}(\lambda_1 - \lambda_3 + 3)$.

Let us now consider the case where $A = (A_1, A_2)$ and $\mu = (\mu_1, \mu_2)$ are of the form $(A_1 + d, A_2 - d)$ where $1 \leq d \leq A_2$. Identifying the weight submodules with homomorphisms as before, one can see that the complex $\text{Hom}_A(\mathbb{D}(\lambda), K(\mu))$ has the form

$$\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}^d \xrightarrow{\beta} \mathbb{Z}(\frac{1}{2})$$

in dimensions zero through two and that $\alpha$ is the $1 \times d$ matrix $[a_1 \ldots a_d]$ where

$$a_k = \left(\lambda_1 - \lambda_2 + d + k \atop k\right)$$

for $1 \leq k \leq d$. The homology $\ker(\beta)/\text{im}(\alpha)$ is then the group $\text{Ext}_A^1(K(\lambda), K(\mu))$, and we observed in the proof of Theorem 8.2 that $\text{Ext}_A^i(K(\lambda), K(\mu))$ is torsion for all $i > 0$. Therefore the rank of $\ker(\beta)$ equals the rank of $\text{im}(\alpha)$ which is free cyclic with generator $(a_1, ..., a_d)$. Moreover, if $(b_1, ..., b_d)$ is a generator of the free cyclic group $\ker(\beta)$ then

$$(a_1, ..., a_d) = m(b_1, ..., b_d)$$

for some integer $m$. Also, $\ker(\beta)$ is a summand of $\mathbb{Z}^d$ because $\beta$ is a map of finitely generated free abelian groups. Therefore the entries of the generator $(b_1, ..., b_d)$ must be relatively prime. It follows immediately that if we take $m$ to be greatest common divisor of the entries of $(a_1, ..., a_d)$, then
(b_1, ..., b_d) = (a_1, ..., a_d)/m is a generator of ker(\beta). One can then conclude that $\text{Ext}_A^1(K(\lambda), K(\mu))$ is cyclic of order $m$ where

$$m = \text{g.c.d.} \left\{ \left( \lambda_1 - \lambda_2 + d + k \right) \right\} \bigg| k = 1, ..., d \right\}.$$  

(14)

So far our discussion has been exclusively over the integers. Keeping to the notation of Section 5 we will let $\tilde{A}_p = \mathbb{Z}_p \otimes_\mathbb{Z} A$ denote the Schur algebra over $\mathbb{Z}_p = \mathbb{Z}/(p)$ obtained by reducing the Schur algebra $A$ over $\mathbb{Z}$ modulo a prime $p$. Similarly we let $\tilde{K}_p(v)$ denote the Weyl module $\mathbb{Z}_p \otimes_\mathbb{Z} K_p(v)$ obtained by reducing the $A$-module $K_p(v)$ modulo $p$. Suppose now that the group $\text{Ext}_A^1(K(\lambda), K(\mu))$ over $\mathbb{Z}$ is known to be cyclic, as it is in the examples described above, of finite order $m$. If $\lambda$ and $\mu$ are distinct partitions, then $\text{Hom}_A(K(\lambda), K(\mu)) = 0$ because $K(\lambda)$ and $K(\mu)$ are $\mathbb{Z}$-forms of non-isomorphic simple modules over the Schur algebra $\mathbb{Q} \otimes_\mathbb{Z} A$. Applying Theorem 5.3 to this situation, with $i = 1$, we obtain an isomorphism

$$\text{Hom}_{\tilde{A}_p}(\tilde{K}_p(\lambda), \tilde{K}_p(\mu)) \cong \text{Tor}_1^A(\mathbb{Z}_p, \text{Ext}_A^1(K(\lambda), K(\mu))).$$

Since $\text{Ext}_A^1(K(\lambda), K(\mu))$ is cyclic of order $m$, we can conclude that

$$\text{Hom}_{\tilde{A}_p}(\tilde{K}_p(\lambda), \tilde{K}_p(\mu)) = \begin{cases} 0 & \text{if } p \nmid m \\ \mathbb{Z}_p & \text{if } p | m. \end{cases}$$  

(15)

There has been a great deal of interest in recent years in the groups $\text{Hom}_{\tilde{A}_p}(\tilde{K}_p(\lambda), \tilde{K}_p(\mu))$ and various results have been obtained about their vanishing behaviour. A good deal of attention has focused on pairs of partitions $\lambda$ and $\mu$ which are related to each other in the following way,

$$\mu_i = \lambda_i + d, \quad \mu_j = \lambda_j - d, \quad \mu_h = \lambda_h \quad \text{for } h \neq i, j,$$

(16)

for some $i < j$. It is proved in [4] that for such a pair of partitions $\lambda$, $\mu$, the group $\text{Hom}_{\tilde{A}_p}(\tilde{K}_p(\lambda), \tilde{K}_p(\mu))$ is not zero if there exists a positive integer $e$ such that

$$d < p^e \quad \text{and} \quad p^e \nmid (\lambda_i - \lambda_j + j - i + d).$$

(17)

This result contains as a special case an earlier nonvanishing result in [3, 4]. It is also noted in [3, p. 231] that J. C. Jantzen is able to compute the dimension over $\mathbb{Z}_p$ of $\text{Hom}_{\tilde{A}_p}(\tilde{K}_p(\lambda), \tilde{K}_p(\mu))$ to be equal to one in many cases.

Characteristic $p$ and from the computations done earlier in this section on some of the groups $\text{Ext}_A^1(K(\lambda), K(\mu))$ over $\mathbb{Z}$, it was tempting to make
the following guess. If \( \lambda \) and \( \mu \) are a pair of partitions related by the formula in (16), then \( \text{Ext}^1_\mathfrak{A} (K(\lambda), K(\mu)) \) over \( \mathbb{Z} \) is cyclic of order \( m \) where

\[
m = \text{g.c.d.} \left\{ \binom{\lambda_i - \lambda_j + j - i + d - 1 + k}{k} \mid 1 \leq k \leq d \right\}.
\]  

(18)

It can be shown that this number is

\[
m = \frac{\lambda_i - \lambda_j + j - i + d}{\text{g.c.d.} \{ \lambda_i - \lambda_j + j - i + d, \text{l.c.m.}\{1, \ldots, \alpha\} \}}.
\]  

(19)

Another way of describing this number is as follows. If \( p_1^{e_1} \cdots p_r^{e_r} \) is the prime factorization of \( \lambda_i - \lambda_j + j - i + d \) then the integer \( m \) in (19) equals the product \( p_1^{f_1} \cdots p_r^{f_r} \) where

\[
f_i = \min \{ 0, e_i - \text{ord} p_i (\text{l.c.m.}\{1, \ldots, d\}) \}.
\]  

(20)

However, further computations with \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) and \( \mu = (\lambda_1 + 2, \lambda_2, \lambda_3 - 2) \) tell us that the picture is more complicated than indicated in the above discussion. For, in this case it turns out that \( \text{Ext}^1_\mathfrak{A} (K(\lambda), K(\mu)) \) is cyclic of order \( m' \) where

\[
m' = (\lambda_1 - \lambda_3 + 4)/\text{g.c.d.}\{ \lambda_1 - \lambda_2 + 3, \lambda_2 - \lambda_3 + 1, 2 \}
\]  

(21)

which differs from the number \( m \) gives in (19) when \( \lambda_1 - \lambda_2 \) and \( \lambda_2 - \lambda_3 \) are both even. The best general guess we can offer at the moment is that the extension group \( \text{Ext}^1_\mathfrak{A} (K(\lambda), K(\mu)) \) is cyclic when \( \lambda \) and \( \mu \) are as in (16) and that the order should be

\[
p_1^{f_1} \cdots p_r^{f_r},
\]  

(22)

where \( f_i \leq g_i \leq e_i \) and \( f_i \) as in (20).

In order to compute the groups \( \text{Ext}^1_\mathfrak{A} (K_\mathfrak{A}(\lambda), K_\mathfrak{A}(\mu)) \) one must know more about the initial terms of the projective resolutions \( \mathfrak{D}(\lambda) \) of the Weyl modules \( K(\lambda) \) for general partitions \( \lambda \). Of course one is ultimately interested in all the higher extension groups \( \text{Ext}^1_\mathfrak{A}(K(\lambda), K(\mu)) \) over the integers and in characteristic \( p \). Therefore one needs an explicit description of the resolutions \( \mathfrak{D}(\lambda) \), similar to the description given in [1, 7.4] for partitions \( \lambda \) of length 2, at least up to the global dimension of the relevant Schur algebra over \( \mathbb{Z} \). We address the problem of obtaining explicit information about the general resolution \( \mathfrak{D}(\lambda) \) in the next section.
10. Towards a Description of the Resolution $D(\alpha)$

In [1] we succeeded in writing down an explicit description of the resolutions $D(\alpha)$ and $\Lambda(\alpha)$ for any 2-rowed skew-shape $\alpha$. This was accomplished in two steps. First we established the exact sequences

$$\begin{align*}
0 & \rightarrow K(\gamma) \rightarrow K(\beta) \rightarrow K(\alpha) \rightarrow 0 \\
0 & \rightarrow L(\gamma) \rightarrow L(\beta) \rightarrow L(\alpha) \rightarrow 0,
\end{align*}$$

where if $\alpha = (\lambda_1, \lambda_2)/(\mu_1, \mu_2)$, we define $\beta$ to be $(\lambda_1, \lambda_2 - 1)/(\mu_1, \mu_2 - 1)$ and $\gamma$ to be $(\lambda_1, \lambda_2 - (\mu_1 - \mu_2) - 2)/(\mu_2 - 1, \mu_2 - 1)$. Then, by induction on $j(\alpha)$, we assumed the explicit form of the resolutions $D(\gamma)$ and $D(\beta)$ ($\Lambda(\gamma)$ and $\Lambda(\beta)$), described an explicit comparison map between those resolutions, and showed that the mapping cone of this map had the desired explicit form for $D(\alpha)$ ($\Lambda(\alpha)$). When we went to skew-shapes having three or more rows, we were led to the family $J$ of shapes to establish analogues of the exact sequences in (1) (see [1, 6.16, 7.5]).

In order to find the explicit form of our resolutions $D(\alpha)(\Lambda(\alpha))$, we could either hypothesize the exact form of such a resolution for each shape in our family $J$, and proceed as in the 2-row case, or we could confine our attention to resolutions of skew shapes, search for canonical exact sequences of skew-shapes analogous to (1), and then make an assumption first about resolutions of skew-shapes. Neither approach has led as yet to the explicit descriptions we are seeking, although we have some fairly concrete conjectures. However, the exact sequences of skew-shapes analogous to (1) do give us explicitly the initial terms of the general resolutions of skew-shapes and therefore theoretically permit the computation of $\text{Ext}^1(K(\lambda), K(\mu))$ for partitions of the type discussed in Section 9. We say "theoretically" because the combinatorics involved in the computation of the general case still present an obstacle. Nevertheless, because of the interest of this problem, we will sketch here the results we have obtained so far. We will not include too much detail or any proof but leave that for a later paper in which, we expect, we can offer a complete solution to this problem.

The first result we shall describe is the exact sequence of skew-shapes generalizing (1). We will treat the 3-row case first as that is a bit less complicated than, yet contains the ingredients of, the general situation.

Our class $J_2$ of 3-row shapes contains only two essentially different types of shapes, namely those of the form $\alpha(0, 0)$ and $\alpha(1, 0)$ where $\alpha$ is a skew-shape. If we write

$$\alpha = (\lambda_1, \lambda_2, \lambda_3)/(\mu_1, \mu_2, \mu_3),$$
then we have the exact sequences
\[ 0 \rightarrow L(\gamma) \rightarrow L(\beta) \rightarrow L(\alpha) \rightarrow 0 \quad (2)_L \]
\[ 0 \rightarrow K(\gamma) \rightarrow K(\beta) \rightarrow K(\alpha) \rightarrow 0, \quad (2)_K \]
where \( \beta = \alpha(0; 1) = (\lambda_1, \lambda_2, \lambda_3 - 1)/(\mu_1, \mu_2, \mu_3 - 1) \) and \( \gamma = \alpha(1; 0) = (\lambda_1, \lambda_2, \lambda_3 - 1)/(\mu_1, \mu_3 - 1, \mu_2). \)

(As it makes no difference to our formal discussion whether we are talking about Schur or Weyl modules, we will usually write our exact sequences without \( L \)'s and \( K \)'s. Thus \( (2)_L \) and \( (2)_K \) can really be written
\[ 0 \rightarrow \gamma \rightarrow \beta \rightarrow \alpha \rightarrow 0. \quad (2) \]

Since the shape \( \beta \) is skew, what is needed is a "resolution" of the shape \( \gamma \) in terms of skew-shapes. We will therefore focus our attention on a typical "bad" shape \( \gamma \) and, for the sake of convenience, write it as
\[ \gamma = (\lambda_1 + 1, \lambda_2 + 1, \lambda_3)/(\mu_1 + 1, \mu_3, \mu_2 + 1), \]
where \( (\lambda_1, \lambda_2, \lambda_3) \) and \( (\mu_1, \mu_2, \mu_3) \) are partitions with \( \mu \subseteq \lambda. \)

We let
\[ \gamma_1 = (\lambda_1 + 1, \lambda_2 + 1, \lambda_3 - (\mu_2 - \mu_3) - 1)/(\mu_1 + 1, \mu_3, \mu_3) \]
and
\[ \gamma(l) = (\lambda_1 + 1, \lambda_2 + 1, \lambda_3 - (\mu_1 - \mu_3) - 2 - l)/(\mu_2 - l, \mu_3, \mu_3) \]
where, as usual, we set these terms equal to 0 when they do not yield real skew-shapes. Clearly \( \gamma_1 \) is a skew-shape provided \( \lambda_3 - \mu_2 - 1 \geq 0 \), and \( \gamma(l) \) is a skew-shape provided \( \mu_2 - l \geq \mu_3 \) and \( \lambda_3 - \mu_1 - 2 - l \geq 0 \).

Now define the modules \( M_1(\gamma) = \gamma_1 \) (meaning the Schur or Weyl module associated to the shape \( \gamma_1 \)) and
\[ M_{v+2}(\gamma) = \sum_{l \geq 0} \mathbb{V} \left[ \begin{array}{c} \mu_1 - \mu_2 + 1 + l \\ l \end{array} \right] \otimes \gamma(l) \quad \text{for } v \geq 0. \]
These modules will be the terms of a resolution of \( \gamma \),
\[ \ldots \rightarrow M_3(\gamma) \xrightarrow{\delta_1} M_2(\gamma) \xrightarrow{\delta_2} M_1(\gamma) \xrightarrow{\delta_1} \gamma \rightarrow 0, \quad (3) \]
with the maps \( \delta_i \) remaining to be defined.

The map \( \delta_1 \) is induced by the identity map on the generators of \( \gamma_1 \). That this is indeed a well-defined map follows from considerations in [1, 6.11]. Essentially the map \( \delta_1 \) is obtained by pushing the last row of \( \gamma \) to the left
until it is flush with the second row. To define the map $\gamma(l) \to \gamma(1)$ for all $l \geq 0$. If we let $X(a)$ denote either $A^a$ or $D_a$, we define

$$X(\lambda_1 - \mu_2 + l + 1) \otimes X(\lambda_2 - \mu_3 + 1) \otimes X(\lambda_3 - \mu_1 - l - 2)$$

as the composition

$$X(\lambda_1 - \mu_1) \otimes X(\mu_1 - \mu_2 + l + 1) \otimes X(\lambda_2 - \mu_3 + 1)$$

$$\otimes X(\lambda_3 - \mu_1 - l - 2)$$

$$X(\lambda_1 - \mu_1) \otimes X(\lambda_2 - \mu_3 + 1) \otimes X(\lambda_3 - \mu_2 - 1),$$

where the first map is the indicated diagonalization, and the second is the appropriate multiplication. It is easy to check that the map above induces well-defined maps from $\gamma(l)$ to $\gamma_1$.

For $v > 2$, we define the map $\delta_v : \sum_{l \geq 0} \mathbb{K}[\mu_1 - \mu_2 + 1 + l]_v \otimes \gamma(l) \to \sum_{l \geq 0} \mathbb{K}[\mu_1 - \mu_2 + 1 + l]_{v-3} \otimes \gamma(l)$ on each summand as follows. For $e^{\ell_1}_1 A \cdots A e^{\ell_{v-2}}_{v-2}$ a basis element of $\mathbb{K}[\mu_1 - \mu_2 + 1 + l]_{v-2}$, and $x \otimes y \otimes z \in X(\lambda_1 - \mu_2 + l + 1) \otimes X(\lambda_2 - \mu_3 + 1) \otimes X(\lambda_3 - \mu_1 - l - 2)$, we set

$$\delta_v(e^{\ell_1}_1 A \cdots A e^{\ell_{v-2}}_{v-2}) \otimes x \otimes y \otimes z = \delta^v(e^{\ell_1}_1 A \cdots A e^{\ell_{v-2}}_{v-2}) \otimes x \otimes y \otimes z$$

$$\pm e^{\ell_{v-3}-i_1}_1 \cdots e^{\ell_{v-2}-i_{v-2}}_{v-2} \otimes x(\lambda_1 - \mu_2 + l + 1 - i_1) \otimes y \otimes x'(i_1) \otimes z,$$

where we have diagonalized the $x$ term as indicated by the degrees in parentheses. The map $\delta_v$ induces a map from $\mathbb{K}[\mu_1 - \mu_2 + 1 + l]_{v-2} \otimes \gamma(l)$ to $\mathbb{K}[\mu_1 - \mu_2 + 1 + l]_{v-3} \otimes \gamma(l) \oplus \mathbb{K}[\mu_1 - \mu_2 + i - i_1] \otimes \gamma(l - i_1)$. The map $\delta^v$ is the boundary map of the complex $\mathbb{K}[\mu_1 - \mu_2 + 1 + l]$.

The proof that the complex (3) is an acyclic complex over $\gamma$ proceeds by induction on $j(\gamma)$ as in the 2-rowed case. Namely, given $\gamma$, we know that there is a short exact sequence

$$0 \to \tau \to \sigma \to \gamma \to 0 \tag{4}$$

with $j(\tau), j(\sigma) < j(\gamma)$. It may happen that $\sigma$ is a skew-shape, but this can occur if and only if $\mu_2 = \mu_3$, in which case it is easily seen that $\sigma = M_1(\gamma)$, $\tau = M_2(\gamma)$, and $M_i(\gamma) = 0$ for $i > 3$. Thus (3) reduces to (4) and we are done. If $\sigma$ is not a skew-shape, one writes the appropriate complexes for $\tau$ and $\sigma$,.
one finds a canonical map from the one into the other, and the mapping cone gives us the complex (3). The proof is straightforward and rests mainly on the mapping cone properties of the complexes $\mathbb{K}[?]$ described in [1, 4].

To go from the complex (3) to the resolution of our original shape $\alpha$ is now trivial. One simply splices the complex (3) with the exact sequence (2) and, setting $M_0 = \beta$, we have the complex

$$\ldots \to M_2 \to M_1 \to M_0 \to \alpha \to 0$$

which is a resolution of the skew-shapes with fewer overlaps than $\alpha$. Notice that when $\alpha$ is a partition, i.e., when $\mu_1 = \mu_2 = \mu_3$, the sequence (5) is the fundamental exact sequence attached to a partition [1, 1.16].

The above discussion becomes considerably more complicated when $\alpha$ is a skew-shape with $n + 1$ rows, $n > 2$. We still, of course, have our exact sequence (2) with $\beta = \alpha(0; 1)$ and $\gamma = \alpha(1; 0)$, but there are many more types of shapes in $J_n$ than those of type $\alpha(1; 0)$. In fact, the shapes in $J_n$ run through all shapes $\alpha(t; 0)$ with $0 \leq t \leq n - 1$ and $\alpha$ an arbitrary skew-shape. In order, to follow the line of proof used when $n = 2$, one must write down the terms of a conjectured complex for all shapes $\alpha(t; 0)$, $1 \leq t \leq n - 1$, and use exact sequences (4) and mapping cone properties to prove that these complexes are exact. We will simply state what this complex is for $\gamma = \alpha(1; 0)$.

Again, for convenience, we will write

$$\alpha = (\lambda_1 + 1, \ldots, \lambda_n + 1, \lambda_{n+1} + 1)/((\mu_1 + 1, \ldots, \mu_{n+1})$$

$$\beta = (\lambda_1 + 1, \ldots, \lambda_n + 1, \lambda_{n+1})/((\mu_1 + 1, \ldots, \mu_{n+1})$$

$$\gamma = (\lambda_1 + 1, \ldots, \lambda_n + 1, \lambda_{n+1})/((\mu_1 + 1, \ldots, \mu_{n+1} + 1, \mu_{n+1}, \mu_n + 1).$$

Define

$$M_1 = \gamma_1 = (\lambda_1 + 1, \ldots, \lambda_n + 1, \lambda_{n+1} - (\mu_n - \mu_{n+1}) - 1)/$$

$$(\mu_1 + 1, \ldots, \mu_{n-1} + 1, \mu_{n+1}, \mu_{n+1})$$

and

$$\gamma^k(l_{n-k}, l_{n-k+1}, \ldots, l_{n-1})$$

$$= (\lambda_1 + 1, \ldots, \lambda_n + 1, \lambda_{n+1} - (\mu_{n-k} - \mu_{n+1})$$

$$- k - 1 - l_{n-k} - \cdots - l_{n-1})/$$

$$(\mu_1 + 1, \ldots, \mu_{n-k-1} + 1, \mu_{n-k+1} - l_{n-k}, \mu_{n-k+2}$$

$$- l_{n-k+1}, \ldots, \mu_{n} - l_{n-1}, \mu_{n+1}, \mu_{n+1}),$$

where $k = 1, \ldots, n - 1.$
Now define

\[ M_{v+2} = \sum_{k=1}^{n-1} \sum_{l_k \geq 0} (\otimes \mathbb{K} \left[ \frac{\mu_{n-k} - \mu_{n-k+1} + 1 + l_k}{l_k} \right] \otimes \mathbb{K} \left[ \frac{\mu_{n-k} - \mu_{n-k+2} + 2 + l_k}{l_k} \right] \otimes \cdots \otimes \mathbb{K} \left[ \frac{\mu_{n-k} - \mu_{n-k+l} + \cdots + l_{k-1}}{l_k + l_{k-1} + \cdots + l_1} \right])_{v-k+1} \otimes \gamma^k(l_{n-k}, \ldots, l_k). \]

Notice that the various \( \gamma^k \) terms enter the complex one dimension at a time. That is, \( M_2 \) just involves the \( \gamma^1 \) terms, \( M_3 \) the \( \gamma^1 \) and \( \gamma^2 \) terms, etc. Recall, too, that we always keep in force the convention that the shapes \( \gamma^k \) are \( \mathcal{O} \) if they are not skew-shapes.

To define the boundary map from \( M_1 \) to \( \gamma \) is easy; again it is induced by the identity map on the generators. For the definition of the boundary maps from \( M_{v+2} \) to \( M_{v+1} \), we must introduce some additional notation. We set

\begin{enumerate}
  \item \( e_{J^s} \) is a basis element of \( \mathbb{K}[J^s] \) with \( J^s \) an increasing sequence of indices. We write \( e_{J^s} \) if \( J^s = \mathcal{O} \);
  \item \( e_{J^s, j_\alpha} \) means the basis element corresponding to the index set \( J^s \) with \( j_\alpha \in J^s \) removed;
  \item \( e_{J^s, a_i} \) means \( e_{J^s} \) where \( k_\alpha = j_\alpha - \lambda_\alpha, j_\alpha \in J^s, \lambda_\alpha \in A^s \), and \( A^s \) is any sequence of the same length as \( J^s \);
  \item \( A^s < J^s \) means \( \lambda_\alpha < j_\alpha \) for all \( \alpha \) unless \( J^s = \mathcal{O} \), in which case \( A^s = \mathcal{O} \) and \( J^s = A^s \).
\end{enumerate}

If

\[ e_{J^s, a_i}^k \otimes \cdots \otimes e_{J^s, a_i}^k \otimes x \text{ is in } \mathbb{K} \left[ \frac{\mu_{n-k} - \mu_{n-k+1} + 1 + l_k}{l_k} \right] \otimes \cdots \otimes \mathbb{K} \left[ \frac{\mu_{n-k} - \mu_{n-k+l} + \cdots + l_{k-1}}{l_k + l_{k-1} + \cdots + l_1} \right] \otimes \gamma^k(l_{n-k}, \ldots, l_k), \tag{6} \]

we will assume that \( x \) is the image of \( x_1 \otimes \cdots \otimes x_{n+1} \), where the \( x_i \) are in suitable exterior or divided powers. That is, \( x_i \) is of degree \( \lambda_i - \mu_i \) for \( i = 1, \ldots, n-k-1 \), \( x_{n-k+j} \) is of degree \( \lambda_{n-k+j} - \mu_{n-k+j+1} + l_k \) for \( j = 0, \ldots, k-1 \), \( x_n \) is of degree \( \lambda_n - \mu_{n+1} + 1 \), and \( x_{n+1} \) is of degree \( \lambda_{n+1} - \mu_{n-k} - l_k \). Define
If $P = a$ and $s < k$, then we set the term = 0. If $P = 0$, and $s = k$, we replace $x_{ik = k}$ by $\sum_{i=1}^{k} - k_{s-1}$ above, and delete $\prod_{i=1}^{k} - k_{s-1}$.

In the expression (\*), the terms $\binom{j}{k}$ are multinomial coefficients. The terms

$$x_{n-s}(-j^{n-s}_{1}) \otimes x_{n-s}(k_{1}) \otimes \cdots \otimes x_{n-s}(j^{n-s}_{s-1}) \otimes x_{n-s}(j^{n-s}_{s}) - |a_{1}|$$

mean that $x_{n-s}$ has been subjected to an $(s+1)$-fold diagonalization where the degree of the first factor is the degree of $x_{n-s} \cdot j^{n-s}_{1}$ decreased by $j^{n-s}_{1}$, $x_{n-s}(k_{u})$ is the $(u+1)$st factor and is of degree $k_{u}$, while $x_{n-s}(j^{n-s}_{s} - |a_{1}|)$ is the $(s+1)$st factor and has, necessarily the degree $j^{n-s}_{1} - k_{1} - k_{2} - \cdots - k_{s-1}$.

With this notation in place, we define the boundary map from $M_{n+2}$ to $M_{n+1}$ on an element of the form (6) to be

$$\partial(e_{m-n-k}^{s} \otimes \cdots \otimes e_{m-n-1}^{s}) \otimes x + \sum_{s=1}^{k} \pm e_{m-n-k}^{s} \otimes \cdots \otimes e_{m-n-s}^{s} \otimes \delta_{m-n-s}(e_{m-n-s}^{s} \otimes \cdots \otimes x),$$

where $\partial(e_{m-n-k}^{s} \otimes \cdots \otimes e_{m-n-1}^{s})$ means the usual boundary map in the tensor product of complexes.

The next result we can describe is the resolution in dimensions 0, 1, and 2 of the skew-shape $\alpha = (\lambda_{1}, \ldots, \lambda_{n+1})/(\mu_{1}, \ldots, \mu_{n+1})$. Again we shall not distinguish between Schur and Weyl modules or between divided and exterior powers. To simplify notation, we set

$$\alpha_{i} = \lambda_{i} - \mu_{i}, \quad i = 1, \ldots, n+1$$

$$t_{i} = \mu_{i} - \mu_{i+1} + 1, \quad i = 1, \ldots, n.$$

For any sequence of integers $a_{1}, \ldots, a_{n+1}$, we denote by $(a_{1}, \ldots, a_{n+1})$ either
the tensor product of the divided powers $D_{a_1} \otimes \ldots \otimes D_{a_{n+1}}$ or exterior powers $\Lambda^{a_1} \otimes \ldots \otimes \Lambda^{a_{n+1}}$.

By $[\mathbb{F}]_v$ we mean the chains of degree $v$ of the complex $\mathbb{K}[\mathbb{F}]$. With these conventions, we can write down the first terms of our resolution.

Degree 0:

$$(\alpha_1, \ldots, \alpha_{n+1})$$

Degree 1:

$$\sum_{l \geq 0} \left[ \begin{array}{c} t_1 + l \\ l \end{array} \right]_0 \otimes (\alpha_1 + t_1 + l, \alpha_2 - t_1 - l, \alpha_3, \ldots, \alpha_{n+1})$$

$$\ldots \oplus \sum_{l \geq 0} \left[ \begin{array}{c} t_n + l \\ l \end{array} \right]_0 \otimes (\alpha_1, \ldots, \alpha_n, \alpha_n + t_n + l, \alpha_{n+1} - t_n - l)$$

Degree 2:

$$\sum_{l \geq 1} \left[ \begin{array}{c} t_1 + l \\ l \end{array} \right]_1 \otimes (\alpha_1 + t_1 + l, \alpha_2 - t_1 - l, \alpha_3, \ldots, \alpha_{n+1})$$

$$\ldots \oplus \sum_{l \geq 1} \left[ \begin{array}{c} t_n + l \\ l \end{array} \right]_1 \otimes (\alpha_1, \ldots, \alpha_n, \alpha_n + t_n + l, \alpha_{n+1} - t_n - l)$$

$$\oplus \sum_{(i,i+1), 1 \leq i \leq n-1} \sum_{l,k} \left[ \begin{array}{c} t_i + l \\ l \end{array} \right]_0 \otimes \left[ \begin{array}{c} t_i + t_{i+1} + l + k \\ k \end{array} \right]_0$$

$$\oplus (\alpha_1, \ldots, \alpha_i + t_i + l, \alpha_{i+1} + t_{i+1} + k, \alpha_{i+2} - t_i - t_{i+1} - l - k, \ldots, \alpha_{n+1})$$

$$\oplus \sum_{(2,i-1), 2 \leq i \leq n} \sum_{l,k} \left[ \begin{array}{c} t_i + l \\ l \end{array} \right]_0 \otimes \left[ \begin{array}{c} t_{i-1} + t_i + l + k \\ k \end{array} \right]_0$$

$$\oplus (\alpha_1, \ldots, \alpha_{i-1} + t_{i-1} + t_i + l + k, \alpha_i - t_{i-1} - k, \alpha_{i+1} - t_i - l, \ldots, \alpha_{n+1})$$

$$\oplus \sum_{(i,j), 3 \leq i \leq n, i-j \neq 2} \sum_{l,k} \left[ \begin{array}{c} t_i + l \\ l \end{array} \right]_0 \otimes \left[ \begin{array}{c} t_j + k \\ k \end{array} \right]_0$$

$$\oplus (\alpha_1, \ldots, \alpha_j + t_j + k, \alpha_j + t_j + l, \alpha_{i+1} - t_i - l, \ldots, \alpha_{n+1}).$$

The map from degree 1 to degree 0 is just the standard $\square$ map in the presentation of Schur or Weyl modules [2, II.2, 3]. From degree 2 to degree 1 we essentially have four types of maps: Those from the single summation terms, and those from each of the double summation terms.
The map from the $i$th single summation terms goes as follows:

$$
\epsilon_i^l \otimes x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_{n+1}
$$

$$
\mapsto \left( t_i + l \right) \epsilon_0^l \otimes x_1 \otimes \cdots \otimes x_{n+1} - \epsilon_0^{l-\lambda} \otimes x_1
\otimes \cdots \otimes x_i(\alpha_i + t + l - \lambda) \otimes x_i(\lambda) x_{i+1} \otimes \cdots \otimes x_{n+1},
$$

where $\epsilon_2^l$ is a basis element of $[t_i + l]'_1$ and $\epsilon_0^l$ is the identity or basis element of $[t_i + l]'_0 = \mathbb{Z}$. The integer $(t_i + l)$ is the binomial coefficient. By $x_i(\alpha_i + t_i + l - \lambda) \otimes x_i(\lambda)$ we mean the diagonalization of $x_i$ into its components of bidegree $(\alpha_i + t_i + l - \lambda, \lambda)$.

The map from the first double summation term is

$$
\epsilon_0^l \otimes \epsilon_0^k \otimes x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes x_{i+2} \otimes \cdots \otimes x_{n+1}
$$

$$
\mapsto \epsilon_0^l \otimes x_1 \otimes \cdots \otimes x_i \otimes x_{i+1}(\alpha_{i+1} - t_i - l)
\otimes x_i^l(t_i + t_{i+1} + l + k) x_{i+2} \otimes \cdots \otimes x_{n+1}
\sum_{u \geq 0} (-1)^u \epsilon_0^{k+u} \otimes x_1 \otimes \cdots \otimes x_i(\alpha_i) \otimes x_{i+1} x_j(u)
\otimes x_i^l(t_i + l - u) x_{i+2} \otimes \cdots \otimes x_{n+1},
$$

where by $x_i(\alpha_i) \otimes x_j(u) \otimes x_i^l(t_i + l - u)$ we mean the three-fold diagonalization of $x_i$ into its components of indicated tridegree.

The map from the second double summation term is

$$
\epsilon_0^l \otimes \epsilon_0^k \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_{n+1}
$$

$$
\mapsto \epsilon_0^l \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes x_i \otimes x_{i-1}(\alpha_{i-1}) \otimes x_i(t_i - t_i + l + k) x_i
\otimes x_{i+1} \otimes \cdots \otimes x_{n+1}
\sum_{u \geq 0} \epsilon_0^{k+u} \otimes x_1 \otimes \cdots \otimes x_{i-1}(\alpha_{i-1} + t_{i-1} + k + u)
\otimes x_i(\alpha_i - t_{i-1} - k - u) \otimes x_{i-1}(t_i + l - u) x_j(u) x_{i+1} \otimes \cdots \otimes x_{n+1}.
$$

The map from the third double summation term is

$$
\epsilon_0^l \otimes \epsilon_0^k \otimes x_1 \otimes \cdots \otimes x_j \otimes x_{j+1} \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_{n+1}
$$

$$
\mapsto \epsilon_0^l \otimes x_1 \otimes \cdots \otimes x_j(\alpha_j) \otimes x_j(t_j + k) x_{j+1}
\otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_{n+1} - \epsilon_0^k \otimes x_1 \otimes \cdots \otimes x_j \otimes x_{j+1}
\otimes \cdots \otimes x_i(\alpha_j) \otimes x_j(t_j + l) x_{i+1} \otimes \cdots \otimes x_{n+1}.
$$
APPENDIX

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REFERENCES