The aim of the Schur–Cohn algorithm is to compute the number of roots of a complex polynomial in the open unit disk, each root counted with its multiplicity. Unfortunately, in its original form, it does not work with all polynomials. Using technics similar to those of the sub-resultants, we construct a new sequence of polynomials, the Schur–Cohn sub-transforms. For this, we propose an algorithm in only $O(d^2)$ arithmetical operations ($d$ being the degree of the polynomial studied), which is well adapted to computer algebra and supports specialization. We then show how to use bezoutians and hermitian forms to compute the number of roots in the unit disk with the help of the Schur–Cohn sub-transforms we have built.

1. Introduction

Let $P = a_d X^d + \cdots + a_0$ be a complex polynomial. The object of the Schur–Cohn algorithm is to compute the number, $\# P$, of roots of $P$ contained in the open unit disk $D$. Each root is counted with its multiplicity. This problem is involved in the more general problem of isolation of the roots of a polynomial, but also in the problem of the stability of discrete dynamic system; it is particularly useful in the theory of stability of filter-basis in the signal theory. It was described for the first time by Schur (1918) and Cohn (1922).

It works as follows.

Let $T(1)P = \bar{a}_0 P - a_d P^*$ be the first Schur–transform of $P$ where $P^* = X^d \bar{P}(1/X) = \bar{a}_0 X^d + \cdots + \bar{a}_d$. $T(1)P$ is considered as a polynomial of exact degree $d - 1$. It has real constant term and, applying Rouche’s theorem, assuming that $P$ has no root on the unit circle, we can say that $\# P = \# T(1)P$ if $T(1)P(0) > 0$ and that $\# P = d - \# T(1)P$ if $T(1)P(0) < 0$. We can iterate the process and compare $\# P$ successively with $\# T(1)P$, then with $\# T(2)P$ where $T(2)P = T(1)(T(1)P)$, and so on, as long as we do not meet a $T(i)P(0) = 0$, in which case no more comparison between $\# P$ and $\# T^{(i)}P$ is possible.

As the degrees of the $T^{(i)}Ps$ are strictly decreasing, the algorithm ends up, the last transform of $P$ being a constant polynomial, and furnishes the number of roots of $P$ in the case where no $T^{(i)}P(0) = 0$; otherwise the algorithm fails. For a complete description of this elementary method, we refer the reader to Henrici (1974).

It is easy to verify that, when we use exact arithmetic on a computer, the length of the coefficients of the Schur transforms, $T^{(k)}P$, are approximately multiplied by 2 at each step and therefore, the algorithm is in exponential complexity in the degree of $P$. 

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In a previous article, Saux Picart (1993), we introduced the concept of the Schur–Cohn sub-transform. The aim was to replace the $T^{(k)} P$ by other polynomials, proportional to them, with much smaller coefficients. The number of arithmetic operations is always in $O(d^2)$, but with this trick, we reduced the total complexity to $O(d^4 t^2)$ in the case where $P \in \mathbb{Z}[X]$ (here $t$ is the maximal length of the coefficients of $P$).

The Schur–Cohn sub-transforms are defined as follows. Let us denote by $\text{Sylv}_k(P, P^*)$ the matrix of Sylvester of order $k$ constructed with $P$ and $P^*$; it is the matrix $2k \times k + d$ defined by

$$
\text{Sylv}_k(P, P^*) = \begin{pmatrix}
    a_0 & a_1 & \cdots & \cdots & a_d \\
    a_0 & \cdots & \cdots & \cdots & a_d \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    \ddots & \ddots & \ddots & \ddots & \ddots \\
    \bar{a}_d & \bar{a}_{d-1} & \cdots & \cdots & \bar{a}_0 \\
    \bar{a}_d & \cdots & \cdots & \cdots & \bar{a}_0 \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}^{k},
$$

Then, the $k$-th Schur–Cohn sub-transform, $sT^{(k)} P$, is the polynomial of degree $d - k$, whose coefficient of $X^i$ is given by the determinant of the matrix $2k \times 2k$ extracted from $\text{Sylv}_k(P, P^*)$ by using the columns $1, 2, \ldots, k - 1, k + i, d + 1, \ldots, d + k$ in this order. In our previous article (Saux Picart, 1993), we proved the following result.

**Theorem 1.1. (Schur–Cohn sub-transforms)** If we note $a_0^{(i)} = T^{(i)} P(0)$, then we have for $k \geq 2$,

$$
a_0^{(1)k-2} a_0^{(2)k-3} \cdots a_0^{(k-2)} sT^{(k)} P = T^{(k)} P.
$$

As a direct consequence, we obtain the fact that, if $a_0^{(i)} \neq 0$ for $i = 1, \ldots, k - 2$, we have

$$
sT^{(k)} P = \frac{T^{(1)} sT^{(k-1)} P}{a_0^{(k-2)}}.
$$

This gives us a nice way to compute the sub-transforms without having to use their determinantal definition. However, this result does not show how to compute $sT^{(k)} P$ from $T^{(k)} P$ if some $a_0^{(i)}$ appear to be zero for $1 \leq i \leq k - 2$.

Furthermore, we showed that, in the case where no constant term is found to be zero, we can compute $\# P$ by considering the variation of signs of the sequence of these constant terms:

$$
\# P = \mathcal{V}(1, sT^{(1)} P(0), \ldots, sT^{(d)} P(0)).
$$

However, this result does not make sense when one of the $sT^{(i)} P(0)$ vanishes. In order to exemplify our problems let us consider a very elementary polynomial $R = X^4 - 8X^3 + 21X^2 + 9X - 1$; we have

$$
T^{(1)} R = X^3 + 42X^2 + X \\
T^{(2)} R = -X^2 - 42X - 1 \\
T^{(3)} R = 0
$$

†We draw the attention of the reader on the fact that we use a definition of $\text{Sylv}_k(P, P^*)$ slightly modified from the one used in Saux Picart (1993). This is done to simplify some results, but it introduces some minor modifications from that article.
but, we can compute the Schur–Cohn sub-transforms using the definition:

\[
\begin{align*}
ssT^{(1)} R &= -X^3 - 42X^2 - X \\
ssT^{(2)} R &= -X^2 - 42X - 1 \\
ssT^{(3)} R &= -2121X - 101 \\
ssT^{(4)} R &= 4488440.
\end{align*}
\]

The questions are: how to compute \( ssT^{(3)} R \) without using Theorem 1.1 or the definition and then, what is the value of \( \# R \)?

In the present article, we will define the concept of Schur–Cohn sub-transforms in a more general way (not only for polynomials with complex coefficients, but for polynomials with coefficients in a suitable ring), and give an algorithm which computes the sequence of polynomials \( ssT^{(i)} P \) in any situation. It will work nicely under a specialization process in order to study, for example, polynomials depending on parameters. It does not use the direct definition of the \( ssT^{(i)} P \) but only polynomial computations and it needs \( \mathcal{O}(d^2) \) operations in the ring of the coefficients. Then, using bezoutians and signature of hermitian forms, we will show how to use the sequence of \( ssT^{(i)} P(0) \), \( i = 1, \ldots, d \) to obtain \( \# P \) even when some \( ssT^{(i)} P(0) \) vanish.

2. Some Generalities

We consider an integral ring \( A \), and we suppose that on this ring we can define an isomorphic involution, we will call a conjugation as in \( C \). If \( a \in A \), we will denote \( \bar{a} \) its conjugate. We can also extend this involution to the quotient field of \( A \), \( \mathcal{K}(A) \), by putting

\[
\frac{a}{b} = \frac{\bar{a}}{\bar{b}}.
\]

Typically, \( A \) will be \( C[Y_1, \ldots, Y_k] \), or \( R[Y_1, \ldots, Y_k] \) or \( Z[[Y_1, \ldots, Y_k]] \). We will denote \( |a|^2 = a\bar{a} \), as in \( C \). It is clear that, if \( a \) and \( b \) are different from 0 in \( A \), \( |a|^2 = |b|^2 \) is true if and only if there exists \( u \in \mathcal{K}(A) \) such that \( a = ub \) with \( |u|^2 = 1 \).

Then, let us consider a polynomial \( P \in A[X] \). We note \( d \) its degree:

\[
P = a_dX^d + \cdots + a_0.
\]

We continue to denote by \( P^* \) the reciprocal polynomial of \( P \) in \( A[X] \); \( P^* = X^d \bar{P}(1/X) = \bar{a}_dX^d + \cdots + \bar{a}_0 \). Let us mention two obvious little results.

**Lemma 2.1.** Let \( P = P_1X^k \in A[X] \). Then \( P^* = P_1^* \).

Let \( \deg P = d \), and let us consider \( \bar{P} = 0X^{d+k} + \cdots + 0X^{d+1} + P \). \( \bar{P} \) is simply \( P \) viewed as a polynomial of degree \( d + k \). Then, \( \bar{P}^* = X^kP^* \).

2.1. Some Properties of the Schur Transforms

We continue to denote \( TP = T^{(1)} P = \bar{a}_dP - a_dP^* \). Its degree is at most \( d - 1 \). More generally, \( T^{(k)}P = T(T^{(k-1)}P) \). It is a well-known fact, that the sequence of \( T^{(k)}P \) is related to the gcd of \( P \) and \( P^* \), \( P \wedge P^* \) computed in \( \mathcal{K}(A)[X] \).

**Proposition 2.1.** If, for \( i = 1, \ldots, k \), we have \( T^{(i)}P(0) \neq 0 \), and if \( T^{(k+1)}P \) is identically zero, then \( P \wedge P^* = T^{(k)}P \).
(We omit the proof of this proposition, which can be found in Barnett (1983).)

We have to notice that the condition on the $T^{(1)}P(0)$ is essential. For example, $R \land R^\ast = 1$ in $\mathbb{C}$ (the resultant $\text{Res}(P, P^\ast) = ssT^{(1)}R \neq 0$); then $T^{(3)}R = 0$ does not imply that $T^{(2)}R = P \land P^\ast$. So the Schur–Cohn algorithm is not realistic to compute $P \land P^\ast$.

We now particularly look at these special cases where the constant term of $TP$ vanishes.

We call a polynomial $P$ such as $TP(0) = 0$ defective. In some cases, the valuation of $T^{(1)}P$ can be strictly more than 1. For example, let $S = X^8 - 12X^7 - 4X^6 + 10X^3 - 72X^2 + 31X^3 - 4X^2 - 12X + 1$; then $T^{(1)}S = -21X^2 + 21X^3$ with valuation 3. In fact, the next proposition will show that the examples $R$ and $S$ are the generic cases.

**Proposition 2.2.** Let $P = a_dX^d + \cdots + a_0$ be defective, with $a_da_0 \neq 0$. Put $v = \text{val}(TP)$, the valuation of $P$. Then

$$\text{deg}(TP) = d - v.$$  

**Proof.** We have $TP(0) = 0 \iff |a_0| = |a_d|$. The coefficient of $X^i$ in $T(P)$ is $\bar{a}_0a_i - a_da_{d-i}$. Then we have:

$$\bar{a}_0a_i - a_da_{d-i} = 0 \iff |a_0|^2\bar{a}_da_i - a_0|a_d|^2a_{d-i} = 0 \iff a_0a_i - a_0a_{d-i} = 0.$$  

But this term is the opposite of the conjugate of the coefficient of degree $d - i$ of $T^{(1)}P$. □

If $P$ is defective, let us write $ua_d = \bar{a}_0$ where $u$ is in $\mathcal{F}(A)$ such that $|u|^2 = 1$. If $v = \text{val}(TP)$, we see from the proposition above that for $i = 1, \ldots, v$, we have $ua_{d-i} = \bar{a}_i$. Hence, a defective polynomial can be written as $P = FX^{d-v+1} + X^vC + uF^\ast$ where $F$ is a polynomial with $\text{deg}(F) = v - 1$ and $C$ is a polynomial of less or equal degree to $d - 2v$ such that $\text{lc}(C) \neq uC(0)$, where $\text{lc}(C)$ is the leading coefficient of $C$. We will say more precisely in this case that $P$ is $v$-defective.

**Proposition 2.3.** If $P$ is $v$-defective, unitary, then we have, using the notation described above,

$$(TP)^\ast = -u\frac{T(P)}{X^v}.$$  

**Proof.** With our notations, we have $TP = \bar{u}P - P^\ast = b_{d-i}X^{d-v} + \cdots + b_vX^v$. The coefficient of degree $i$ of $TP$ is $b_i = \bar{u}a_i - \bar{a}_{d-i}$, and that of degree $d - i$, $b_{d-i} = ua_{d-i} - \bar{a}_i$. Then we have $b_i = -u\bar{a}_{d-i}$. □

It is easy to verify that a polynomial $P$ is such as $T^{(1)}P \equiv 0$ if and only if it exists $u \in \mathcal{F}(A)$, $|u|^2 = 1$, such as for every $i = 0, \ldots, d$ we have $a_{d-i} = u\bar{a}_i$. We will say that such a polynomial is symmetrical.

### 2.2. Schur sub-transforms

We will note $I_k$ the identity matrix of order $k$ and $V_{i,j}^k$ the matrix $(d - k + i + j + 1) \times (i + j + 1)$ defined by

$$V_{i,j}^k = \begin{pmatrix} I_i & 0 \\ 0 & V_k \\ 0 & I_j \end{pmatrix}$$
where $V_k$ is the transposed of the vector 

$$(0, 0, \ldots, 0, 1, X, X^2, \ldots, X^{d-k}, 0, \ldots, 0),$$

with $i$ zeros at the beginning and $j$ at the end.

$P$ being a polynomial in $A[X]$, we use the same definition of $\text{Sylv}_k(P, P^*)$ as in the introduction. We will use also more general definitions: if $P$ and $Q$ are two polynomials in $A[X]$, we denote $\text{Sylv}_{k,k}(P, Q)$ the $(k + k') \times \sup(\deg P + k, \deg Q + k')$ matrix whose rows are composed of the coefficients of the polynomials $P, XP, \ldots, X^{k-1}P, Q, XQ, \ldots, X^{k'-1}Q$. We will also write $\text{Sylv}_k(P)$ to denote the $k \times (d + k)$ matrix whose rows are composed of the coefficients of the polynomials $P, XP, \ldots, X^{k-1}P$.

We can then write $ssT^{(k)}P$ in a condensed shape as a polynomial determinant:

$$ssT^{(k)}P = \det(\text{Sylv}_k(P, P^*) \cdot V_{k-1,k}^k).$$

It is not difficult to verify that $ssT^{1}P = TP$ and we see from the definition that $ssT^{(d)}P$ is the resultant of $P$ and $P^*$, $\text{Res}(P, P^*)$.

If we replace $P$ by $\alpha P$ where $\alpha \in \mathcal{F}(A)$, then we have

$$ssT^{(k)}(\alpha P) = |\alpha|^{2k} ssT^{(k)}P.$$

So, throughout this article, we will consider that $P$ is an unitary polynomial for simplicity.

Another simple case is considered in the following result.

**Proposition 2.4.** If $P = a_0X^d + \cdots + a_\alpha X^\alpha = X^\alpha P_1$ in $A[X]$ ($\alpha \geq 1$), then, for $k$ such that $1 \leq k \leq \alpha$, we have

$$ssT^{(k)}P = (-1)^k |a_d|^{(2k-1)} a_d P_1^*$$

and for $k \in [\alpha, d]$,

$$ssT^{(k)}P = (-1)^\alpha |a_d|^{2\alpha} ssT^{(k-\alpha)}P_1.$$

**Proof.** We expand the determinant defining $ssT^{(k)}P$, using the first and last columns where only one coefficient is different from 0. We have for $k = 1$

$$ssT^{(k)}P = (-1)|a_d|^2 \det(\text{Sylv}_{k-1,k-1}(X^{\alpha-1}P_1, P_1^*) \cdot V_{k-2,k-1}^{d-k}).$$

Iterating this process $k - 1$ times in the case where $k \leq \alpha$, we obtain

$$ssT^{(k)}P = (-1)^{k-1} |a_d|^{2(k-1)} \det(\text{Sylv}_{1,1}(X^{\alpha-k+1}P_1, P_1^*) \cdot V_{0,1}^{d-k}).$$

Hence the result is obtained by computing the $2 \times 2$ determinant. If $k > \alpha$, we obtain, using the same trick $\alpha$ times,

$$ssT^{(k)}P = (-1)^\alpha |a_d|^{2\alpha} \det(\text{Sylv}_{k-\alpha}(P_1, P_1^*) \cdot V_{k-\alpha-1,k-\alpha}^{d-k}).$$

A symmetrical result is achieved when the real degree of the polynomial is less than $d$.

**3. Computation of Schur–Cohn Sub-transforms**

The aim of this section is to design an algorithm to compute the sequence of the Schur–Cohn sub-transforms of a polynomial of $A[X]$. 
3.1. A FIRST RESULT IN THE NON-DEFECTIVE CASE

Proposition 3.1. For all $i = 2, 3, \ldots, d$, we have, whenever $\deg ssT^{(i-1)}P = d - i + 1$,
$$ssT^{(i-2)}P(0)ssT^{(i)}P = T(ssT^{(i-1)}P)$$ (3.1)
(here we write $ssT^{(0)}P(0) = 1$). Furthermore, for $i = 2, 3, \ldots, d-1$ and $j = 1, 2, \ldots, d-i$,
we have
$$ssT^{(i)}(ssT^{(i)}P)ssT^{(i)}P(0)^j ssT^{(i)+1}P.$$ (3.2)

Proof. First, we generalize the relation of Theorem 1.1. We will need some more general concept of sub-transform. Let $U = U_0 + \cdots + U_dX^d$ be a polynomial of degree $d$
in $\mathbb{Z}[A_0, A_d, B_0, \ldots, B_d][X]$, where $A_0, A_d, B_0, \ldots, B_d$ are independent indeterminates. If $V$ is an element of $\mathbb{Z}[A_0, \ldots, A_d, B_0, \ldots, B_d]$, we define $\bar{V}$ as the polynomial obtained
by exchanging in $V$, $A_0$ with $B_0$, $A_1$ with $B_1$, \ldots, $A_d$ with $B_d$. We define also $U^* = U_0^* + \cdots + U_d^*X^d$ as the polynomial of degree $d$
such that $U_0^* = \bar{U}_d$. We can then define an extended concept of Schur–Cohn sub-transform:
$$ssT^{(i)}_Z(U) = \det(Sylv_i(U, U^*) \cdot V^{d-i}).$$

It is a polynomial in $\mathbb{Z}[A_0, \ldots, A_d, B_0, \ldots, B_d][X]$. Let us define
$$TU = \text{lc}(U^*)U - \text{lc}(U)U^* = U^{(1)},$$
where $\text{lc}(V)$ denotes the leading coefficient of $V$. This polynomial is of exact degree $d - 1$,
otherwise the leading coefficient of degree $d - 1$ would vanish and then define a relation
of dependency on the variables $A_0, \ldots, A_d, B_0, \ldots, B_d$. We can then iterate the process, defining
$$T^{(i)}U = T(T^{(i-1)}U) = U^{(i)}.$$ We will denote $U^{(i)} = U_0^{(i)} + U_1^{(i)}X + \cdots + U_d^{(i)}X^{d-i}$; the coefficients of this polynomial
are polynomials of $\mathbb{Z}[A_0, \ldots, A_d, B_0, \ldots, B_d]$ and we observe that their constant terms
are all different from 0, because of the independence of the indeterminates.

We observe also that $ssT^{(i)}_Z(U) = TU$. Particularly, we have $U_0^{(1)} = \bar{U}_dU_0 - \bar{U}_dU_d$ and
more generally,
$$U_0^{(i)} = U_0^{(i-1)}U_0^{(i-1)} - U_d^{(i-1)}U_d^{(i-1)}.$$ Moreover, $\bar{U}_d^{(i)} = \bar{U}_d^{(i)}$. And we see that, if $W$ is an element of $\mathbb{Z}[A_0, \ldots, B_d]$, then
$T(WU) = WWT(U)$. It is clear that, if we use $U = A_0 + \cdots + A_dX^d$, we can consider the Schur–Cohn
sub-transform $ssT^{(k)}(P)$, as a specialization of $ssT^{(k)}_Z(U)$ (defined by sending the $A_i$ on
$a_i$ and the $B_i$ on $a_i$). The specialization of $TU$ is $TP$ whenever $\deg U = \deg P$.

Then we show that
$$U_0^{(k-2)}U_0^{(k-3)} \cdots U_0^{(k-2)} \cdot \frac{ssT^{(k)}(U)}{U^{(k)}} = U^{(k)}.$$ We define the $2k \times 2k$ matrix, $\Lambda_{k,i}$, by
$$\Lambda_{k,i} = \begin{pmatrix}
U_0^{(i)}I_k & -U_{d-1}^{(i)}I_k \\
-U_{d-i}^{(i)}I_k & U_0^{(i)}I_k
\end{pmatrix}.$$
We have det($A_{k,0}$) = $U_0^{(1)}$ ≠ 0. Then $ssT_{Z}^{(k)}(U)$ verifies

$$U_0^{(1)} \cdot ssT_{Z}^{(k)}(U) = \det(A_{k,0} \cdot \text{Sylv}_k(U, U^*) \cdot V_{k-1,k}^k)$$

$$= \det(\text{Sylv}_k(U^{(1)}, U^{(1)*}) \cdot V_{k-1,k}^k),$$

here, in $\text{Sylv}_k(U^{(1)}, U^{(1)*})$, the polynomials $U^{(1)}$ and $U^{(1)*}$ are considered as polynomials of degree $d$.

We can expand the determinant with respect to the first and the last column, where only one term is not 0 (do not forget that the determinant is of even dimension $2k \times 2k$):

$$U_0^{(1)} \cdot ssT_{Z}^{(k)}(U) = \det(\text{Sylv}_{k-1}(U^{(1)}, U^{(1)*}) \cdot V_{k-2,k-1}^k).$$

We iterate this process $k-2$ times, multiplying successively by the matrices $A_{k-i,i}$, whose determinant is $U_0^{(1)i-k-i} \neq 0$. Hence

$$U_0^{(1)i-k-k} \cdot ssT_{Z}^{(k)}(U) = \det(\text{Sylv}_1(U^{(k-1)}, U^{(k-1)*}) \cdot V_{0,1}^k) = U^{(k)}.$$

It is then an easy task to verify by recurrence that

$$ssT_{Z}^{(k-2)}(U)(0) \cdot ssT_{Z}^{(k)}(U) = \mathcal{T}(ssT_{Z}^{(k-1)}(U)).$$

We specialize, as previously described, to obtain the first result of the proposition.

We demonstrate the second relation by recurrence on $j$. For simplicity’s sake, we put $S_i = ssT_{Z}^{(j)}(U)$. For $j = 1$ and all $i \geq 2$, the relation is the previous formula:

$$ssT_{Z}^{(1)}(S_i) = \mathcal{T}S_i = S_{i-1}(0)S_{i+1}.$$

Suppose then the relation is true up to $j$:

$$ssT_{Z}^{(j)}(S_i) = S_{i-1}(0)^{j}S_{i}(0)^{j-1}S_{i+j}.$$

We have

$$ssT_{Z}^{(j-1)}(S_i)(0) \cdot ssT_{Z}^{(j+1)}(S_i) = \mathcal{T}(ssT_{Z}^{(j)}(S_i))$$

$$= \mathcal{T}(S_{i-1}(0)^{j}S_{i}(0)^{j-1}S_{i+j})$$

$$= S_{i-1}(0)^{j+1}S_{i}(0)^{j-1} \cdot \mathcal{T}(S_{i+j})$$

$$= S_{i-1}(0)^{j+1}S_{i}(0)^{j-1}S_{i+j-1}(0)S_{i+j+1}.$$

Using the relation of recurrence once again, we can also write

$$ssT_{Z}^{(j-1)}(S_i)(0) \cdot ssT_{Z}^{(j+1)}(S_i) = S_{i-1}(0)^{j-1}S_{i}(0)^{j-2}S_{i+j-1}(0)ssT_{Z}^{(j+1)}(S_i).$$

Comparing these two results, we obtain

$$ssT_{Z}^{(j+1)}(S_i) = S_{i-1}(0)^{j+1}S_{i}(0)^{j}S_{i+j+1}.$$

The second relation of the proposition is produced by specialization as above. □

### 3.2. Schur–Cohn sub-transforms of a defective polynomial

Let us now attack the more technical part of this article. If the polynomial $P$ is defective we can no longer use Proposition 3.1 to compute the sequence of $ssT^{(k)}P$ because $TP(0) = 0$; we need then some other results!
**Theorem 3.1.** Let $P \in \mathbb{A}[X]$ be a $v$-defective unitary polynomial. We write $P = FX^{d-v+1} + X^vC + uF^*$ as described before, and we note $TP = b_dX^{d-v} + \cdots + b_0X^v$ ($b_v \neq 0$). Then we have

$$ssT^{(1)}P = TP,$$

For $k = 2, \ldots, 2v-1$, $ssT^{(k)}P \equiv 0$.

$$ssT^{(2v)}P = (-1)^v b_v |b_v|^{2(v-1)} \frac{TP}{X^v}.$$  \hfill (3.3)

And, finally, for $l = 1, \ldots, d-2v$,

$$ssT^{(2v+l)}P = \frac{(-1)^{v+l}}{|b_v|^{2(v-1)}} ssT^lQ$$  \hfill (3.4)

where

$$Q = \left(\frac{-b}{X}\right)^v \left[ |b_v|^{2v}P^* + TP \left( b_{d-v} A^* \frac{X^v}{V} - \tilde{b}_{d-v} A X \right) \right]$$  \hfill (3.5)

with $A$ denoting the pseudo-quotient of $P^*$ by $XTP$. Here $Q$ is considered of degree exactly $d-2v$.

**Proof.** As the proof is rather long, we subdivide it in several parts, one for each case. First, let us show a prerequisite formula. Multiplying each of the first $k$ rows of $\text{Sylv}_k(P, P^*)$ by $\overline{u}$, we obtain

$$ssT^{(k)}P = \det(\text{Sylv}_k(P, P^*) \cdot V_{k-1,k}^k) = \overline{u}^{-k} \det(\text{Sylv}_{k,k}(\overline{u}P, P^*) \cdot V_{k-1,k}^k).$$

Then, adding to the $i$-th row the opposite of the $k+i$-th ($i = 1, \ldots, k$), we obtain

$$ssT^{(k)}P = \overline{u}^{-k} \det(\text{Sylv}_{k,k}(TP, P^*) \cdot V_{k-1,k}^k).$$ \hfill (3.6)

For $P$ $v$-defective with $v \geq 1$, the matrix $\text{Sylv}_{k,k}(TP, P^*)$ has a very special shape: the first $v$ and the last $v$ rows have only 0 on the first $k$ rows; we will use this fact to begin in each situation. The first relation announced, the value of $ssT^{(1)}P$, is a simple remark already made. Therefore, we go on with the second case. \(\square\)

**Proof for** $2 \leq k \leq 2v-1$. We start with the formula (3.6); if $k$ is even, we expand the determinant using, in this order, the rows $k+1, k+2, \ldots, k+\left[\frac{k}{2}\right]$ and then the rows $2k, 2k-1, \ldots, 2k - \left[\frac{k}{2}\right] + 1$, because, at each step, the first or last column contains only one term different from 0. Since we delete $\left[\frac{k}{2}\right]$ columns on the right of the matrix and since $\left[\frac{k}{2}\right] < v$, there remains at least one column on the right which contains only 0, and, therefore, the determinant vanishes. If $k$ is odd, we expand the determinant using the rows of order $k+1, k+2, \ldots, k+\left[\frac{k}{2}\right] + 1$ and $2k, 2k-1, \ldots, 2k - \left[\frac{k}{2}\right] + 1$, and obtain the same result. \(\square\)

**Proof for** $k = 2v$. If $k = 2v$, the development of the determinant using the formula (3.6) gives:

$$ssT^{(2v)}P = \overline{u}^{-2v} \det(\text{Sylv}_{2v,2v}(TP, P^*) \cdot V_{2v-1,2v}^{2v})$$

$$= \overline{u}^{-v} \det \left( \text{Sylv}_{2v} \left( \frac{TP}{X^v} \right) \cdot V_{v-1,v}^{2v} \right),$$
because, removing line by line, we delete the 2\(v\) rows where \(P^*\) appears.

Now, how can we obtain the coefficients of the polynomial \(ssT^{(2v)}(P^*)\)? The coefficient of \(X^i\) is obtained by multiplying \(\bar{a}^{-v}\) by the minor of the matrix \(\text{Sylv}_{2v} \left( \frac{TP}{X^v} \right)\) constructed by using the columns of order \(1, 2, \ldots, v-1, v+i, d-v+1, \ldots, d\) in this order; its shape looks like this:

\[
\begin{vmatrix}
 b_v & \cdots & b_{2v-2} & \times & 0 & 0 \\
 \vdots & \ddots & \vdots & \times & \vdots & \vdots \\
 b_v & \times & b_{v+i} & 0 \\
 \times & \vdots & \vdots & \vdots & \times & 0 \\
 \times & b_{d-v} & \vdots & \vdots & \vdots & \vdots \\
 \times & \vdots & \vdots & \times & b_{d-2v+1} & \cdots & b_{d-v}
\end{vmatrix}
\]

But this determinant is exactly \(b_v^{-1}b_{d-v}^v b_{d+i}\). Then we have

\[
ssT^{(2v)}(P^*) = \bar{a}^{-v}b_v^{-1}b_{d-v}^v \sum_{i=0}^{d-2v} b_{v+i}X^i = (-1)^v b_v |b_v|^{2(v-1)} T P \frac{X^v}{X^d}.
\]

This is the second relation we had to prove. \(\square\)

**Proof for** \(k = 2v + l\) (\(l = 1, 2, \ldots, d-2v\)). We start once again from the formula 3.6. Using the same process as before, we delete \(2v\) of the last \(2v + l\) columns in the matrix \(\text{Sylv}_{2v+1,2v+l}(TP, P^*)\). We obtain:

\[
ssT^{(2v+l)}(P^*) = (-1)^v b_v^{-1} \det \left( \text{Sylv}_{2v+1,l} \left( \frac{TP}{X^v}, P^* \right) \cdot \mathbf{v}^{2v+l}_{v+l-1,v+l} \right).
\]

Let us define the polynomial \(A\) as pseudo-quotient of \(P^*\) by \(XTP\). It is a polynomial of degree \(v-1\). Let us call \(Q_1\) the associated pseudo-remainder:

\[
Q_1 = b_v^v - A.XTP.
\]

The coefficients of \(Q_1\) are obtained as linear combinations of the coefficients of \(XTP, X^2TP, \ldots, X^vTP\) and of \(P^*\), the coefficients of linearity being those of \(Q_1\). More generally, for \(1 \leq i \leq l-1\), \(X^iQ_1 = b_v^{-v}X^iT P^* - A.X^{i+1}TP\) has coefficients which are linear combinations of the coefficients \(X^{i+1}TP, X^{i+2}TP, \ldots, X^{v+i}TP\) and of \(X^iT P^*\). We temporarily denote \(\hat{Q}_1\) the polynomial \(Q_1\) viewed as a polynomial of degree \(d\).

We use the combinations described above in the computation of

\[
\det \left( \text{Sylv}_{2v+l,l} \left( \frac{TP}{X^v}, P^* \right) \cdot \mathbf{v}^{2v+l}_{v+l-1,v+l} \right)
\]

and develop it with respect to the last \(v\) columns, then we obtain

\[
ssT^{(2v+l)}(P^*) = \frac{(-1)^v}{\bar{a}^{v+l}b_v^{v} - d_{v}} \det \left( \text{Sylv}_{2v+l,l} \left( \frac{TP}{X^v}, \hat{Q}_1 \right) \cdot \mathbf{v}^{2v+l}_{v+l-1,v+l} \right) = \frac{1}{\bar{a}^{v+l}b_v^{v(l-1)v}} \det \left( \text{Sylv}_{v+l,l} \left( \frac{TP}{X^v}, Q_1 \right) \cdot \mathbf{v}^{2v+l}_{v+l-1,l} \right).
\]
We can see that \( Q_1(0) \neq 0 \) because \( Q_1 = b_{d,v}P^* - A.X.TP \) where \( P^*(0) \neq 0 \) and \( \text{val}(A.X.TP) \geq 1 \) (val denotes here the valuation of the polynomial). The valuation of \( Q_1 \) is then 0. We consider that it is formally of degree \( d - v \) so that \( \deg Q_1 = d - v \). Let us now define \( Q^* \) and \( A_1 \), respectively, as the pseudo-remainder and the pseudo-quotient of \( Q_1^* \) by \( X(TP)^* \). (We will see later the relation between \( A \) and \( A_1 \).)

As the degree of \((TP)^* \) is exactly \( d - 2v \), \( \deg A_1 = v - 1 \) and \( \deg Q^* \leq d - 2v \). We look at \( Q^* \) and \( Q \) as polynomials of formal degree \( d - 2v \). We have

\[
Q^* = \tilde{b}_v^* Q_1^* - A_1.X.(TP)^*
= \tilde{b}_v^* Q_1^* + uA_1 \frac{TP}{X^{v-1}}.
\]

We use then Lemma 2.1 to compute \( Q = (Q^*)^* \):

\[
Q = \frac{1}{X^v} \left[ \tilde{b}_v^* Q_1^* + \tilde{u} A_1 \left( \frac{TP}{X^{v-1}} \right)^* \right]
= \frac{1}{X^v} \left[ \tilde{b}_v^* Q_1^* - A_1 \frac{TP}{X^v} \right].
\]

By the same reasoning as in the case of \( Q_1 \), we see that for \( i = 0, 1, \ldots, l - 1 \), the coefficients of \( X^{v+i}Q \) can be obtained by linear combinations of those of \( \frac{TP}{X^{v-1}}, \frac{TP}{X^{v-2}}, \ldots, \frac{TP}{X^v} \) and of \( X^i Q_1 \). Using this fact, we obtain

\[
ss T^{(2v+l)} p = \frac{1}{u^{v+l} b_{d-v}^{(l-1)v}} \det \left( \text{Sylv}_{v+l,l} \left( \frac{TP}{X^v}, X^v Q \right) \cdot V_{v+l-1,l} \right)
= \frac{1}{u^{v+l} b_{d-v}^{(l-1)v}} \det \left( \text{Sylv}_{l,l} \left( \frac{TP}{X^v}, Q \right) \cdot V_{l-1,l} \right).
\]

Moreover, \( b_v = -\tilde{u} \tilde{b}_{d-v} \), then

\[
ss T^{(2v+l)} p = \frac{(-1)^{(l-1)v}}{u^{(v+1)|b_v| 2(l-1)v}} \det \left( \text{Sylv}_{l,l} \left( \frac{TP}{X^v}, Q \right) \cdot V_{l-1,l}^2 \right).
\]

We have to describe now, more precisely \( A_1 \) and \( Q \). Note that, if \( U \) and \( V \) are two polynomials in \( A[X] \) and \( \alpha \) is an element of \( A \), the pseudo-quotient satisfies the following relations: \( \text{Pquo}(\alpha U, V) = \alpha \text{Pquo}(U, V) \) and \( \text{Pquo}(U, \alpha V) = \alpha^{\deg U - \deg V} \text{Pquo}(U, V) \). Also, if we multiply both \( U \) and \( V \) by a same power of \( X \), the pseudo-quotient is not modified. Therefore, we have

\[
A_1 = \text{Pquo}(Q_1^*, X.T(P)^*)
= \text{Pquo} \left( \frac{1}{X^v} \left( \tilde{b}_{d-v}^* P + uTP \frac{A^*}{X^v} \right), -uX.TP \right)
= \text{Pquo} \left( \tilde{b}_{d-v}^* P + uTP \frac{A^*}{X^v}, -uX.TP \right).
\]

As \( \deg XTP = d - v + 1 \), the pseudo-quotient depends only on the monomials of higher or equal degree to \( d - v + 1 \), then it in fact depends only on \( \tilde{b}_{d-v}^* FX^{d-v+1} \):

\[
A_1 = \text{Pquo}(\tilde{b}_{d-v}^* FX^{d-v+1}, -uXTP)
= \tilde{b}_{d-v}^* \text{Pquo}(FX^{d-v+1}, -uXTP)
= \tilde{b}_{d-v}^* (-u)^{v-1} \text{Pquo}(FX^{d-v+1}, XTP)
\]
We can use this result to give another expression of $Q^*$:

$$Q^* = b_v Q_1^* + b_d(-1)^{v-1}u v^+ A X T P X v$$

$$= \frac{(-u b_d v)}{X v} \left( b_v v P + u T P A^* \right) + b_d(-1)^{v-1}u v^+ A X T P X v$$

$$= \left( \frac{-u}{X} \right) \left[ b_v v P + u T P \left( b_d^* A^* - b_d A X \right) \right].$$

In the same way, we obtain

$$Q = \left( \frac{-\bar{u}}{X} \right) \left[ b_v^* v P^* + T P \left( b_d^* A^* - b_d A X \right) \right].$$

A very simple relation exists between $Q$ and $Q^*$:

$$\bar{u} \left( \frac{X}{-u} \right)^v Q^* = \left( \frac{X}{-u} \right)^v Q = |b_v|^2v(u P - P^*)$$

so that

$$Q^* = u^{2v+1} Q + |b_v|^2v(-1)^v u v^+ T P X v.$$  \hspace{1cm} (3.7)

Therefore, we see that the coefficients of $Q^*$ are obtained as linear combinations of coefficients of $Q$ and of $\frac{T P X}{X v}$. Hence, it follows that

$$ssT_{2v+1} P = \frac{(-1)^{(2v+1)}}{|b_v|^2v(2v+1) v^+(-1)^v u v^+ T P X v} \times \det(Sylv_{l,i}(Q^*,Q) \cdot V_{l,i}^{2v+1})$$

$$= \frac{(-1)^v}{|b_v|^2v(2v-1)} \det(Sylv_{l,i}(Q^*,Q) \cdot V_{l,i}^{2v+1})$$

$$= \frac{(-1)^{v+i}}{|b_v|^2v(2v-1)} \det(Sylv_{l,i}(Q,Q^*) \cdot V_{l,i}^{2v+i})$$

$$= \frac{(-1)^{v+i}}{|b_v|^2v(2v-1)} ssT_{i} Q$$

because $\deg Q = d - 2v$ formally. □

Looking at the example $S$, we see that $A = P \text{quo}(S, XT S) = 441 X^2 - 5292 X - 1323$; therefore $Q = -1629556299 X^2 + 6689757438 X + 171532242$. The Schur–Cohn sub-transforms given by the theorem are

$$ssT_{1} S = 0 \text{ for } i = 2, \ldots, 5$$

$$ssT_{6} S = 85766121 X^2 - 85766121,$$

$$ssT_{7} S = T Q = 140484906198 X - 30618505197$$

$$ssT_{8} S = \frac{T(ssT_{7} S)}{ssT_{6} S(0)} = \frac{-1879516008962142806395}{-85766121} = 219183469996995$$

that we can verify, using the definition of the Schur–Cohn sub-transforms.
It is also noteworthy that, when $P$ is $v$-defective, $ssT^{(2v)}P(0) \neq 0$ so that the group of zeroes at the beginning of the sequence of the $ssT^{(i)}P(0)$ is always composed of an odd number of $2v - 1$ zeroes. Furthermore, it is easy to verify that $\text{lc}(Q)$ and $Q(0)$ cannot vanish simultaneously, as otherwise we would have $(v + 1)$-defective $P$. As a consequence, we see that $Q$ cannot be 0 when $P$ is not symmetrical. However, we cannot say anything about $ssT^{(2v + 1)}P(0)$, in fact, it can vanish. Let us give the example

$$T = X^8 + X^7 - 10X^6 - 19X^5 + 2X^4 + X^3 - 2X^2 + 5X - 1.$$  

The sequence of $ssT^{(i)}T(0)$ is $0, -36, 0, 0, -1201038336, 365115654144, -33915187429376$. In such a case, the polynomial is sequentially defective: we have in the sequence of the $ssT^{(i)}P(0)$'s several adjacent groups of zeroes separated from one another by one single non-zero term only.

### 3.3. An Algorithm

We give as a consequence of Proposition 3.1 and Theorem 3.1, a brief sketch of an algorithm which output the sequence of the Schur–Cohn sub-transforms without using their determinantal definition, but only polynomial computations.

**Entry:** $P$ a polynomial of degree $d$ with coefficients in $A$.

**Output:** the sequence of $P_i = ssT^{(i)}P$, for $i = 1, 2, \ldots, d$.

**Initialization:** $P_{-1} = 1, P_0 := P; P_1 := T^{(1)}P; i = 1$.

**Loop:** while $i < d$ do

$$P_{i+1} := \frac{T^{(i)}P}{P_{i-1}(0)}; i := i + 1$$

If $\text{val} P_i = v > 0$ then do

$$P_{i+2} := \cdots = P_{i+2v-1} := 0$$

Compute $P_{i+2v}$ by formula 3.3

Compute $Q$ by formula 3.5

$$P_{i+2v+k} := \frac{ssT^{(k)}Q}{P_{i-1}(0)P_{i-1}(0)}$$  

for $k = 1, \ldots, d - 2v - i$.

$i := d$

End

End.

Although we give for simplicity a recursive description of this algorithm, it is clear that an iterative version must be implemented. Instead of computing the $ssT^{(k)}Q$s to obtain subsequently the $ssT^{(k+2v)}P$s, it is preferable to go back to the direct computation of the $ssT^{(j)}P$s as soon as two consecutive $ssT^{(k)}Q$s are found not defective, which is what we did with the example $S$ above. We are then assured of using coefficients of minimal size and avoiding unnecessary divisions. Furthermore, it can be slightly improved by considering what happens when, for some $k$, $T^{(k)}P$ is not defective but presents a degree deflation greater than one, that is when we have $\text{deg} T^{(k)}P = l < d - k$. This is, for example, the case with $X^{100} - 5$ for which the first sub-transform is a constant instead of a polynomial of degree 99. In this case we can save a lot of time by jumping directly to $T^{(d-l)}P$, incorporating the results of Proposition 2.4.
COMPLEXITY

It only needs $O(d^2)$ arithmetical operations in $A$ and, if we consider the special case $A = Z$, computing with classical arithmetic, its complexity is in $O(d^4t^2)$ binary operations, where $t$ is an upper bound of the size of the coefficients of $P$ in $Z$. It can be proved as we did in Saux Picart (1993), using Hadamard’s bound.

SPECIALIZATION

The problem here is to consider a family of polynomials with complex coefficients depending on parameters. If we need to know the number of roots in the unit disk for different values of the parameters, we can ask ourselves whether it is possible to compute the specialization from $A$ to $C$ which fixes each $Y_i$. Then, by (3.5), $D$ also divides $TP$ and its reciprocal. Therefore $O(d^2)$ binary operations in $Z$.

We say that a ring morphism $\varphi$ from $A$ to $C$ is a morphism of specialization if for every $a \in A$, we have $\varphi(\bar{a}) = \varphi(a)$. Such a morphism extends naturally into a morphism from $A[X]$ to $C[X]$ and we have for every polynomial in $A[X]$, $\varphi(f) = \varphi(f^*)$.

For example, if $A = C[Y_1, \ldots, Y_k]$, a morphism of specialization can be defined by using a $C$-morphism which fixes each $Y_i$ on a real $\alpha_i$. If $A = C[Y_1, \ldots, Y_k, Z_1, \ldots, Z_k]$ and the conjugation on $A$ is defined by $\bar{a}(Y_1, \ldots, Y_k, Z_1, \ldots, Z_k) = a(Z_1, \ldots, Z_k, Y_1, \ldots, Y_k)$, the morphism can be constructed by sending each $Z_i$ on a complex $\alpha_i$ and each $Z_i$ on the conjugate $\bar{\alpha}_i$.

Due to the determinantal definition of the Schur–Cohn sub-transforms, we have an obvious proposition.

Proposition 3.2. If $P$ is a polynomial of degree $d$ in $A[X]$ and $\varphi$ a morphism of specialization from $A[X]$ to $C[X]$, then we have for every $k = 1, \ldots, d$

$$\varphi(ssT^{(k)}P) = ssT^{(k)}(\varphi(P)).$$

COMPUTATION OF $P \land P^*$

We have observed in general that we cannot compute the gcd of $P$ and $P^*$ in $F(A)$ with the knowledge of the $T(i)P$ in the case where we encounter a defective polynomial. Here also, we can conclude in every case.

Proposition 3.3. Let $P \in A[X]$. Let $ssT^{(i_0)}P$ be the last non identically zero Schur–Cohn sub-transform of $P$. Then $ssT^{(i_0)}P$ is the gcd of $P$ and $P^*$ in $F(A)$.

Proof. We need only to consider the case of defective polynomials, the other case being trivial. Let us write $D = P \land P^*$ with $v$-defective $P$. As $D$ divides $P$ and $P^*$, it divides $TP$ and its reciprocal. Then, by (3.5), $D$ also divides $Q$ and $Q^*$ and therefore $Q \land Q^*$.

Reciprocally, if $D = Q \land Q^*$, we see that $D$ divides $TP/X^v$ by (3.7) and then also $P$ and $P^*$ by (3.5). Therefore $P \land P^* = Q \land Q^*$. Furthermore, (3.7) shows us that, if $Q$ is symmetrical, $Q$ and $\frac{Q^*}{Q}$ are proportional. In the end, we obtain $P \land P^* = TP \land TP^* =$
\[
\cdots = ssT^{(i_0)}P \land (ssT^{(i_0)})^* \quad \text{(the only is, } 1 \leq i \leq i_0, \text{ mentioned in this sequence of}
\text{equalities are those for which } ssT^{(i)}P \neq 0, i_0 \text{ being the last).} \]

We give now a complement: assume that \( P = P_1(P \land P^*) \); we are interested by
the relation between the Schur–Cohn sub-transforms of \( P \) and those of \( P_1 \). In fact, the relation
is not easy to describe, however, the relation between their constant terms is quite simple
and will be useful in the next section.

**Proposition 3.4.** Let \( P \in \mathcal{A}[X] \) of degree \( d \) be such that \( \deg(P \land P^*) = d_0 \geq 1 \); let us
write \( P = P_1(P \land P^*) \). Then, for \( i = 1, \ldots, d - d_0 \) we have
\[
ssT^{(i)}P(0) = (P \land P^*)(0)^2 ssT^{(i)}P_1(0).
\]

**Proof.** Let us denote \( P \land P^* = g_0 + \cdots + g_{d_0} X^{d_0} \). There exists \( u \in \mathcal{A} \) such that
\( |u|^2 = 1 \) and \( (P \land P^*)^* = u(P \land P^*) \) (\( P \land P^* \) is symmetrical).

The result comes from a multiplicative property of the Sylvester’s matrices we use. We have
\[
Sylv_k(P, \bar{u}P^*) = Sylv_k(P_1, P_1^*) \cdot Sylv_{d-d_0+k}(P \land P^*).
\]

Therefore, we obtain
\[
\bar{u}^k ssT^{(k)}P = \det(Sylv_k(P_1, P_1^*) \cdot Sylv_{d-d_0+k}(P \land P^*) \cdot V_{k-1,k}^k).
\]

In particular, we have
\[
\bar{u}^k ssT^{(k)}P(0) = \det(Sylv_k(P_1, P_1^*) \cdot G)
\]

where \( G = Sylv_{d-d_0+k}(P \land P^*) \cdot V_{k-1,k}^k \) is a \((d - d_0 + k) \times 2k\) matrix with the rows
\( k + 1, \ldots, d - d_0 \) containing only zeroes:
\[
G = \begin{pmatrix}
g_0 & g_1 & \cdots & g_{k-1} & 0 & \cdots & 0 \\
g_0 & & & & \ddots & & \\
& g_0 & & & \ddots & & \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
& & & & & \ddots & & \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
& & & & & & \ddots & \\
& & & & & & & g_{d_0} \\
& & & & & & \ddots & \\
0 & & \cdots & 0 & g_{d_0} & \cdots & g_{d_0} \\
& \ddots & \ddots & & & & \\
0 & \cdots & 0 & g_{d_0-1} & \cdots & g_{d_0-1} & g_{d_0}
\end{pmatrix}
\]

We use the Binet–Cauchy formula (see Gantmacher (1966)) to expand this determinant. In the sum of the formula, only one term does not vanish because we must use
minors of order 2\( k \) of the matrix \( G \) and only one of them can be built without a null
row. We have

\[ \bar{u}^k ssT^{(k)} P(0) = ssT^{(k)} P_1(0) \cdot \det \begin{pmatrix} g_0 & g_1 & \cdots & \vdots & \vdots \\ g_0 & g_1 & \cdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & g_0 \\ g_{d_0} & g_{d_0-1} & \cdots & \ddots & \vdots \\ \cdots & \cdots & \ddots & \ddots & g_{d_0-1} \end{pmatrix} \]

The above matrix is composed of two blocks of the same size, one upper-triangular and one lower-triangular. Then

\[ \bar{u}^k ssT^{(k)} P(0) = ssT^{(k)} P_1(0) g_k^k g_{d_0} \]

Hence the result.

4. Computation of the Number of Roots in the Unit Disk

We turn now to our second task. We suppose that \( P \) is a complex polynomial or the image in \( \mathbb{C}[X] \) of a polynomial by a morphism of specialization. We want to compute \( \# P \), the number of roots of \( P \) in the open unit disk, each root counted with its multiplicity.

4.1. Relation with Bezoutian

We recall in this first section the link between our problem, bezoutian and hermitian forms. Most of these results can be found in the works of Cohn (1922), Fujiwara (1926), or in the more recent survey by Krein and Neimark (1981) and in the books of Jury (1982) and Barnett (1983).

We will say that a complex number \( z \) is a symmetrical root of the polynomial \( P \) with respect to the unit disk if \( 1/\bar{z} \) is also a root of \( P \). In particular, a root on the unit circle is symmetrical. These roots are the roots of \( P \land P^* \). This polynomial is, up to a complex factor, equal to its reciprocal, and its Schur–Cohn sub-transforms are identically null.

Let us define in \( \mathbb{C}[X,Y] \) the polynomial

\[ k(P) = \frac{P^*(X)P^*(Y) - P(X)P(Y)}{1 - XY} = \sum_{0 \leq i,j \leq d-1} k_{i,j}X^iY^j. \]

Then, we consider the matrix \( K(P) = (k_{i,j})_{0 \leq i,j \leq d-1} \) that it generates. It is a \( d \times d \) hermitian matrix which defines an hermitian form \( K(P) \). It has been proved that the rank of \( K(P) \) is \( d - d_0 \), where \( d_0 \) is the degree of \( P \land P^* \), and that its signature, \( \text{sgn}(K(P)) \), is the difference between the number of roots of \( P \) inside and outside the unit disk (not counting the roots on the unit circle)

\[ \text{sgn}(K(P)) = \# P - \# P^*. \]

If we write this form as a sum of squares, the number \( \pi \) of plus signs appearing in the
sum is just the number of roots of $P$ inside the open unit circle which are not symmetrical. We can then write

$$\# P = \pi + \#(P \land P^\ast).$$

If $\pi$ is known and if $P \land P^\ast \neq 1$, we must use another result of Cohn (1922), to reach the goal (see Marden (1966)).

**Proposition 4.1.** If $P$ is a symmetrical polynomial, then

$$\# P = \#(P')^\ast.$$

Therefore, to compute $\text{sgn}(K(P))$ and to compute $\# P$ are two equivalent problems. Furthermore the relation between the principal minors of $K(P)$ and the constant terms of the Schur–Cohn sub-transforms has already been studied.

**Proposition 4.2.** Let $K_i$ be the principal minor of order $i$ of $K(P)$, then

$$\text{ssT}^{(i)}P(0) = (-1)^i K_i.$$

If $P = P_1 P \land P^\ast$, we have seen (Proposition 3.4) that, for $i = 1, \ldots, d - d_0$, the $\text{ssT}^{(i)}P(0)s$ are equal to the $\text{ssT}^{(i)}P_1(0)s$, up to a positive factor, so that the study $K(P)$ is that of $K(P_1)$. As the algorithm of Section 2 furnishes the $\text{ssT}^{(i)}Ps$ and $P \land P^\ast$ too, we are able to study $K(P_1)$, and, if needed, $P \land P^\ast$ separately.

We shall not consider polynomials which are prime with their reciprocals.

We can therefore give, as a simple consequence, a new version of the rule known from Cohn.

Let us denote $V(u_1, \ldots, u_n)$ the number of changes of signs in a sequence of real non-zero numbers $u_1, \ldots, u_n$. We will denote also $V^l_i = V(\text{ssT}^{(i)}P(0), \ldots, \text{ssT}^{(l)}P(0))$ with the convention that no $\text{ssT}^{(i)}P(0)$ in the sequence is null ($i \leq l \leq j$) and that $\text{ssT}^{(0)}P(0) = 1$.

**Proposition 4.3.** If $P$, a complex polynomial of degree $d$, is such that $P \land P^\ast = 1$ and no $\text{ssT}^{(i)}P(0)$ vanish for $i = 1, \ldots, d$, then we have

$$\# P = V^d_0.$$

This comes from the well-known rule to compute the signature of an hermitian matrix with its principal minors (see Gantmacher (1966)). However, it does not work when some minors vanish, i.e. in the case where at least one Schur–Cohn sub-transform is defective.

### 4.2. The Number of Roots of a Defective Polynomial

The principal result of this section is the following one.

**Theorem 4.1.** Let $P \in \mathbb{C}[X]$ be a $v$-defective polynomial. Let $Q$ be defined as in Theorem 3.1. Then

$$\# P = v + \# Q^\ast.$$
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PROOF. As a first step, we consider the relation between \( \mathcal{K}(P) \) and \( \mathcal{K}(Q) \). In order to simplify our notations, let us note \( W \) the rational function defined by

\[
W = b_{d-v}^v \frac{A^*}{X^v} - \tilde{b}_{d-v}^v A X
\]

where \( A \) is defined in Theorem 3.1. Then we have

\[
Q = \frac{(-\bar{u})^v}{X^v} (|b_v|^2 P^* + TP \bar{W}).
\]

Therefore, an easy computation gives us

\[
X^v Y^v (1 - XY) k(Q) = |b_v|^{4v}(P(X)\bar{P}(Y) - (P^*(X)\bar{P}^*(Y))
+ |b_v|^{2v} TP(X)W(X)(\bar{u}\bar{P}(Y) - \bar{P}^*(Y))
+ |b_v|^{2v} TP(Y)\bar{W}(Y)(\bar{u}P(X) - P^*(X))
\]

and, then

\[
- |b_v|^{4v} k(P) = X^v Y^v k(Q) - |b_v|^{2v} \frac{TP(Y)\bar{W}(Y)TP(X) + TP(X)W(X)\bar{P}^*(Y)}{1 - XY}
\]

Our task is now to decompose \( \mathcal{K}(P) \) in a sum of squares. We look at the second term in the above sum:

\[
X^v Y^v (W(X) + \bar{W}(Y)) = b_{d-v}^v A^*(X)Y^v - \tilde{b}_{d-v}^v A(X)X^{v+1} Y^v
+ \tilde{b}_{d-v}^v \bar{P}(Y)X^v - \bar{b}_{d-v}^v \bar{A}(Y)Y^{v+1} X^v
= \tilde{b}_{d-v}^v \sum_{k=0}^{v-1} a_k X^v Y^{v-1-k}(1 - X^{k+1} Y^{k+1})
+ \bar{b}_{d-v}^v \sum_{k=0}^{v-1} \bar{a}_k X^v Y^{v-1-k}(1 - X^{k+1} Y^{k+1}).
\]

Therefore,

\[
\frac{X^v Y^v (W(X) + \bar{W}(Y))}{1 - XY} = \sum_{i=0}^{v-1} X^i Y^i \left( \sum_{k=0}^{v-i} \tilde{b}_{d-v}^v a_k X^v Y^{v-k-1} + \bar{b}_{d-v}^v \bar{a}_k X^{v-k-1} Y^v \right).
\]

Let us write \( W_i(X) = b_{d-v}^v \sum_{k=0}^{v-1} \tilde{a}_k X^v Y^{v-1-k} \). It is a simple fact that we have

\[
W_i(X)Y^v + \bar{W}_i(Y)X^v = \frac{1}{2} [(W_i(X) + X^v)(\bar{W}_i(Y) + Y^v) - (W_i(X) - X^v)(\bar{W}_i(Y) - Y^v)].
\]

Then,

\[
\frac{TP(Y)\bar{W}(Y)TP(X) + TP(X)W(X)\bar{P}^*(Y)}{1 - XY}
= \frac{1}{2} \sum_{i=0}^{v-1} \frac{TP(X) TP(Y)}{X^{v-i} Y^{v-i}}
\times [(W_i(X) + X^v)(\bar{W}_i(Y) + Y^v) - (W_i(X) - X^v)(\bar{W}_i(Y) - Y^v)].
\]

Now, we can consider the hermitian forms \( \mathbf{K}(P) \) and \( \mathbf{K}(Q) \). Since \( P \wedge P^* = Q \wedge Q^* \), the rank of the form \( \mathbf{K}(P) \) is \( d - d_0 \) and that of \( \mathbf{K}(Q) \) is \( d - 2v - d_0 \).
If $U = \sum_{i=0}^{d-1} u_i X^i$ is a polynomial in $C[X]$, we denote by $L(U)$ the linear form defined on $C^d$ by $L(U)(x_0, \ldots, x_{d-1}) = \sum_{i=0}^{d-1} u_i x_i$.

We define $L_{2k-1}$ as $L\left(\frac{T_P}{X^{k-1}}(W_{k-1}(X) + X^v)\right)$ and $L_{2k}$ as $L\left(\frac{T_P}{X^{k-1}}(W_{k-1}(X) - X^v)\right)$.

Formula (4.1) translated in terms of hermitian forms on $C^d$ gives

$$-|b_v|^2K(P) = K(X^vQ) - |b_v|^2v \sum_{k=1}^v (|L_{2k-1}|^2 - |L_{2k}|^2).$$

Assuming that $K(X^vQ)$ is already written as a sum of $d-2v-d_0$ squares of independent linear forms, we have obtained $K(P)$ as a sum of $d-d_0$ squares of linear forms. Because the rank of $K(P)$ is just $d-d_0$, these forms are linearly independent. Therefore, the number of plus signs in the signature of $K(P)$ is just the number of minus signs in the decomposition of $K(X^vQ)$, which is the same as the number of plus sign in the decomposition of $K(Q^*)$ (the factor $X^v$ inducing only a translation of the indices) increased by $v$. Hence the result. □

We can now compute $\#P$ when $P$ is defective and when only one group of zeroes is encountered in the sequence of the $ssT^{(1)}P(0)$. This is the object of the following proposition. Just before, we give a simple lemma.

**Lemma 4.1.** If $u_1, \ldots, u_n$ is a sequence of real, non-zero numbers, and if $u_1', \ldots, u_n'$ is the sequence defined by $u_i' = (-1)^i u_i$, then we have

$$V(u_1, \ldots, u_n) + V(u_1', \ldots, u_n') = n - 1.$$  

**Proof.** It is well known that

$$V(u_1, \ldots, u_n) = \sum_{i=1}^{n-1} \frac{1}{2} (1 - \text{sgn}(u_i u_{i+1})).$$

But here $u_i u_{i+1} = -u_i' u_{i+1}'$, so that $\text{sgn}(u_i u_{i+1}) + \text{sgn}(u_i' u_{i+1}') = 0$. Hence the result. □

Therefore, we have

**Proposition 4.4.** Let $P$ be a complex polynomial of degree $d$, $v$-defective such that the sequence of the $ssT^{(1)}P(0)$ contains only one group of zeroes (just at its beginning). Let us assume that $P \wedge P^* = 1$. Then we have

$$\#P = v + V_{2v}^{d}.$$  

**Proof.** We use the polynomial $Q$, defined before. Its Schur–Cohn sub-transforms are proportional to those of $P$ of order $2v + 1, 2v + 2, \ldots, d$; therefore neither $Q$ nor some of its sub-transforms are defective until the rank $d - 2v$. Then, from the previous results, we obtain

$$\#P = v + (d - 2v - \#Q) = v + (d - 2v - V(1, ssT^{(1)}Q(0), \ldots, ssT^{(d-2v)}Q(0))).$$
As \(\frac{(-1)^{i+1}}{|u_i|^{i+1}}\) \(ssT(i)Q(0) = ssT^{2v+i}P(0)\), the variations of signs satisfy
\[
\mathcal{V}(1, ssT^{(1)}Q(0), \ldots, ssT^{(d-2v)}Q(0))
\]
\[
= \mathcal{V}(1, (-1)^v ssT^{(2v+1)}P(0), (-1)^v ssT^{(2v+2)}P(0), \ldots, (-1)^{d-v} ssT^{(d-2v)}Q(0))
\]
\[
= \mathcal{V}((-1)^v, -ssT^{(2v+1)}P(0), ssT^{(2v+2)}P(0), \ldots, (-1)^{d-2v} ssT^{(d)}P)
\]
because we do not change the variation of signs of a sequence if we multiply all the elements of the sequence by the same number. Furthermore, \((-1)^v\) is just the sign of \(ssT^{2v}P(0)\) (see Theorem 3.1). Hence, using the previous lemma, it follows that
\[
\mathcal{V}(1, ssT^{(1)}Q(0), \ldots, ssT^{(d-2v)}Q(0))
\]
\[
= \mathcal{V}(ssT^{2v}P(0), -ssT^{(2v+1)}P(0), ssT^{(2v+2)}P(0), \ldots, (-1)^d ssT^{(d)}P)
\]
\[
= d - 2v - \mathcal{V}^d_{2v}. \square
\]

For example, if we look at the polynomial \(S\) (Sections 2.1 and 4.2), we obtain \#\(S = 3 + \mathcal{V}(-85766121, -30618505197, 219183469996995) = 4\) which can be verified numerically.

### 4.3. General Case

Finally, we have to consider polynomials for which the sequence of the constant terms of their Schur–Cohn sub-transforms presents several groups of zeroes. We need a lemma which enables us to jump from one group of zeroes to the next one.

**Lemma 4.2.** We suppose that \(T^{(i)}P(0) \neq 0\) for \(i = 1, \ldots, k\). Let us call \(A_{k1}^{k2} = \prod_{i=k1}^{k2} a_{0}^{(i)}\). Then, we have
\[
\#P = \begin{cases} 
\mathcal{V}_0^k + \#ssT^{(k)}P & \text{if } A_{k1}^k > 0, \\
\mathcal{V}_0^k + d - k - \#ssT^{(k)}P & \text{if } A_{k1}^k < 0.
\end{cases}
\]

**Proof.** We give a rapid sketch of it because it is only a simple extension of the results of Henrici (1974) and Saux Picart (1993). If \(k_1, \ldots, k_l\) denote the values of \(i\) between 1 and \(k\) (included) where \(T^{(i)}(P)(0) < 0\), then we have by Rouché’s theorem
\[
\#P = d - k_1 + 1 - \#T^{(k_1)}P
\]
\[
= d - k_1 + 1 - (d + 1 - k_2 - \#T^{(k_2)}P)
\]
\[
= \ldots
\]
\[
= \sum_{j=1}^{l} (-1)^{j-1}(d + 1 - k_j) + (-1)^{l} \#T^{(k_l)}P
\]
\[
= \sum_{j=1}^{l} (-1)^{j-1}(d + 1 - k_j) + (-1)^{l} \#T^{(k)}P,
\]
the last equality being satisfied because \#\(T^{(k_i)}(P) = \#T^{(k)}(P)\), as \(a_0^{(i)}\) is positive for \(k_1 < i \leq k\). Then we notice that
\[
\sum_{j=1}^{l} (-1)^{j-1}(d + 1 - k_j) = \sum_{j=1}^{l} (-1)^{j} k_j
\]
if $l$ is even; otherwise, we have
\[ \sum_{j=1}^{l} (-1)^{j-1} (d + 1 - k_j) = d + 1 + \sum_{j=1}^{l} (-1)^j k_j. \]

But we can show by the same argument as in Saux Picart (1993) that, when $l$ is even,
\[ Y_0^k = \sum_{j=0}^{l} (-1)^j k_j \]

and when $l$ is odd
\[ Y_0^k = \sum_{j=0}^{l} (-1)^j k_j + k + 1. \]

To end with the proof, we have to observe that $l$ is even if and only if $A_1^{k_1} > 0$. □

In fact, a purely algebraic proof of this lemma (without the use of Rouche’s theorem) can be done: it is rather technical, using $K(sT^{(k)}(P))$ and we skip it for brevity’s sake.

We are now ready to state our last results which will take into account the sequentially defective polynomials.

**Theorem 4.2.** Let $P$ be a complex polynomial of degree $d$ such that $P \wedge P^* = 1$. We suppose that $sT^{(k)}P$ is $(v_k)$-defective, for some $k = k_1, k_2, \ldots, k_s$ where $k_1 \geq 0$, $k_{i+1} > k_i + 2v_{k_i}$ and $d > k_s + 2v_s$. Let us write $k_{s+1} = d$ and $V_i^k = 0$. Then,
\[ \# P = \sum_{i=1}^{s} (v_i + V_i^{k_{i+1}}). \]

**Proof.** By recurrence over the number of groups of zeroes in the sequence of the $sT^{(k)}P(0)$.

We assume that $s = 1$, and that $P$ is not defective, (otherwise the result is given by Proposition 4.4). We show first that $sT^{(k_1-1)}P(0)sT^{(k_1)}P(0)$ has the same sign as $A_1^{k_1}$. Using Theorem 1.1, we have
\[ sT^{(k_1-1)}P(0)sT^{(k_1)}P(0) = \prod_{i=1}^{k_1-3} a_0^{(i)} a_0^{(k_1)} a_0^{(k_1-2)} a_0 \prod_{i=1}^{k_1} a_0^{(i)} a_0^{(k_1-2)} a_0^{(k_1-1)} a_0. \]

Therefore,
\[ \text{sgn}(sT^{(k_1-1)}P(0)sT^{(k_1)}P(0)) = \text{sgn} \left( \prod_{i=1}^{k_1-3} a_0^{(i)} a_0^{(k_1-2)} a_0^{(k_1-1)} a_0 \right) = \text{sgn}(A_1^{k_1}). \]

Let us assume that $A_1^{k_1} > 0$. From our previous lemma, we obtain
\[ \# P = Y_0^{k_1} + \# sT^{(k_1)}P. \]
By Proposition 4.4, we have
\[
#ssT^{(k_1)} P = v_1 + \mathcal{V}(ssT^{(2v_1)}(ssT^{(k_1)} P)(0), \ldots, ssT^{(d - k_1)}(ssT^{(k_1)} P)).
\]

But the variation of signs in the above formula is just \(\mathcal{V}_{k_1 + 2v_1}^d\) because \(ssT^{(k_1 - 1)} P(0)\) \(\times ssT^{(k_1)} P(0)\) is positive (Proposition 3.1), hence
\[
#P = \mathcal{V}_{k_1}^0 + v_1 + \mathcal{V}_{k_1 + 2v_1}^d.
\]

The case when \(A_1^{k_1} < 0\) is treated in the same way by showing that the variation \(\mathcal{V}(ssT^{(2v_1)}(ssT^{(k_1)} P)(0), \ldots, ssT^{(d - k_1)}(ssT^{(k_1)} P))\) is equal to \(d - 2v_1 - \mathcal{V}_{2v_1}^d\) because one sign over two is changed when \(ssT^{(k_1 - 1)} P(0)ssT^{(k_1)} P(0)\) is negative.

We use the same tricks to go from a polynomial with \(s\) groups of zeroes among the constant terms of its Schur–Cohn sub-transforms to one with \(s + 1\) groups of zeroes. □

Looking at the example \(T\) (Section 3.3), we see that
\[
#T = 1 + 2 + \mathcal{V}(-1201038336, 365115654144, -33915187429376) = 4
\]
which is the right result.

5. Conclusion

We have answered a question open since 1921: how to solve the singularities of the algorithm of Schur–Cohn; we have then produced an algorithm with complexity in \(Od^2\) arithmetic operations which computes in any case the number of roots of a complex polynomial in the open unit disk. It supports specialization and can be used to study polynomials which depend on parameters. It is well adapted to computer algebra.

The similarity of our method with the method of the sub-resultants is evident. The definitions of sub-resultants and of Schur–Cohn sub-transforms are themselves very similar and the formula of Theorem 4.2 looks like the rule of signs used in the Sturm–Habicht sequences. The parallelism between the computation of the number of roots of a polynomial in an half-plane (methods of Routh–Hurwitz and Hermite), in an interval of \(\mathbb{R}\) (method of Sturm) and in the unit circle has been described long ago by Fujiwara (1926), and more recently by Krein and Neimark (1981) or Jury (1982).

We used signature of hermitian forms, an algebraic tool, to study an algebraic object, a polynomial, walking in the footsteps of Hermite. One point is noteworthy: the computation of the signature of an hermitian form can be done with the variation of signs of the sequence of its principal minors only in the case when there is no zero inside the sequence. However, two exceptions exist: one for the Hankel forms for which we can use a rule due to Frobenius (see Gantmacher (1966)), another for the Toeplitz forms for which an analogous rule due to Iohvidov (1982) is available. It can be shown that \(K(P)\) is congruent to a Toeplitz matrix so that our last result, Theorem 4.2, can be viewed as an application of Iohvidov’s rule and its proof as a new demonstration of this rule in a special case.

References


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