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Counterexamples to three conjectures concerning perfect graphs

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Abstract

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We will present counterexamples to a conjecture of Hoàng on alternately orientable graphs, a conjecture of Hertz and de Werra on even pairs and to a conjecture of Reed on Berge graphs. All these three conjectures are related to perfect graphs.

1. Introduction

A graph G is called *perfect* iff for every induced subgraph $H \subseteq G$ the chromatic number of H equals the size of a largest clique in H. A graph is called *Berge* iff neither G nor $\bar{G}$ (i.e. the complement of G) contains an odd induced cycle of length greater than three. In 1960 Berge [2] posed the following conjecture which is still open.

The Strong Perfect Graph Conjecture. A graph is perfect iff it is Berge.

This conjecture (called SPGC for short) has motivated several people to work on perfect graphs and many partial results were obtained. Probably the most important

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result is the following theorem of Lovász [15] which was conjectured by Berge together with the SPGC.

The Perfect Graph Theorem. *A graph is perfect iff its complement is perfect.*

In the following three sections we will present counterexamples to three conjectures concerning perfect graphs.

2. A conjecture on alternately orientable graphs

One of the first classes of graphs which has been shown to be perfect is the class of transitively orientable graphs (also called comparability graphs). An orientation of the edges of a graph is called *transitive* iff no induced directed path on three vertices exists and it is called *acyclic* if no directed cycle exists. A graph G is called *transitively orientable* if it admits an orientation which is transitive and acyclic. The perfection of transitively orientable graphs has been established by Berge [1]. The following theorem of Ghouila-Houri [7] shows that a graph is already transitively orientable if it admits a transitive orientation.

Theorem. *A graph admits a transitive orientation iff it admits an orientation that is transitive and acyclic.*

A graph is called *alternately orientable* if it admits an orientation of the edges that alternates on every induced cycle of length greater than three. Immediately from the definition it follows that every transitively orientable graph is alternately orientable. The class of alternately orientable graphs was introduced by Hoàng [11] who proved that this class of graphs is perfect. In the same paper he conjectured that a statement similar to the above theorem of Ghouila-Houri might hold.

Conjecture 1. A graph admits an alternating orientation iff it admits an orientation that is alternating and acyclic.

The construction of a counterexample to this conjecture is based on the graph H shown in Fig. 1.

It is easy to see that this graph is alternately orientable and that the edge ac may be oriented arbitrarily in every such orientation while the edges ab and bc form a directed P_3. (The arrows in the above picture show one of the two possible orientations of those edges which lie in an induced cycle of length greater than three. The unoriented edges may be oriented arbitrarily since they lie in no such cycle.) Let H' be a copy of H and a', b' and c' the copies of a, b and c in H'. Now let G be the graph obtained from H and H' by identifying a' with b, b' with c and c' with a. Since H and H' are identified in a clique every induced cycle of G is completely contained in H or H'. Therefore G is

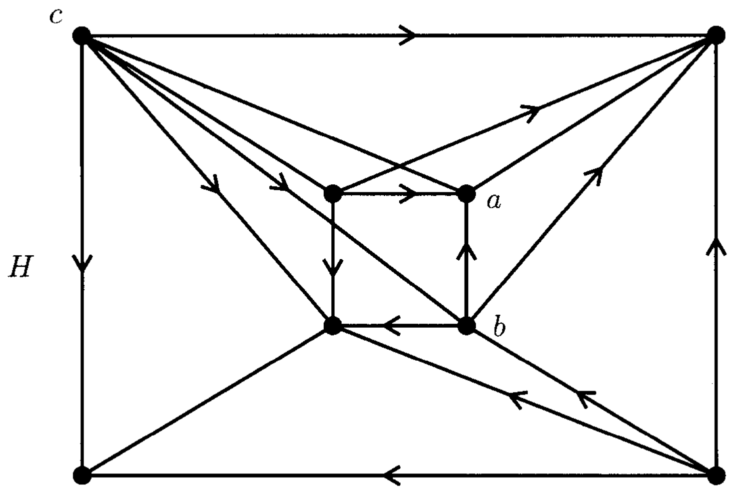

Fig. 1.

also alternately orientable. But the triangle abc is now forced to have a cyclic orientation. Thus G is a counterexample to Conjecture 1.

3. A conjecture on even pairs

A graph is called *minimal imperfect* if the graph itself is not perfect but all of its proper induced subgraphs are. Two vertices in a graph G are called an *even pair* if every induced path connecting these two vertices has even length. Fonlupt and Uhry [6] have proved that contracting an even pair in a perfect graph yields again a perfect graph. This result was essential in the proof of the following lemma of Meyniel [16].

The Even Pair Lemma. *No minimal imperfect graph contains an even pair.*

For several classes of perfect graphs (e.g. weakly triangulated graphs [8], perfectly orderable graphs [10] and quasi-Meyniel graphs [9]) there exist polynomial algorithms to find an optimal coloring of these graphs which are based on the contraction of even pairs. Hertz and de Werra observed that in these algorithms the two vertices of the contracted even pair are always at distance two from each other. This motivated them to pose the following conjecture [5, 18].

Conjecture 2. If a graph contains an even pair then it has an even pair at distance two.

Fig. 2 shows a counterexample to this conjecture.

The two vertices a and b have distance four and they are the only even pair in the graph. To see this look at the graph which is obtained by removing the vertices a and b.

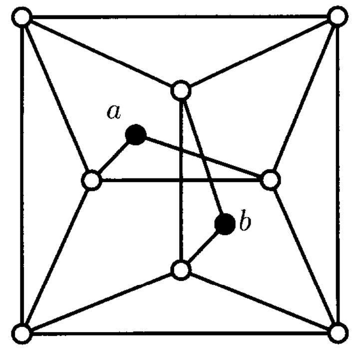

Fig. 2.

This graph is a well-known even-pair-free graph [5]. Thus by symmetry it remains to check that the only even pair containing the vertex a is the pair (a, b). This is easily done.

We are also able to construct examples of graphs where the minimum distance between any two even pairs is arbitrary large as follows: Take the linegraph of an $n \times n$ grid ($n \geqslant 4$, n even). This graph contains no even pair (this follows from a characterization of even-pair-free linegraphs of bipartite graphs given in [13]) and has exactly four edges which do not lie in a triangle. For each of these four edges add a vertex to the graph and connect it to the both endpoints of this edge. The resulting graph has exactly four even pairs and all have distance n.

Conjecture 2 remains false even if it is restricted to the class of strict quasi parity graphs (for a definition see [16]). de Werra [20] conjectured that at least in this case the conjecture should be true. We can construct a counterexample as follows: Take two copies of the $\bar{C}_6$ and choose in both graphs an edge ab resp. $a'b'$ that does not lie in a triangle. If one adds the edges aa' and bb' and removes the edges ab and $a'b'$ then the resulting graph is strict quasi parity with all even pairs at distance at least four.

4. A conjecture concerning Berge graphs

Besides the Even Pair Lemma of Meyniel there exists another often used property of minimal imperfect graphs which was established by Chvátal in 1985 [3] and uses the notion of a star-cutset. A *star-cutset* of a graph G is a cutset C of G that contains a vertex x that is connected to all other vertices of C.

The Star-Cutset Lemma. *No minimal imperfect graph contains a star-cutset.*

From a theorem of König [14] on bipartite graphs one can easily derive the following lemma.

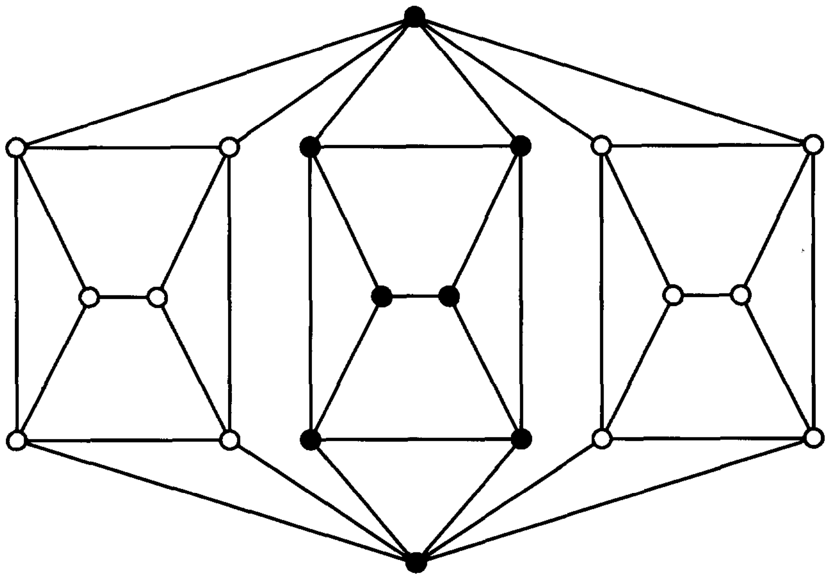

Fig. 3.

Lemma. *No minimal imperfect graph is the linegraph of a bipartite graph.*

These lemmas motivated Reed in 1986 [5, 17] to state the following conjecture.

Conjecture 3. Every Berge graph G satisfies at least one of the following conditions:
(1) G or $\bar{G}$ contains a star-cutset;
(2) G or $\bar{G}$ contains an even pair;
(3) G or $\bar{G}$ is the linegraph of a bipartite graph.

From the Star-Cutset Lemma, the Even Pair Lemma, the lemma of König and the Perfect Graph Theorem it follows that this conjecture would imply the SPGC. With the graph G of Fig. 3 we give a counterexample to Reed's conjecture.

To verify that this graph is a counterexample regard the graph induced by the black vertices. It is easy to see that this graph is Berge, contains no even pair and that all induced paths between the top and the bottom vertex have odd length. From two lemmas in [13] it follows that G therefore is also Berge and has no even pair. (This of course can also be verified directly, but it is quite tedious.) Since every two adjacent vertices in G are the midpoints of an induced P_4 there exists an induced path of length three between any two nonadjacent vertices in $\bar{G}$. Using a characterization of star-cutsets given by Chvátal [3] one can easily verify that neither G nor $\bar{G}$ contains a star-cutset. Since G and $\bar{G}$ contain both a $K_{1,3}$ they are both not linegraphs of a bipartite graph.

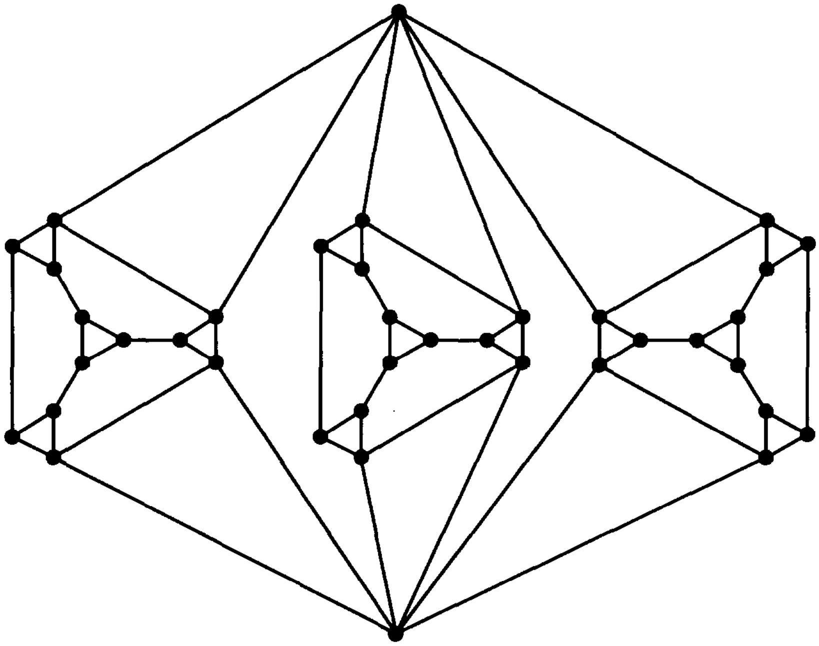

Fig. 4.

The SPGC has been proved for all F-free graphs (i.e. graphs which do not contain F as an induced subgraph) where F is any graph on four vertices except the C_4 and it's complement. It was suggested by Chvátal [4] that one might prove the SPGC for C_4-free graphs by proving Reed's conjecture for this class of graphs. Fig. 4 shows that the conjecture of Reed remains false even for the class of C_4-free graphs.

The verification of this counterexample can be done in a similar way as for the preceding one.

The lemmas in [13] mentioned above give a general method to construct counterexamples to Reed's conjecture. It was observed by Hoàng [12] that every such counterexample is diamond-free. Since the SPGC has been proved for diamond-free graphs [19] and every linegraph of a bipartite graph is diamond-free Hoàng suggested to replace the third condition in Reed's conjecture by the condition that G or $\bar{G}$ is diamond-free. We do not know whether this modification of Reed's conjecture is true.

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