Notes on the Hilbert–Kunz function

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Abstract
We note certain properties of the Hilbert–Kunz function and Hilbert–Kunz multiplicity, including a strengthened inequality between Hilbert–Kunz and Hilbert–Samuel multiplicities and a characterization of a mixed Hilbert function incorporating both ordinary and Frobenius powers of ideals.

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1. Introduction

In this paper we study functions associated to an $m$-primary ideal $I$ of a local Noetherian ring $(R, m)$ of Krull dimension $d$. The Hilbert–Samuel function $H_I : \mathbb{N} \to \mathbb{N}$ is given by $H_I(t) = \ell(R/It)$, where $\ell(M)$ represents the length of an Artinian $R$-module $M$. It is well known from the work of Hilbert and Samuel that $H_I(t)$ is polynomial of degree $d$; in other words, there exists a polynomial $P_I$ of degree $d$ such that $H_I(t) = P_I(t)$ for all sufficiently large $t$. We define the (Samuel) multiplicity $e(I; R)$ of $R$ with respect to $I$ to be $d!$ times the leading coefficient of $P_I$. Alternatively, the multiplicity may be defined asymptotically as

$$e(I; R) = \lim_{t \to \infty} \frac{d! \cdot \ell(R/It)}{t^d}$$

When the ring $R$ is understood, we will sometimes just write $e(I)$ for the multiplicity. The multiplicity of $R$ with respect to its maximal ideal $m$ is sometimes denoted by $e(R)$.

If the ring $R$ has positive prime characteristic $p$, then we may similarly define the Hilbert–Kunz multiplicity $c(I; R)$ of an $m$-primary ideal $I$. For any ideal $I$ of $R$ and any

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integer $e > 0$, we denote by $I^{[p^e]}$ the ideal generated by all $p^e$th powers of elements of $I$.

Because of the characteristic, the ideal is in fact generated by the $p^e$th powers of any set of generators for $I$.) The Hilbert–Kunz function $HK_I : \mathbb{N} \rightarrow \mathbb{N}$ of the ring $R$ with respect to the $m$-primary ideal $I$ is then defined by

$$HK_I(e) = \ell(R/I^{[p^e]}).$$

The fundamental properties of the Hilbert–Kunz function have been developed by Kunz [7,8] and Monsky [10]. Kunz shows that, unlike the Hilbert–Samuel function, $HK_I(e)$ is not necessarily polynomial in $p^e$. However, the main result of Monsky’s paper [10] says that the Hilbert–Kunz function nonetheless induces a multiplicity $c(I; R)$.

**Theorem 1.1** (Monsky [10, Theorem 1.8]). Let $(R, m)$ be a $d$-dimensional local ring of positive prime characteristic $p$, and let $I$ be an $m$-primary ideal of $R$. Then there exists a positive real number $c(I; R)$, called the Hilbert–Kunz multiplicity of $R$ with respect to $I$, such that

$$HK_I(e) = c(I; R)p^{ed} + O(p^{e(d-1)}).$$

In particular, the limit

$$\lim_{e \to \infty} \frac{\ell(R/I^{[p^e]})}{(p^e)^d}$$

exists and is equal to $c(I; R)$.

As above, we will usually just write $c(I)$ when the ring is understood or $c(R)$ for the Hilbert–Kunz multiplicity with respect to the maximal ideal.

Computation of Hilbert–Kunz multiplicities has proved quite difficult; it is not even known whether $c(R)$ must always be a rational number. However, if $(R, m)$ is a $d$-dimensional local ring of positive prime characteristic $p$, and if $I$ is any $m$-primary ideal, then the multiplicities $e(I)$ and $c(I)$ are related by the inequalities

$$e(I)/d! \leq c(I) \leq e(I). \quad (1)$$

Since $I^{[q]} \subseteq I^q$ for any $q = p^e$, the first inequality follows from a comparison of the leading (asymptotic) coefficients of the Hilbert–Samuel and Hilbert–Kunz functions. The inequality $c(I) \leq e(I)$ is obtained by enlarging the residue field of $R$ so as to know that $I$ possesses a minimal reduction $J$. Now $e(I) = e(J)$, and it follows from a result of Lech [9, Corollary to Theorem 2] that $e(J) = c(J)$, since $J$ is generated by a system of parameters.

Thus, since $J \subseteq I$, we see that $e(I) = c(J) \geq c(I)$.

It has been shown that in any Noetherian local ring $(R, m)$ there exist $m$-primary ideals $I$ for which the ratio of $e(I)$ to $d! \cdot c(I)$ is arbitrarily close to one. In fact, it has been proved

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independently by the author [4] and by Watanabe and Yoshida [13] that for any \( m \)-primary ideal \( I \),

\[
\lim_{t \to \infty} \left( \frac{e(I^t)}{d! \cdot c(I^t)} \right) = 1.
\]

However, the arguments do not show that equality between \( e(I) \) and \( d! \cdot c(I) \) can be achieved for any ideal \( I \), and it has been proposed by Watanabe and Yoshida [12] that in fact such equality is not possible when \( d \geq 2 \). In the next section we will give two different proofs of this statement, as well as some applications.

In the final section we introduce a mixed Hilbert function

\[
H_I(e, t) = \ell \left( \frac{R}{(I^e)^t} \right)
\]

associated to an \( m \)-primary ideal \( I \) of a local ring \((R, m)\) of prime characteristic. We explore the relation of this function to certain Hilbert–Samuel and Hilbert–Kunz functions and prove that the mixed function is, in a certain sense, at least as well-behaved as the Hilbert–Kunz function (Theorem 3.2).

2. Relations between Hilbert–Samuel and Hilbert–Kunz multiplicities

We begin this section with a proof that \( d! \cdot c(I) > e(I) \) for any \( m \)-primary ideal \( I \) of a \( d \)-dimensional local ring \((R, m)\) of positive prime characteristic, provided that \( d \geq 2 \). Although the proof of Theorem 2.2 is quite general and satisfying, it does not provide any effective lower bound for the difference \( c(I) - e(I)/d! \). We have therefore taken an alternative approach in Theorem 2.4 which produces a lower bound for the difference in terms of the dimension \( d \) and the minimal number of generators \( \nu(I) \) of the ideal \( I \).

For the proof of Theorem 2.2 we require the following result of [4], which has been proved independently by Watanabe and Yoshida [12].

**Proposition 2.1** [4, Proposition II.1]. Let \( J \subseteq I \) be any \( n \)-primary ideals of a local ring \((S, n)\) of prime characteristic \( p > 0 \). Then for any \( q = p^e \),

\[
\ell(S/J^{[q]}) \leq \ell(S/I^{[q]}) + \ell(S/n^{[q]}) \cdot \ell(I/J).
\]

Consequently, \( c(J) \leq c(I) + c(n) \cdot \ell(I/J) \).

In particular, \( c(I) \leq \ell(S/I) \cdot c(n) \) for any \( n \)-primary ideal \( I \).

We proceed to the main theorem.

**Theorem 2.2.** Let \( I \) be any \( m \)-primary ideal of a local ring \((R, m)\) of prime characteristic \( p \), and assume that \( d = \dim(R) > 1 \). Then \( e(I) < d! \cdot c(I) \).
Proof. We may reduce to the case that \((R, m)\) is a complete local ring with infinite coefficient field, by passing to the completion at the maximal ideal of \(R[z]/mR[z]\) and the expansion of \(I\). This is a faithfully flat extension of the original ring \(R\), of the same dimension, and it is easy to see that lengths mod the expansions of \(m\)-primary ideals remain unchanged.

We may now let \(J = (x_1, \ldots, x_d)\) be a minimal reduction of \(I\). We note that there exists a fixed \(k > 0\) so that \(I^{k+t} \subseteq J^t\) for all \(t > 0\). Since \(I^q \subseteq I^q\) for all \(q = p^r\), and since \(v(I^q) \leq v(I)\) for all \(q \neq p^r\), it follows that

\[
\ell \left( \frac{I^q + J^q}{J^q} \right) \leq v(I) \cdot \ell \left( R/I^k \right),
\]

for all \(q\). We set \(M = v(I) \cdot \ell (R/I^k)\), a fixed constant.

For any \(q = p^r\),

\[
c(I) \cdot q^d = c(I^q) \geq c(I^q + J^q).
\]

Moreover, since the length of \((I^q + J^q)/J^q\) is bounded by \(M\), it follows from Proposition 2.1 that

\[
c(I^q + J^q) \geq c(J^q) - M \cdot c(m),
\]

for any \(q = p^r\). (2) and (3) together imply that the difference \(c(J^q) - c(I) \cdot q^d\) is bounded above by a fixed constant \(M \cdot c(m)\).

Finally, let \(K\) be a coefficient field for \(R\), and let \(A_J = K[x_1, \ldots, x_d]\) be the complete regular subring of \(R\) generated by the \(x_i\)’s. Notice that

\[
\text{rank}_{A_J} R = e(J; R) = e(I),
\]

since \(I\) is contained in the integral closure of \(J\). Consequently,

\[
c(J^q; R) = \text{rank}_{A_J} R \cdot c(J^q; A_J) = e(I) \cdot c(J^q; A_J),
\]

and since \(A_J\) is a power series ring with maximal ideal \(J\), we have by the work of Kunz [7] that

\[
c(J^q; A_J) = \ell \left( \frac{A_J}{J^q} \right) = \binom{q + d - 1}{d}.
\]

Putting the results of Eqs. (2)–(5) together, we get that

\[
c(I) \cdot q^d \geq e(I) \cdot \binom{q + d + 1}{d} - M \cdot c(m),
\]

for all \(q = p^r\). We now have only to note that the right side of (6) is a degree \(d\) polynomial in \(q\) with leading term \((e(I)/d!)q^d\), and that the coefficient on \(q^{d-1}\) is positive as long as
$d \geq 2$. Thus, provided that $d \geq 2$, the expression on the right side of (6) will be strictly greater than its leading term $(e(I)/d!)q^d$ for all sufficiently large $q$. Dividing through by $q^d$ in (6) therefore shows that $c(I) > e(I)/d!$, as claimed. 

**Corollary 2.3.** Let $I$ be any $m$-primary ideal of a local ring $(R, m)$ of prime characteristic $p$, and assume that $d = \dim(R) > 1$. Then $HK_1(e) > \frac{e(I)}{d!}(p^e)^d$ for all sufficiently large $e$.

Theorem 2.2 shows that $c(I) > e(I)/d!$ whenever $d \geq 2$, but it is difficult to apply this result without some lower bound on the difference between the two constants. Our next result will give such a bound in terms of the dimension $d$ and the minimal number of generator $\nu(I)$ of the ideal $I$. The idea here is to note that $I^{[q]} \subseteq I^q$ for any $q = p^e$. It follows immediately from this that $HK_1(e) \geq H_I(q)$, as discussed in the introduction. However, if $d \geq 2$, then the number of generators of $I^q$ tends to infinity, whereas the number of generators of $I^{[q]}$ is at most $\nu(I)$ for all $q$. This leads us to believe that the contribution of $I^q/I^{[q]}$ to $HK_1(e)$ should not be insignificant, as is proved in the following theorem.

**Theorem 2.4.** Let $I$ be any $m$-primary ideal of a local ring $(R, m)$ of prime characteristic $p$, and assume that $d = \dim(R) > 1$. Then

$$c(I) \geq \frac{e(I)}{d!} \cdot \frac{\nu(I)}{(d^2 - \sqrt{\nu(I)} - 1)^{d-1}}.$$ 

**Proof.** We note that $I^{[q]} \subseteq I^q$ for all $q = p^e$, and that $\nu(I^{[q]}) \leq \nu(I)$ for all $q$. It follows that, for any $q = p^e$ and any $s \in \mathbb{N}$,

$$\ell\left(\frac{I^{[q]} + I^q s}{I^q + s}\right) \leq \ell(I) \cdot \ell(R/I^s).$$

Therefore, for any $q = p^e$ and any $s \in \mathbb{N}$, we have

$$\ell\left(\frac{R}{I^{[q]}}\right) \geq \ell\left(\frac{R}{I^{[q]} + I^q s}\right) \geq \ell\left(\frac{R}{I^q + s}\right) - \ell(I) \cdot \ell(R/I^s).$$ (7)

Moreover, using the theory of Samuel functions, we know that

$$\ell(R/I^s) = \frac{e(I)}{d!} s^d + O(d^{d-1}).$$ (8)

The issue is to wisely choose the parameter $s$.

For the approximation of (7) to yield new results on multiplicities, we need $s$ to grow proportionately to $q$. Supposing that this is the case, and setting $s = qh$, we may apply Eqs. (7) and (8) in order to obtain that

$$\ell\left(\frac{R}{I^{[q]}}\right) \geq \frac{e(I)}{d!} [(q + qh)^d - \nu(I)(qh)^d] - O(d^{d-1}).$$
Now, if we ignore the $O(qd^{-1})$ term and derivate in order to maximize the function on the right in terms of $h$, we obtain $h = 1/(d - \sqrt{\nu(I)} - 1)$. The greatest lower bound for $c(I)$ given by Eq. (7) is therefore achieved by setting

$$s = \frac{q}{d - \sqrt{\nu(I)} - 1}.$$  

We do have to deal with the fact that $s$ must be an integer. We therefore set $h = 1/(d - \sqrt{\nu(I)} - 1)$ and apply Eq. (7) with $s = \lceil qh \rceil$. Note that $h > 0$, since $\nu(I) \geq d \geq 2$, and that $s$ therefore grows proportionately to $q$. Also note that we may write $s = q(h - \varepsilon)$, where $\varepsilon < 1/q$.

Applying Eqs. (7) and (8) with this value of $s$ gives us that

$$\ell(R/I[q]) \geq \frac{e(I)}{d!}[(q + s)^d - \nu(I)s^d] - O(qd^{-1})$$

$$\geq \frac{e(I)}{d!} q^d [(1 + h - \varepsilon)^d - \nu(I)(h - \varepsilon)^d] - O(qd^{-1}).$$  

(9)

Dividing through by $qd$ in (9), and letting $q$ tend toward infinity (so that $\varepsilon$ tends toward 0), we obtain the estimate

$$c(I) \geq \frac{e(I)}{d!} [(1 + h)^d - \nu(I)h^d].$$  

(10)

It is easily calculated that

$$(1 + h)^d - \nu(I)h^d = \frac{\nu(I)}{(d - \sqrt{\nu(I)} - 1)^d - 1},$$

which completes the proof of the theorem. □

**Remark 2.5.** In general, the approximation $\ell(I^{[q]})/(I^{[q]} + I^{[q]+}) \leq \nu(I) \cdot \ell(R/I^q)$ is quite crude; these numbers will often differ significantly. In fact, the question of determining the maximal Hilbert function of an ideal with a given number of generators of given degrees is interesting in itself (see, e.g., [1, 6]). On the other hand, since we know that $c(I)$ can be as great as $e(I)$, the fact that our estimate can be way off should come as no surprise.

Of course, for a fixed dimension $d$ the constant $\nu(I)/(d - \sqrt{\nu(I)} - 1)^d - 1$ tends toward one for large values of $\nu(I)$. However, if we make some assumptions on the dimension or on the number of generators of $I$, Theorem 2.4 does yield some interesting corollaries. Corollary 2.6, which gives the statement of the theorem for two-dimensional rings, has already appeared in work of Watanabe and Yoshida [13], but also follows quite easily from Theorem 2.4.

**Corollary 2.6 (cf. [13, Lemma 2.1]).** For any $m$-primary ideal $I$ of a two-dimensional local ring $(R, m)$ of positive prime characteristic, $c(I) \geq e(I)\nu(I)/(2\nu(I) - 2)$.
Corollary 2.7. If \( I \) is an \( m \)-primary ideal of a \( d \)-dimensional local ring \((R, m)\) of positive prime characteristic, and if \( \nu(I) \leq 2^{d-1} \), then \( c(I) \geq e(I)2^{d-1}/d! \).

**Proof.** The statement is trivial when \( d = 1 \). Otherwise, we apply Theorem 2.4, noting that the function

\[
F(x) = \frac{x}{(d-1/\sqrt{x} - 1)^{d-1}} = \left(1 + \frac{1}{(d-1/\sqrt{x} - 1)}\right)^{d-1}
\]

is decreasing for \( x > 1 \), and that \( F(2^{d-1}) = 2^{d-1} \).  \( \square \)

Corollary 2.8. Let \((R, m)\) be a \( d \)-dimensional hypersurface ring of prime characteristic, where \( d \geq 3 \). Then \( c(m) \geq e(m)2^{d-1}/d! \).

**Proof.** Apply the previous corollary, noting that \( \nu(m) = d + 1 \leq 2^{d-1} \) for \( d \geq 3 \).  \( \square \)

Theorem 2.4 and its corollaries should help to strengthen several extant approximation results. For example, it is a theorem of Lech [9] that

\[
e(I) \leq d! \ell(R/I)e(m),
\]

for any \( m \)-primary ideal \( I \) of a \( d \)-dimensional local ring \((R, m)\). It has been noted in earlier work [4] that this inequality can be recovered from Proposition 2.1 for rings of prime characteristic, since \( e(I) \leq d!c(I) \) and \( c(m) \leq e(m) \). But we now see that the former inequality can be replaced by the one given by Theorem 2.4. We therefore obtain the following corollary.

Corollary 2.9. Let \((R, m)\) be a \( d \)-dimensional local ring of prime characteristic, and let \( I \subseteq R \) be any \( m \)-primary ideal. Then

\[
e(I) \leq \frac{d!(d-1/\sqrt{\nu(I)} - 1)^{d-1}}{\nu(I)}\ell(R/I)c(m).
\]

**Proof.** We simply apply Theorem 2.4 to the ideal \( I \), and then note that by Proposition 2.1, \( c(I) \leq \ell(R/I)c(m) \).  \( \square \)

As before, Corollary 2.9 is most relevant when the number of generators of \( I \) is relatively small. If, for example, \( \nu(I) \leq 2^{d-1} \), we obtain the inequalities

\[
e(I) \leq \frac{d! \ell(R/I)c(m)}{2^{d-1}} \leq \frac{d! \ell(R/I)e(m)}{2^{d-1}}.
\] (11)

In fact, all of the bounds on \( e(I) \) obtained in [4] are achieved by combining a bound on \( c(I) \) with the inequality \( e(I) \leq d! \cdot c(I) \). We may therefore apply Theorem 2.4 in order to strengthen the inequalities. The resulting bounds are given by the following two theorems (which correspond to Propositions III.5 and III.8 of [4]).
Theorem 2.10. Let \((R, m)\) be a \(d\)-dimensional local ring of prime characteristic \(p\), and let \(I\) be an \(m\)-primary ideal. Then for any (minimal) reduction \(J\) of \(m\),
\[
e(I) \leq d! \cdot \left( \frac{(d-1)^{d-1}}{\nu(I)} - 1 \right) \left( \frac{d}{\epsilon} \right) \cdot c(m).
\]

Theorem 2.11. Let \((R, m)\) be a local ring of dimension \(d\) and positive prime characteristic \(p\), and let \(I\) be an \(m\)-primary ideal with \(I \subseteq m^t\). Then
\[
e(I) \leq d! \cdot \left( \frac{(d-1)^{d-1}}{\nu(I)} - 1 \right) \left( \frac{d}{\epsilon} \right) \cdot c(m) \cdot \left( \frac{t}{d} \right).\]

Remark 2.12. As in [4], we may replace the term \(c(m)\) by \(e(m)\) in each of the three preceding results, since \(c(m) \leq e(m)\), and then we may show by a reduction to characteristic \(p\) that the resulting inequalities are equally valid over rings of equal characteristic zero. Since these reductions will be essentially the same as those given in [4] (all that is required is that we also preserve the number of generators of \(I\) while making the reduction), we do not carry them out here.

As a simple example, we offer the following.

Example 2.13. Let \(R\) be the hypersurface ring
\[
R = \mathbb{Z}/3\mathbb{Z}[X_1, \ldots, X_5] / (X_1^2 + \ldots + X_5^2).
\]
Han and Monsky, who have a general algorithm for computing the Hilbert–Kunz multiplicities of diagonal hypersurfaces [3], compute \(c(R) = 23/19\). Since \(\dim(R) = 4\) and \(e(m) = 2\), the general inequality \(e(I) \leq d! \cdot \epsilon(R/I) e(m)\) yields that \(e(I) \leq 48 \epsilon(R/I)\) for any ideal \(I\) of finite colength in \(R\).

Applying Corollary 2.9 certainly gives a better bound. In particular, if \(I\) is an ideal with \(\nu(I) \leq 8\), we see that
\[
e(I) \leq \frac{d!}{8} \cdot \epsilon(R/I) c(m) = \frac{69}{19} \epsilon(R/I).
\]

We may give one last application of Theorem 2.4, or even of Theorem 2.2, for rings of dimension two. Note that if \((R, m)\) is a complete local ring of characteristic \(p\), and if the residue field \(R/m\) is perfect, then the extension rings \(R^{1/q}\) are finite \(R\)-modules for all \(q = p^e\), \(e(m; R^{1/q}) = q^e e(m; R)\) for all \(q\), and \(c(R) = \lim_{e \to \infty} (\nu(R^{1/q})/q^d)\). The statement that \(c(m) > e(m)/d!\) is therefore equivalent to the statement that \((d!)\nu(R^{1/q}) > e(R^{1/q})\) for all sufficiently large \(q\). This is of interest in light of the following result of Ulrich [11].

Theorem 2.14 (Ulrich). Let \(R\) be a local Cohen–Macaulay ring, and suppose that there exists a finitely generated \(R\)-module \(M\) with positive rank such that
(i) $2\nu(M) > e(R) \text{rank}(M)$, and

(ii) $\Ext^i_R(M, R) = 0$ for $1 \leq i \leq \dim(R)$.

Then $R$ is Gorenstein.

When the dimension is two, we obtain the following corollary.

**Corollary 2.15.** If $R$ is a two-dimensional complete local Cohen–Macaulay domain of positive prime characteristic $p$, with perfect residue field $k$, then $R$ is Gorenstein if and only if $\Ext^1_R(R^{1/q}, R) = \Ext^2_R(R^{1/q}, R) = 0$ for all $q = p^s$.

This result bears close resemblance to work of Goto [2], who proved a similar result under weaker conditions in all dimensions, but with the assumption that $R^{1/q}$ be self-dual as an $R$-module for some $q = p^s$. Other applications and extensions of Ulrich’s theorem have recently been studied in [5].

3. Characterization of a mixed Hilbert function

In this section we seek to characterize a “mixed” Hilbert function given by colengths of Frobenius powers of ordinary powers of an Artinian ideal. For this, of course, it is required that the ring have nonzero prime characteristic. Considering both ordinary and Frobenius powers of ideals simultaneously, we are quite naturally led to the following definition.

**Definition 3.1.** Let $(R, m)$ be a $d$-dimensional local ring of prime characteristic $p$, and let $I \subseteq R$ be an $m$-primary ideal. We define the mixed Hilbert function $H_I : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ by

$$H_I(e, t) = \ell \left( \frac{R}{(I^{q^e})^t} \right).$$

One should note that, for any $q = p^s$ and any $t$, 

$$(I^{q^e})^t = (I^t)^{q^e}.$$ 

The order in which we take ordinary and Frobenius powers is therefore irrelevant and we may write $I^{q^e}$, or $I^{q^e}$, without causing any confusion. The mixed Hilbert function recovers various Hilbert–Kunz and Hilbert–Samuel functions associated to the ideal $I$. In particular,

1. For any fixed $e_0$ and $q_0 = p^{e_0}, H_I(e_0, t) = H_{I^{q_0}}(t)$.
2. For any fixed $t_0, H_I(e, t_0) = HK_{I^{q_0}}(e)$.

Since $e(I^{q^e}) = q^e e(I)$ for any $q = p^s$, we see that if $e_0$ is fixed, then the mixed Hilbert function must take the form:

$$H_I(e_0, t) = H_{I^{q_0}}(t) = \frac{e(I)}{d!} (p^{e_0})^d t^d + O(t^{d-1}).$$

(12)
Likewise, for fixed $t_0$, we have by Theorem 1.1 that

$$H_I(e, t_0) = HK_{I_0}(e) = c(t_0^n) \cdot (p^n)^d + O((p^n)^{d-1}). \quad (13)$$

We do not in general have a very good grasp of the leading coefficient $c(t_0^n)$; however, it is known in general that

$$c(t_0^n) \approx \frac{e(I^n)}{d!} = \frac{e(I)}{d!} t^d,$$

as well as that

$$\lim_{t \to \infty} \left( \frac{e(I^n)}{d! \cdot c(I^n)} \right) = 1 \quad (14)$$

(as discussed in the introduction). This leads one to surmise that the function $H_I(e, t)$ should be asymptotic to a polynomial in $q = p^e$ and $t$, of degree $d$ in each variable, with leading term $(e(I)/d!) q^d t^d$. The following theorem makes this more precise.

**Theorem 3.2.** Let $(R, m)$ and $I$ be as in Definition 3.1. Then

$$H_I(e, t) = \frac{e(I)}{d!} (p^e)^d t^d + O((p^e)^{d-1}).$$

More specifically, there exist positive real numbers $a$ and $b$ such that

$$\frac{e(I)}{d!} (p^e)^d t^d - a(p^e)^{d-1} t^{d-1} \leq H_I(e, t) \leq \frac{e(I)}{d!} (p^e)^d t^d + b(p^e)^d t^{d-1}$$

for all positive integers $e$ and $t$.

We make a couple of remarks before proceeding to the proof of the theorem.

**Remark 3.3.** The earlier result that

$$\lim_{t \to \infty} \frac{e(I^n)}{d! \cdot c(I^n)} = 1$$

(cf. [4,13]) easily follows from the statement of Theorem 3.2.

**Remark 3.4.** We remark that, according to Theorem 3.2, the degree $d$ coefficient on $H_I(e, t)$, as a function of $t$, is $(e(I)/d!) (p^e)^d$. In other words,

$$\lim_{t \to \infty} \frac{H_I(e, t)}{t^d} = \frac{e(I)}{d!} (p^e)^d,$$

for any fixed $e$. But we already knew from Eq. (12) that, if $H_I(e, t)$ could be expressed in the way given by Theorem 3.2, then this must be the case.
Likewise, the coefficient on $(p^e)^d$ must be $c(I^t)$ for any fixed $t$, which is always greater than or equal to $(e(I)/d!) \cdot t^d$, with the inequality generally being strict (see Section 2). In particular, for $d \geq 2$ it is not true that

$$H_I(e, t) = \frac{e(I)}{d^d} (p^e)^d t^d + O((p^e)^{d-1}).$$

We even know that for any fixed $t$, there exists some $\epsilon_t > 0$ such that $HI(e, t)/q^d \geq (e(I)/d!) t^d + \epsilon_t$ for all sufficiently large $q = p^r$ (by Theorem 2.2 or 2.4).

**Proof of Theorem 3.2.** For any $q = p^r$ and any $t$, one easily sees that

$$H_I(e, t) = \ell \left( \frac{R}{(I^q)^t} \right) \geq \ell \left( \frac{R}{I^q} \right) = H_I(tq),$$

and it is well known that

$$H_I(tq) = \frac{e(I)}{d!} (tq)^d + O((tq)^{d-1}).$$

This shows that the mixed Hilbert function is bounded below by a formula of the required type.

To prove the other inequality, we may reduce to the case that $R$ is a complete local ring with infinite coefficient field $K$, since completing at the maximal ideal and extending the residue field does not change any of the finite lengths used in determining the various Hilbert-type functions. In this case, we let $J = (x_1, \ldots, x_d) \subseteq I$ be a minimal reduction of the ideal $I$, and we set $A = K[x_1, \ldots, x_d] \subseteq R$, the complete regular local subring generated by the $x$’s. If $r$ is the torsion-free rank of $R$ over $A$, then we may consider a short exact sequence

$$0 \to A^r \to R \to C \to 0$$

of $A$-modules, where $C$ is a torsion module. It follows that, for all $t$ and all $q = p^r$,

$$H_J(e, t) \leq r \cdot \ell \left( \frac{A}{(J^t)^q} \right) + \ell \left( \frac{C}{(J^t)^q C} \right).$$

(15)

We now remark that:

1. $H_I(e, t) \leq H_J(e, t)$ for all $e$ and $t$, since $J \subseteq I$.
2. $r = e(J; R) = e(I; R)$.
3. Since $A$ is regular, for any $t$ and $q = p^r$,

$$\ell \left( \frac{A}{(J^t)^q} \right) = q^d \cdot \ell \left( \frac{A}{J^t} \right) = q^d \cdot \binom{t + d - 1}{d}.$$

4. $\ell(C/(J^t)^q C) = O((tq)^{d-1}).$
Note that \( \dim(C) < d \), and that one may assume point (4) by an induction on the dimension. Alternatively, one may simply note that

\[
\ell(C/(J^t)^{[q]} C) \leq \ell(C/(J^{dq^t}) C)
\]

and use well-known results on the Hilbert–Samuel function. Combining Eq. (15) and notes (1)–(4) yields the inequality:

\[
H_t(e,t) \leq e(I) \cdot q^d \cdot \binom{t + d - 1}{d} + O(tq)^{d-1}
\]

\[
\leq \frac{e(I)}{d!} q^d t^d + q^d O(t^{d-1}) + O(tq)^{d-1}.
\]

From this it is clear that we may choose \( b \) as in the statement of the theorem, and the proof is complete. \( \square \)

The theorem is, theoretically, quite satisfying, in that it shows that we may associate to the mixed Hilbert function a well-defined leading coefficient, i.e., a multiplicity. In fact, unlike the Hilbert–Kunz multiplicity, the leading coefficient of the mixed Hilbert function is a familiar constant that is rational and can quite often be calculated. On the other hand, the results presented here do suggest some limits upon attempts to use the Hilbert–Kunz multiplicity as a fundamentally “new” invariant in examples, or to improve existing approximations/inequalities by reference thereto (as, for example, in our previous work [4] or in Section 2). Such methods may still give nice results, but only if one does not apply them to high powers of ideals, where the new invariant does not significantly vary from a familiar one.

References