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E_{11}, SL(32) and central charges

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Abstract

We show that the E_{11} representation that contains the space–time translation generators also contains the rank two and five totally anti-symmetric representations of A_{10}. By studying the behaviour of these latter A_{10} representations under SL(32), which we argue is contained in the Cartan involution invariant sub-algebra of E_{11}, we find that the rank two and five totally anti-symmetric representations must be identified with the central charges of the eleven dimensional supersymmetry algebra.

1. Introduction

It has been shown that the entire bosonic sector of eleven-dimensional supergravity, can be formulated as non-linear realisation [1]. The algebra, denoted G_{11}, used for this non-linear realisation was not a Kac–Moody algebra. However, it has been suggested [2] that this theory can be reformulated, or extended, in such a way that it can be described as a non-linear realisation based on a Kac–Moody algebra. Assuming this to be the case, it was shown [2] that this Kac–Moody symmetry should contain very extended E_8, i.e., E_{11}. Substantial fragments of this symmetry, as well as other evidence for it, has been presented [2–4]. Analogous results were also found for IIA [2] and IIB supergravity [5] where the corresponding Kac–Moody algebras were also found to be E_{11} in each case. An account of work on symmetries in dimensionally reduced supergravity and string theories is given in Ref. [15].

The algebra G_{11} contained the usual translation generators whose role was to introduce space–time into the theory, however, these generators did not play any role in subsequent discussions on E_{11}. In this Letter we will explain how the translation generators can be incorporated into a non-linear realisation based on E_{11}. The translation generators carry one lower index and transform in the corresponding representation of A_{10}. Since E_{11} contains A_{10} we will enlarge this A_{10} representation to one belonging to E_{11}. We will find that the translations occur in this E_{11} representation together with a second rank and fifth rank anti-symmetric tensor representations of A_{10} as well as an infinite number of other tensors.

The gravity sector of eleven-dimensional supergravity arises as the non-linear realisation of the IGL(11) sub-algebra of G_{11}, the I being the translations. Taking the non-linear realisation of IGL(11) with the Lorentz algebra SO(1,10) as the local sub-algebra does not uniquely lead to gravity. However, if one takes the simultaneous non-

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linear realisation of IGL(11) and the conformal group SO(2,11) one finds to Einstein's theory of gravity [1,6] essentially uniquely. In Ref. [1] the non-linear realisation based on G_{11} used for eleven-dimensional supergravity was combined with a simultaneous non-linear realisation of the conformal algebra and this did lead to the unique bosonic field equations of eleven-dimensional supergravity, up to one undetermined constant [1]. Subsequent papers did not address the question of how this conformal algebra combined with E_{11}. However, it was argued in Ref. [1] that the presence of the fermionic extension of bosonic sector of eleven-dimensional supergravity considered above implied that the full theory would possess an $\mathrm{Osp}(1/64)$. This algebra contains GL(32) which rotates the spinor index on the supercharges and so this must also be a symmetry of M theory. In fact, the algebra GL(32) had previously been proposed as a symmetry of M theory as part of it occurred as a symmetry of the fivebrane equations of motion [8]. It was explained [8] that the SL(32) symmetry was a brane rotating symmetry and was the natural generalisation of the local spin or Lorentz algebra when branes were present.

The local sub-algebra used in formulating the E_{11} non-linear realisation was taken to be the one which is invariant under the Cartan involution. We will argue that SL(32) is indeed part of this local sub-algebra and we calculate the transformations of the second and fifth rank anti-symmetric tensor representations mentioned above under SL(32). We will find that these objects transform in such a way that they should be identified with central charges of the eleven-dimensional supersymmetry algebra.

2. E_{11} at low levels

We invite the reader to draw the Dynkin diagram of E_{11} by drawing ten dots in a horizontal line labeled from one to ten from left to right and connected by a single line. Then place another dot, labeled eleven, above the third node (labeled eight) from the right. We consider E_{11} as a member of the class of Kac–Moody algebras discussed in Section 3 of Ref. [7], namely, an algebra whose Dynkin diagram possess at least one node such that deleting it leads to a finite-dimensional semi-simple Lie algebra. If we delete node eleven in the Dynkin diagram of E_{11}, the remaining algebra is A_{10}. The preferred simple root is α_{11} and the simple roots of A_{10} are α_i, $i = 1, \ldots, 10$. We may write [7]

$$\alpha_{11} = -\lambda_8 + x, \tag{2.1}$$

where x is a vector in a space orthogonal to the roots of A_{10} and λ_i are the fundamental weight vectors of A_{10}. The simple roots have length squared two and so $x^2 = 2 - \lambda_8^2 = -\frac{2}{11}$.

A root α of E_{11} can be written as

$$\alpha = l\alpha_{11} + \sum_i m_i\alpha_i = lx - l\lambda_8 + \sum_{jk} A^f_{jk}\lambda_k, \tag{2.2}$$

where A^f_{jk} is the Cartan matrix of A_{10}. We define the level, denoted l, [4] of the roots of E_{11} to be the number of times the root α_{11} occurs in its decomposition into simple roots given in the equation above. The generators of E_{11} can also be classified according to their level which is just the level of the root associated with the generator.

The E_{11} algebra contains the generators $K^a{}_b$ at level 0 and the generators

$$R^{a_1a_2a_3}, \quad R^{a_1a_2\ldots a_6}, \quad R^{a_1a_2\ldots a_8,b} \tag{2.3}$$

at levels zero, 1, 2 and 3, respectively [2,4], as well as the generators

$$R_{a_1a_2a_3}, \quad R_{a_1a_2\ldots a_6}, \quad R_{a_1a_2\ldots a_8,b} \tag{2.4}$$

at levels -1, -2, -3. The generators of E_{11} at higher levels are listed in Refs. [4,16].

The corresponding Borel sub-algebra up to, and including, level 3 obeys the commutation relations [2]

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b, \tag{2.5}$$

$$[K^a{}_b, R^{c_1\ldots c_6}] = \delta^{c_1}_b R^{ac_2\ldots c_6} + \cdots, \qquad [K^a{}_b, R^{c_1\ldots c_3}] = \delta^{c_1}_b R^{ac_2c_3} + \cdots, \tag{2.6}$$

$$[K^a{}_b, R^{c_1\ldots c_8,d}] = (\delta^{c_1}_b R^{ac_2\ldots c_8,d} + \cdots) + \delta^d_b R^{c_1\ldots c_8,a}, \tag{2.7}$$

$$[R^{c_1\ldots c_3}, R^{c_4\ldots c_6}] = 2R^{c_1\ldots c_6}, \qquad [R^{a_1\ldots a_6}, R^{b_1\ldots b_3}] = 3R^{a_1\ldots a_6[b_1b_2,b_3]}, \tag{2.8}$$

where '$+\cdots$' means the appropriate anti-symmetrisation. The level 0 and negative level generators obey the relations

$$[K^a{}_b, R_{c_1\ldots c_3}] = -\delta^a_{c_1} R_{bc_2c_3} - \cdots, \qquad [K^a{}_b, R_{c_1\ldots c_6}] = -\delta^a_{c_1} R_{bc_2\ldots c_6} - \cdots, \tag{2.9}$$

$$[K^a{}_b, R_{c_1\ldots c_8,d}] = -(\delta^a_{c_1} R_{bc_2\ldots c_8,d} + \cdots) - \delta^a_d R_{c_1\ldots c_8,b}, \tag{2.10}$$

$$[R_{c_1\ldots c_3}, R_{c_4\ldots c_6}] = 2R_{c_1\ldots c_6}, \qquad [R_{a_1\ldots a_6}, R_{b_1\ldots b_3}] = 3R_{a_1\ldots a_6[b_1b_2,b_3]}. \tag{2.11}$$

Finally, the commutation relations between the positive and negative generators of up to level three are given by

$$[R^{a_1\ldots a_3}, R_{b_1\ldots b_3}] = 36\delta^{[a_1a_2}_{[b_1b_2} K^{a_3]}{}_{b_3]} - 4\delta^{a_1a_2a_3}_{b_1b_2b_3} D, \qquad [R_{b_1\ldots b_3}, R^{a_1\ldots a_6}] = -\frac{6!}{3!}\delta^{[a_1a_2a_3}_{b_1b_2b_3} R^{a_4a_5a_6]}, \tag{2.12}$$

where

$$D = \sum_b K^b{}_b, \qquad \delta^{a_1a_2}_{b_1b_2} = \frac{1}{2}(\delta^{a_1}_{b_1}\delta^{a_2}_{b_2} - \delta^{a_2}_{b_1}\delta^{a_1}_{b_2}) = \delta^{[a_1}_{b_1}\delta^{a_2]}_{b_2}$$

with similar formulae when more indices are involved.

The above commutators can be deduced, using the Serre relations and from the identification of the Chevalley generators of E_{11} which are given by [2]

$$\begin{aligned} &E_a = K^a{}_{a+1}, \quad a = 1,\ldots,10, \qquad E_{11} = R^{91011}; \\ &F_a = K^{a+1}{}_a, \quad a = 1,\ldots,10, \qquad F_{11} = R_{91011}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} &H_a = K^a{}_a - K^{a+1}{}_{a+1}, \quad a = 1,\ldots,10, \\ &H_{11} = \frac{2}{3}(K^9{}_9 + K^{10}{}_{10} + K^{11}{}_{11}) - \frac{1}{3}(K^1{}_1 + \cdots + K^8{}_8). \end{aligned} \tag{2.14}$$

The sub-algebra which is invariant under the Cartan involution, namely,

$$E_a \to -F_a, \quad F_a \to -E_a, \quad H_a \to -H_a, \tag{2.15}$$

plays an important part in the non-linear realisation of Ref. [2] as it is taken to be the local sub-algebra. As such, its generators do not lead to fields in the non-linear realisation. The Cartan involution is a linear operator and acts on the generators as

$$K^a{}_b \to -K^a{}_a, \quad R^{a_1a_2a_3} \to -R_{a_1a_2a_3}, \quad R^{a_1\ldots a_6} \to R_{a_1\ldots a_6}, \quad R^{a_1\ldots a_8,b} \to -R_{a_1\ldots a_8,b}. \tag{2.16}$$

The sub-algebra invariant under the Cartan involution is generated by $E_a - F_a$ and at low levels it includes the generators

$$J_{ab} = K^c{}_b\eta_{ac} - K^c{}_a\eta_{bc}, \qquad S_{a_1a_2a_3} = R^{b_1b_2b_3}\eta_{b_1a_1}\eta_{b_2a_2}\eta_{b_3a_3} - R_{a_1a_2a_3}, \tag{2.17}$$

$$S_{a_1\ldots a_6} = R^{b_1\ldots b_6}\eta_{b_1a_1}\cdots\eta_{b_6a_6} + R_{a_1\ldots a_6}, \tag{2.18}$$

$$S_{a_1\ldots a_8,c} = R^{b_1\ldots b_8,b}\eta_{b_1a_1}\cdots\eta_{b_8a_8}\eta_{bc} - R_{a_1\ldots a_8,c}. \tag{2.19}$$

The generators J_{ab} are those of the Lorentz algebra SO(1,10) and their commutators with the other generators just express the fact that they belong to a representation of the Lorentz algebra. The $S_{a_1a_2a_3}$ and $S_{a_1\ldots a_6}$ generators obey the commutators

$$\left[S^{a_1a_2a_3}, S_{b_1b_2b_3}\right] = -36\delta^{[a_1a_2}_{[b_1b_2}J^{a_3]}{}_{b_3]} + 2S^{a_1a_2a_3}{}_{b_1b_2b_3}, \tag{2.20}$$

$$\left[S_{a_1a_2a_3}, S^{b_1\ldots b_6}\right] = -2\frac{6!}{3}\delta^{[b_1b_2b_3}_{a_1a_2a_3}S^{b_4b_5b_6]} - 3S^{b_1\ldots b_6}{}_{[a_1a_2,a_3]}. \tag{2.21}$$

3. Translations and an E_{11} representation

The space–time translation generators carry a single lower index and transform in the corresponding A_{10} representation. This is equivalent to a tensor with ten upper anti-symmetrised indices which, in our conventions, is the representation with weight λ_1, or Dynkin index $p_1 = 1$, all other p_i's vanishing. We wish to consider the representation of E_{11} which contain the translation generators. The fundamental weights of E_{11} are given by [7]

$$l_i = \lambda_i + \lambda_8\lambda_i\frac{x}{x^2}, \quad i = 1, \ldots, 10, \qquad l_{11} = \frac{x}{x^2}. \tag{3.1}$$

The E_{11} representation with highest weight $l_1 = \lambda_1 - \frac{3}{2}x$ obviously contains the states in the A_{10} representation λ_1. The highest weight state $|l_1 >$ can be thought of as being at level $-\frac{3}{2}$. It is straightforward to construct the root string associated with the action of the simple negative roots F_a on this highest weight state. One finds

$$l_1, \quad l_1 - \alpha_1, \quad l_1 - \alpha_1 - \alpha_2, \quad \ldots, \quad l_1 - \alpha_1 - \cdots - \alpha_8 - \alpha_{11}, \quad \ldots. \tag{3.2}$$

The last weight written explicitly is the first one in the string where α_{11} enters and it corresponds to the appearance of a new A_{10} representation in the string. In fact,

$$l_1 - \alpha_1 - \cdots - \alpha_8 - \alpha_{11} = \lambda_9 - \frac{5}{2}x, \tag{3.3}$$

which contains the highest weight for the A_{10} representation whose only non-vanishing Dynkin index is $p_9 = 1$ or a second rank anti-symmetric tensor. Continuing in this way we find that the representation l_1 contains the following A_{10} representations

$$\begin{aligned} &p_1 = 1, \left(-\frac{3}{2}\right); \qquad p_9 = 1, \left(-\frac{5}{2}\right); \qquad p_6 = 1, \left(-\frac{7}{2}\right); \qquad p_4 = 1, \qquad p_{10} = 1, \left(-\frac{9}{2}\right); \\ &p_3 = 1, \left(-\frac{9}{2}\right); \quad \ldots, \end{aligned} \tag{3.4}$$

all other p_i's vanishing. The number in brackets is the corresponding level.

Consider any Lie algebra g with a representation $u(A)$. By definition, the linear operators $u(A)$, for each element $A \in g$, obey the relation $u(A_1A_2) = u(A_1)u(A_2)$. If the representation is carried by the states $|X_s\rangle$ it defines the matrices $u(A)|X_s\rangle = (c(A))_s{}^t|\chi_t\rangle$. Clearly, $c(A_1A_2) = c(A_2)c(A_1)$ and so a matrix representation of g is defined by $d(A) = c(A^\ddagger)$ where $\ddagger$ is any operation that inverts the order of the factors in the Lie algebra. The relevant operation for us is to take $A^\ddagger = I^cI^I(A)$, where I^c is the Cartan involution and I^I is the operators which inverts the order of operators and $I^I(A) = -A$. The latter operator is just the operator in the algebra which corresponds to inversion of group elements.

In these circumstance we can define a semi-direct product algebra. We associate with each state in the representation $|X_s\rangle$ a generator X_s and we extend the algebra g to include the new generators by adopting the commutation relation

$$[X_s, A] = d(A)_s{}^rX_r, \quad A \in g. \tag{3.5}$$

This is consistent with the Jacobi identities involving two elements of g and one X_s. The commutator between two elements X_s and X_r must be chosen and it is consistent to choose it to vanish.

Carrying out this procedure for E_{11} and the representation l_1 we introduce the generators

$$P^{a_1\dots a_{10}},\quad W^{a_1a_2},\quad W^{a_1\dots a_5},\quad W^{a_1\dots a_7,b},\quad \dots \tag{3.6}$$

at levels $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, $\frac{9}{2}$, respectively. it is straightforward to find the generator associated with the root string of Eq. (3.2). The entries explicitly given correspond to $P^{2\dots 11}$, $P^{13\dots 11}$, $P^{124\dots 11}$ and R^{91011}. We denote the semi-direct product of E_{11} with its l_1 representation by $E_{11} \oplus_s L_1$.

It is simpler to work with the more familiar generator

$$P_a = \frac{1}{10!}\epsilon_{ba_1\dots a_{10}} P^{a_1\dots a_{10}}. \tag{3.7}$$

The A_{10} generators act on the translations as

$$\left[K^c{}_b, P^{a_1\dots a_{10}}\right] = 10\delta_b^{[a_1} P^{|c|a_2\dots a_{10}]} - \frac{1}{2}\delta_b^c P^{a_1\dots a_{10}}. \tag{3.8}$$

The last term on the right-hand side of the above equation is required as a result of the relation $[H_{11}, P^{a_1\dots a_{10}}] = 0$. The corresponding commutator involving P_a is

$$\left[K^c{}_b, P_a\right] = -\delta_a^c P_b + \frac{1}{2}\delta_b^c P_a. \tag{3.9}$$

The root string of Eq. (3.2) corresponds to the commutators

$$\left[R^{a_1a_2a_3}, P_b\right] = 3\delta_b^{[a_1} W^{a_2a_3]},\qquad \left[R^{a_1a_2a_3}, W^{b_1b_2}\right] = W^{a_1a_2a_3b_1b_2}, \tag{3.10}$$

$$\left[R^{a_1a_2a_3}, W^{b_1\dots b_5}\right] = W^{b_1\dots b_5[a_1a_2,a_3]}, \tag{3.11}$$

which also normalise the new generators.

Using these relationships and the Jacobi identities one deduces that

$$\left[R^{a_1\dots a_6}, P_b\right] = -3\delta_b^{[a_1} W^{a_2\dots a_6]}. \tag{3.12}$$

Using the Jacobi identities and Eqs. (3.10) and (3.11) the commutators involving the negative generators R_{abc} is found to be given by

$$\left[R_{a_1a_2a_3}, P_b\right] = 0,\qquad \left[R_{a_1a_2a_3}, W^{b_1b_2}\right] = 12\delta^{b_1b_2}_{[a_1a_2} P_{a_3]}, \tag{3.13}$$

$$\left[R_{a_1a_2a_3}, W^{b_1\dots b_5}\right] = 120\delta^{[b_1b_2b_3}_{a_1a_2a_3} W^{b_4b_5]}. \tag{3.14}$$

It will be instructive to consider the commutators of the Cartan invariant sub-algebra generator S_{abc} of E_{11} with the generators associated with the representation l_1. Using Eqs. (3.10) and (3.11) and Eqs. (3.13) and (3.14) we find that

$$\left[S_{a_1a_2a_3}, P_b\right] = 3\eta_{b[a_1} W_{a_2a_3]},\qquad \left[S_{a_1a_2a_3}, W^{b_1b_2}\right] = W_{a_1a_2a_3}{}^{b_1b_2} - 12\delta^{b_1b_2}_{[a_1a_2} P_{a_3]}, \tag{3.15}$$

$$\left[S_{a_1a_2a_3}, W^{b_1\dots b_5}\right] = W^{b_1\dots b_5}{}_{[a_1a_2,a_3]} - 120\delta^{[b_1b_2b_3}_{a_1a_2a_3} W^{b_4b_5]}. \tag{3.16}$$

It is interesting to examine what adding the generators corresponding to the l_1 representation means in terms of the weight lattice. The weight $l_1 = -\frac{1}{2}(l + \bar{l})$ in the notation of Ref. [7], Section 5. The root lattice of E_{11} is given by [7]

$$\Lambda_{E_8} \oplus \Pi^{(1,1)} \oplus \{(n,n),\ n \in \mathbf{Z}\}. \tag{3.17}$$

Adding the l_1 representation corresponds to adding the vector $-\frac{1}{2}(1,-1)$ to the last factor in the above decomposition and so one is working on the full weight lattice.

4. GL(32) and central charges

In this section we review and expand the results of Refs. [1,8,9]. The eleven-dimensional supersymmetry algebra is of the form [10]

$$\{Q_\alpha, Q_\beta\} = Z_{\alpha\beta} = \left(\gamma^a C^{-1} P_a + \gamma^{a_1a_2} C^{-1} Z_{a_1a_2} + \gamma^{a_1\dots a_5} C^{-1} Z_{a_1\dots a_5}\right)_{\alpha\beta}, \tag{4.1}$$

$$[Q_\alpha, Z_{\alpha\beta}] = 0, \qquad [Z_{\alpha\beta}, Z_{\gamma\delta}] = 0. \tag{4.2}$$

This algebra admits GL(32) as an automorphism whose action is given by

$$\left[Q_\alpha, T_\gamma{}^\delta\right] = \delta_\alpha{}^\delta Q_\gamma, \qquad \left[Z_{\alpha\beta}, T_\gamma{}^\delta\right] = \delta_\alpha^\delta Z_{\gamma\beta} + \delta_\beta{}^\delta Z_{\alpha\gamma}. \tag{4.3}$$

This automorphism was found to play a role in the symmetries of the fivebrane dynamics [8,9] and later was shown to be a symmetry of M theory [1].

To gain a more familiar set of generators we may expand $T_\gamma{}^\delta$ out in terms of the elements of the enveloping Clifford algebra

$$T_\gamma{}^\delta = \sum_p \sum_{a_1\dots a_p} \frac{1}{32p!}(-1)^{\frac{p(p-1)}{2}} \left(\gamma^{a_1\dots a_p}\right)_\gamma{}^\delta T_{a_1\dots a_p} \quad \text{or} \quad \left(\gamma^{a_1\dots a_p}\right)_\delta{}^\gamma T_\gamma{}^\delta = T^{a_1\dots a_p}. \tag{4.4}$$

The summation only goes over $p = 0, 1, \dots, 4, 6$ since we may use the relation

$$\gamma^{a_1\dots a_p} = \frac{1}{q!}(-1)^{(q-1)p}(-1)^{\frac{q(q+1)}{2}} \epsilon^{a_1\dots a_p b_1\dots b_q} \gamma_{b_1\dots b_q}, \tag{4.5}$$

for $p + q = 11$. We then find that

$$\left[Q_\alpha, T^{a_1\dots a_p}\right] = \left(\gamma^{a_1\dots a_p}\right)_\alpha{}^\beta Q_\beta. \tag{4.6}$$

As such, the Q_α carry a representation of GL(32) with $T^{a_1\dots a_p}$ represented by $\gamma^{a_1\dots a_p}$. The Lorentz generators are represented in the usual way by $J_{ab} = \gamma_{ab}$.

Using the relation

$$\left[\gamma^{a_1\dots a_n}, \gamma_{b_1\dots b_m}\right] = \sum_{s=0}^{n} \frac{n!m!}{s!(n-s)!(m-s)!}(-1)^{s(s+1)/2}(-1)^{sn}\left(1-(-1)^{nm+s}\right)\delta^{[a_1\dots a_s}_{[b_1\dots b_s}\gamma^{a_{s+1}\dots a_n]}{}_{b_{s+1}\dots b_m]}, \tag{4.7}$$

valid for $m > n$, it straightforward to show, for example, that

$$\left[T^{a_1a_2a_3}, T_{b_1b_2b_3}\right] = -36\delta^{[a_1a_2}_{[b_1b_2} J^{a_3]}{}_{b_3]} + 2T^{a_1a_2a_3}{}_{b_1b_2b_3}, \tag{4.8}$$

$$\left[T^{a_1a_2a_3}, T_{b_1\dots b_6}\right] = +36\delta^{[a_1}_{[b_1} T^{a_2a_3]}{}_{b_2\dots b_6]} - 2\cdot 5!\delta^{a_1a_2a_3}_{[b_1b_2b_3} T_{b_4b_5b_6]}. \tag{4.9}$$

Examining the other commutators one finds that SL(32) is generated by multiple commutators of $T^{a_1a_2a_3}$.

For the effect of SL(32) on the central charges was given in Eq. (4.3) which in terms of the new basis becomes

$$\begin{aligned}\left[Z_{\alpha\beta}, T^{a_1\dots a_p}\right] = {} & \left(\gamma^{a_1\dots a_p}\left(\gamma^a P_a + \gamma^{a_1a_2} Z_{a_1a_2} + \gamma^{a_1\dots a_5} Z_{a_1\dots a_5}\right)\right)_\alpha{}^\delta \left(C^{-1}\right)_{\delta\beta} \\ & + (-1)^{p(p+1)/2}\left(\left(\gamma^a P_a + \gamma^{a_1a_2} Z_{a_1a_2} + \gamma^{a_1\dots a_5} Z_{a_1\dots a_5}\right)\gamma^{a_1\dots a_p}\right)_\alpha{}^\delta \left(C^{-1}\right)_{\delta\beta}.\end{aligned} \tag{4.10}$$

As a result we find that [8]

$$\left[P_b, T_{a_1a_2a_3}\right] = -6\eta_{b[a_1} Z_{a_2a_3]}, \qquad \left[Z^{b_1b_2}, T_{a_1a_2a_3}\right] = -5! Z_{a_1a_2a_3}{}^{b_1b_2} + 6\delta^{b_1b_2}_{[a_1a_2} P_{a_3]}, \tag{4.11}$$

$$\left[Z^{b_1\dots b_5}, T_{a_1a_2a_3}\right] = -\frac{1}{4}\epsilon^{b_1\dots b_5}{}_{c_1\dots c_5[a_1a_2} Z_{a_3]}{}^{c_1\dots c_5} + 2\delta^{[b_1b_2b_3}_{a_1a_2a_3} Z^{b_4b_5]}. \tag{4.12}$$

GL(32) first arose in the dynamics of the M theory fivebrane which was found [8] to possess an unexpected symmetry whose generator had three anti-symmetrised eleven-dimensional space–time indices. This generator was shown to obey commutation relations that identified it with a generator in contracted version of SL(32) [8]. Furthermore, when the dynamics of the fivebrane was described by a non-linear realisation the third rank world volume gauge field strength was found to be the Goldstone boson for a rank-three generator in SL(32) which acted on the supercharges as in Eq. (4.3).

It was explained in Ref. [8], that SL(32) is the natural generalisation of $\mathrm{spin}(1,10)$, required for the point particles, to the situation when branes are present in M theory and it acts as a brane rotating symmetry. $\mathrm{Spin}(1,10)$ can be defined is just the group which acts by a transformation on Q_α that takes $(\gamma^a)_\alpha{}^\beta P_a$ into on object of the same form. However, when branes are present this last expression is replaced by the arbitrary matrix $Z_{\alpha\beta}$ and so the natural symmetry to consider is SL(32).

Finally, in Ref. [1] it was argued that GL(32) was a symmetry of M theory. We recall the outline of the argument. Eleven-dimensional supergravity is invariant under the supersymmetry algebra and it is also invariant under the eleven-dimensional conformal group. The latter follows from the formulation of eleven-dimensional supergravity as a simultaneous non-linear realisation of the conformal algebra and G_{11} [1]. However, it has been known [10] for many years that the only algebra that contains the conformal algebra and the supersymmetry algebra is Osp(1/64) and so the result follows.

This invariance is consistent with the much earlier result of Ref. [11] which showed that eleven-dimensional supergravity was invariant under SO(16) provided one took only $\mathrm{SO}(1,2) \times \mathrm{SO}(8)$ as the Lorentz group. This SO(16) is contained in SL(32) in a straightforward way; we write the 32-component spinor in terms of its $\mathrm{spin}(1,2) \times \mathrm{spin}(8)$ decomposition and then the spin(16) acts on the indices associated with the latter.

Very recently SL(32) has been considered in the context of the holonomy of M theory [12–14].

5. E_{11} and the identification of the central charges

It was suggested [8] that the Lorentz algebra should be replaced by SL(32) in the context of M theory. As such, one might anticipate that this algebra should be contained in E_{11} and, in particular, one might expect that SL(32) should be contained in the corresponding local sub-algebra which is the Cartan involution invariant sub-algebra of E_{11}. Such an identification can also be seen from another viewpoint. The world volume gauge field strength that occurs in the fivebrane equation of motion is the Goldstone boson for part of the SL(32) automorphism symmetry [9] and it is also the case that the third rank gauge field of eleven-dimensional supergravity is the Goldstone boson associated with the generator $R^{a_1a_2a_3}$ of E_{11} [1,2]. However, these two fields appear in the world volume dynamics in a linear combination due to their gauge symmetry and as such the generators of the two algebras should be related to each other.

The generators of the Cartan involution invariant sub-algebra of E_{11} are given, at low levels, in Eqs. (2.17)–(2.19) and their commutation relations in Eqs. (2.20) and (2.21). Comparing these with the analogous relations for the SL(32) automorphism algebra of the eleven-dimensional supersymmetry algebra given in Eqs. (4.8) and (4.9) we find the same commutation relations provided we identify

$$T_{a_1a_2a_3} = S_{a_1a_2a_3}, \qquad T_{b_1\dots b_6} = S_{b_1\dots b_6} \quad \text{and} \quad S_{b_1\dots b_6}{}^{[a_1a_2,a_3]} = \frac{1}{2}\delta^{[a_3}_{[b_1}\epsilon^{a_1a_2]}{}_{b_1\dots b_6]c_1\dots c_4}T^{c_1\dots c_4}. \tag{5.1}$$

The reason for the contraction appearing in Ref. [8] is because the symmetry of the fivebrane dynamics considered there involves the generator $R^{b_1b_2b_3}$ rather than $S_{a_1a_2a_3}$. It follows from the above arguments that, even though we have not introduced the supersymmetry supercharges themselves, the SL(32) generators identified in E_{11} in this Letter are the ones that acts on the supercharges as in Eq. (4.3).

At first sight the rank of SL(32) is too large for it to be contained in E_{11}. However, E_{11} does contain very large commuting sub-algebras, certainly much larger than eleven, and to identify Sl(32) we have set to zero some generators in E_{11} and this step could also increase the number of generators allowed in a commuting sub-algebra. The only remaining SL(32) generator that is left to identify is T_a which occurs at the next level and it would be good to extent the calculation given in this paper to the next level and so identify this generator.

Given the above identification of SL(32) one can examine if its action on the central charges given in Eqs. (4.11) and (4.12) plays a role in E_{11}. Indeed, comparing these equations with Eqs. (3.15) and (3.16) we find the same commutators provided we identify the central charges of the eleven-dimensional supersymmetry algebra with the two- and five-rank anti-symmetric tensors that appear in the E_{11} representation with highest weight l_1, in particular we take

$$W^{a_1a_2} = 2Z^{a_1a_2}, \qquad W^{a_1\dots a_5} = 2\cdot 5!Z^{a_1\dots a_5}, \qquad W^{a_1\dots a_7,c} = 60\epsilon^{a_1\dots a_7}{}_{d_1\dots d_4}Z^{cd_1\dots d_4}. \tag{5.2}$$

Since the generator P_a is in common to the supersymmetry algebra and the semi-direct product of E_{11} with its l_1 representation, we are obliged to identify the two- and five-rank anti-symmetric tensors that appear in $E_{11} \oplus_s L_1$.

The conjectured E_{11} symmetry of M theory was uncovered in Ref. [2] by examining the bosonic sector of eleven-dimensional supergravity and it is encouraging that aspects of M theory found outside this sector, i.e., in the fermionic sector and in the fivebrane dynamics lead to symmetries that are contained in semi-direct product of E_{11} with its l_1 representation. This bodes well for the incorporation of supersymmetry and the rest of the conformal group into an extension of $E_{11} \oplus_s L_1$.

The realisation that E_{11} places the central charges in the same symmetry multiplet as the translations generators indicates that the theory which is invariant under E_{11} should involve the usual space–time and coordinates associated with the central charges. In particular, we should construct the non-linear realisation of $E_{11} \oplus_s L_1$ and so consider a group element of the form

$$\begin{aligned} g &= \exp\bigl(x^a P_a + x_{a_1a_2}W^{a_1a_2} + x_{a_1\dots a_5}W^{a_1\dots a_5} + \cdots\bigr)\exp\bigl(h_a{}^b K^a{}_b\bigr) \\ &\quad \times \exp\left(\frac{A_{c_1\dots c_3}R^{c_1\dots c_3}}{3!} + \frac{A_{c_1\dots c_6}R^{c_1\dots c_6}}{6!}\right)\cdots, \end{aligned} \tag{5.3}$$

where '$+\cdots$' stand for higher-level generators in l_1 and the final '$\cdots$' for terms containing higher level generators of E_{11}. The fields should then depend on $x^a, x_{a_1a_2}, x_{a_1\dots a_5}, \dots$. This formulation should be able to describe point particles and branes on an equal footing as it encodes the brane rotating symmetries. Traditionally, non-linear realisation have been used to describe the low energy dynamics of a theory in which a symmetry is spontaneously broken, however, at high energies the same symmetry is expected to be linearly realised. It is to be expected that the same might occur here and that the fundamental theory may have its symmetries realised on different variables. This presumably should include the coordinates.

6. Solutions and Weyl transformations

In a recent paper [15], the Cartan sub-algebra of E_{11}, that is group elements of the form $g = e^{q^m(x)H_m}$, where H_m are the Cartan generators of E_{11}, was considered and the action of the E_{11} Weyl transformations induced on the space–time fields q^m was found. After relating these to the diagonal components of the metric, it was shown that the moduli space of an enlarged set of Kasner solutions carry a representation of these Weyl transformations [15]. In this section we generalise this work and consider group elements that include the space–time translation generators and higher level generators in the l_1 representation. We will find the action of the Weyl transformations also on the space–time and other coordinates at low levels. As a result, we will be able to find the action of Weyl transformations on the solutions themselves and not just their moduli. Our discussion will be rather brief, but we will set out the general procedure and illustrate it in the context of the generalised Kasner solutions.

Restricting the E_{11} part of the group element to be in the Cartan sub-algebra, the group element of Eq. (5.3) takes the form

$$g = \exp\bigl(x^\mu P_\mu + x_{a_1a_2}W^{a_1a_2} + x_{a_1\ldots a_5}W^{a_1\ldots a_5} + \cdots\bigr)\exp\bigl(q^m H_m\bigr)$$
$$= \exp\bigl(x^\mu P_\mu + x_{a_1a_2}W^{a_1a_2} + x_{a_1\ldots a_5}W^{a_1\ldots a_5} + \cdots\bigr)\exp\bigl(p_a K^a{}_a\bigr). \tag{6.1}$$

For E_{11} the Cartan sub-algebra is expressed in terms of the generators $K^a{}_a$ which belong to GL(11) [2]. A detailed explanation of the change from the q^m to the p_a variables is given in Section 2 of Ref. [15]. The theory resulting from the non-linear realisation involving such a restriction only possess a diagonal metric whose components are given by $g_{\mu\mu} = e^{2p_a}\eta_{a\mu}$.

Although the Cartan sub-algebra of a Kac–Moody algebra g carries a representation of the Weyl group this is not the case for the representations of g. However, an extension of the Weyl group by a cyclic group does have an action on representations of g. The effect of this extension is to introduce various signs into the action of the Weyl group. In this first account we will omit the, possibly significant, effects of the centre and give the effect of Weyl transformations are only up to signs. The Weyl reflections, S_{α_a} corresponding to the simple roots α_a on the weights w of a representation are given by $S_{\alpha_a} w = w - (\alpha_a, w)\alpha_a$. Carrying this out on the root string of Eq. (3.2) and using the correspondence between the weights and the generators, we find that S_a, $a = 1, \ldots, 10$, does not introduce any α_{11} and so they do not change the level of the generator on which they act. They act on the space–time translation generators as

$$S_a(P_a) = P_{a+1}, \qquad S_a(P_{a+1}) = P_a, \qquad S_a(P_b) = P_b, \quad b \neq a, a+1. \tag{6.2}$$

The Weyl transformation S_{11} can change the level as it can introduce a α_{11} in its reflection. Its action on the translations is given by

$$S_{11}(P_a) = P_a, \quad a \neq 9, 10, 11, \qquad S_{11}(P_9) = W^{1011}, \qquad S_{11}(P_{10}) = W^{911}, \qquad S_{11}(P_{11}) = W^{910}, \tag{6.3}$$

while on the second rank central charge we find that

$$S_{11}\bigl(W^{1011}\bigr) = P_9, \qquad S_{11}\bigl(W^{911}\bigr) = P_{10}, \qquad S_{11}\bigl(W^{910}\bigr) = P_{11}, \qquad S_{11}\bigl(W^{ab}\bigr) = W^{ab}, \quad a \leqslant 8, \quad b > 8. \tag{6.4}$$

S_{11} takes the remaining components of the two-form central charge into the five-form central charge. Examining the group element of Eq. (6.1) we find that the Weyl transformations S_a, $a = 1, \ldots, 10$, act on the space–time coordinates and variables p_a as

$$x^a \leftrightarrow x^{a+1}, \quad p_a \leftrightarrow p_{a+1}. \tag{6.5}$$

It is important to remember that the p_a depend on the space–time and other coordinates and the change in these must also be carried out when evaluating the transformed variable. The Weyl transformation S_{11} mixes the space–time coordinates with the central charge coordinates, for example $x^c \to x^c$, $c = 1, \ldots, 8$, $x^9 \leftrightarrow x_{1011}$, $x^{10} \leftrightarrow x_{911}$ and $x^{11} \leftrightarrow x_{910}$.

The generalised Kasner solutions can be labeled by the space–time variable that the metric depends on. The x^b-Kasner solution has a metric of the form

$$g_{\mu\mu} = \eta_{\mu a} e^{2\tilde{p}_a x^b}, \tag{6.6}$$

where $\tilde{p}_a$ are constants which must satisfy

$$\sum_{c, c\neq b} \tilde{p}_c = \tilde{p}_b, \qquad \sum_{c, c\neq b} (\tilde{p}_c)^2 = (\tilde{p}_b)^2, \tag{6.7}$$

in order to obey Einstein's equations. The usual Kasner solution has $b = 0$, i.e., depends on a time variable.

Using Eq. (6.5) we can carry out the Weyl transformations for S_a, $a = 1, \dots, 10$, on the generalised Kasner solutions and we find that the solutions are indeed exchanged under the action of these Weyl transformations. It takes the x^b-Kasner solution into the x^b-Kasner solution if $b \neq a, a+1$, and it swops the x^a-Kasner solution with the x^{a+1}-Kasner solution. We observe that this includes the swopping of the temporal and spatial generalised Kasner solutions if $a = 1$.

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References

[1] P.C. West, JHEP 08 (2000) 007, hep-th/0005270.
[2] P. West, Class. Quantum Grav. 18 (2001) 4443, hep-th/0104081.
[3] M. Gaberdiel, P. West, JHEP 0208 (2002) 049, hep-th/0207032.
[4] P. West, Class. Quantum Grav. 20 (2003) 2393, hep-th/0212291.
[5] I. Schnakenburg, P. West, Phys. Lett. B 517 (2001).
[6] A. Borisov, V. Ogievetsky, Teor. Mat. Fiz. 21 (1974) 329.
[7] D. Olive, M. Gaberdiel, P. West, Nucl. Phys. B 645 (2002) 403, hep-th/0205068.
[8] O. Barwald, P. West, Phys. Lett. B 476 (2000) 157, hep-th/9912226.
[9] P. West, JHEP 0002 (2000) 024, hep-th/0001216.
[10] J. van Holten, A. van Proeyen, J. Phys. A 15 (1982) 376.
[11] B. de Wit, H. Nicolai, Nucl. Phys. B 274 (1986) 363;
B. de Wit, H. Nicolai, Phys. Lett. B 155 (1985) 47;
H. Nicolai, Phys. Lett. B 187 (1987) 316.
[12] M. Duff, J. Liu, hep-th/0303140.
[13] C. Hull, hep-th/0305039.
[14] G. Papadopoulos, D. Timpsis, hep-th/0306117.
[15] F. Englert, L. Houart, A. Taormina, P. West, hep-th/0304206.
[16] H. Nicolai, T. Fischbacher, hep-th/0301017.