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Linear transformations between matrix spaces that map one rank specific set into another ☆

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Abstract

We characterize, in several instances, those linear transformations from the linear space of $m \times n$ matrices into the linear space of $p \times q$ matrices that map the set of matrices having a fixed rank into the set of matrices having a fixed rank. Examples are given showing that, in contrast with the case of linear transformations on the linear space of $m \times n$ matrices mapping a rank specific set into itself, in the more general case of linear transformations between two full matrix spaces, often one cannot expect neat and predictable results.

Keywords: Linear preserver; Rank

1. Introduction

Let $\mathbb{F}$ be a field. Let $M_{p\times q}(\mathbb{F})$ be the linear space of $p \times q$ matrices with entries in $\mathbb{F}$. We study linear transformations

$$\phi : M_{m\times n}(\mathbb{F}) \to M_{p\times q}(\mathbb{F}) \tag{1.1}$$

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(with p, q, m, n fixed) that are specific to certain matrix properties related to the rank. As a general formulation that encompasses many problems, we state the following:

Problem 1.1. Fix positive integers k and s. Describe all linear transformations (1.1) that satisfy one of the following properties (a)–(e):
(a) $A \in M_{m\times n}(\mathbb{F})$, $\operatorname{rank} A = k \Rightarrow \operatorname{rank}\phi(A) = s$.
(b) $A \in M_{m\times n}(\mathbb{F})$, $\operatorname{rank} A = k \Leftrightarrow \operatorname{rank}\phi(A) = s$.
(c) $A \in M_{m\times n}(\mathbb{F})$, $\operatorname{rank} A \leqslant k \Rightarrow \operatorname{rank}\phi(A) \leqslant s$.
(d) $A \in M_{m\times n}(\mathbb{F})$, $\operatorname{rank} A \leqslant k \Leftrightarrow \operatorname{rank}\phi(A) \leqslant s$.
(e) $A \in M_{m\times n}(\mathbb{F})$, $\operatorname{rank} A = k \Rightarrow \operatorname{rank}\phi(A) \leqslant s$.

If $(p, q) = (m, n)$ many results solving many problems in the spirit of Problem 1.1 are known, most often assuming that $k = s$, see [10, Chapter 2].

In this paper, we consider the cases when $(p, q) \neq (m, n)$, and study those linear transformations ϕ that satisfy one of the properties (a)–(e) of Problem 1.1. In contrast with the case $(p, q) = (m, n)$, here one need not assume $k = s$ to obtain meaningful results. Examples show that in full generality Problem 1.1 is probably intractable, and we confine ourselves here to a few particular instances when we were able to obtain a complete description of such maps ϕ.

There is an extensive literature concerning linear transformations on a full matrix algebra that preserve certain matrix properties, such as determinants, ranks, norms, numerical ranges, etc. Only recently there appeared works concerning structure of linear preservers between different full matrix spaces. We mention here [3,4] (on preservers of unitary matrices, norms, numerical ranges, and other related properties), and [6] (on invertibility preserving maps).

We denote by A^{t} the transpose of A.

2. Linear maps on rank-one matrices

Structure of invertible rank-one nonincreasing linear maps on $M_{m\times n}(\mathbb{F})$ was described in [8]. All such maps have the form

$$A \mapsto PAQ \quad \text{or} \quad A \mapsto PA^{\mathrm{t}}Q,$$

where P and Q are matrices of appropriate sizes. In the spirit of this result, in the next theorem we consider rank-one preserving linear maps between different (generally speaking) full matrix spaces. We do not assume nondegeneracy.

Theorem 2.1. *Let $\phi : M_{m\times n}(\mathbb{F}) \to M_{p\times q}(\mathbb{F})$ be a linear transformation such that*

$$A \in M_{m\times n}(\mathbb{F}), \quad \operatorname{rank} A = 1 \;\Rightarrow\; \operatorname{rank}\phi(A) = 1. \tag{2.1}$$

Then there exist invertible matrices $P \in M_{p\times p}(\mathbb{F})$ and $Q \in M_{q\times q}(\mathbb{F})$ such that one of the following four alternatives holds:

(1)

$$m \leqslant p, \quad n \leqslant q, \quad \phi(A) = P \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} Q.$$

(2)

$$m \leqslant q, \quad n \leqslant p, \quad \phi(A) = P \begin{bmatrix} A^{\mathrm{t}} & 0 \\ 0 & 0 \end{bmatrix} Q.$$

(3)

$$\phi(A) = P \begin{bmatrix} \psi(A) & 0 \end{bmatrix} Q,$$

where $\psi : M_{m\times n}(\mathbb{F}) \to M_{p\times 1}(\mathbb{F})$ *is a linear transformation such that* $\psi(A) \neq 0$ *for every* $A \in M_{m\times n}(\mathbb{F})$ *having rank* 1.

(4)

$$\phi(A) = P \begin{bmatrix} \psi(A) \\ 0 \end{bmatrix} Q,$$

where $\psi : M_{m\times n}(\mathbb{F}) \to M_{1\times q}(\mathbb{F})$ *is a linear transformation such that* $\psi(A) \neq 0$ *for every* $A \in M_{m\times n}(\mathbb{F})$ *having rank* 1.

We need a lemma to prove the theorem.

Lemma 2.2. *If* $U, V \in M_{p\times q}(\mathbb{F})$ *are such that for every vector* $w \in M_{q\times 1}(\mathbb{F})$ *the vectors* Uw *and* Vw *are linearly dependent, then either the ranges of both* U *and* V *are contained in the same one-dimensional subspace of* $M_{p\times 1}(\mathbb{F})$, *or* U *and* V *are linearly dependent.*

Proof. We consider separately the case when U and V are both of rank 1. Thus, let $U = x_1y_1^{\mathrm{t}}$, $V = x_2y_2^{\mathrm{t}}$. Let w be such that $y_1^{\mathrm{t}}w \neq 0$, $y_2^{\mathrm{t}}w \neq 0$. Then $Uw = (y_1^{\mathrm{t}}w)x_1$ and $Vw = (y_2^{\mathrm{t}}w)x_2$. By the hypotheses of the lemma, x_1 and x_2 are scalar multiples of each other.

Now assume that U or V, say V, has rank at least 2. Let $v_1, \ldots, v_q$ be the columns of V. Multiplying U and V on the right by the same invertible matrix, we may assume that each pair of columns in the following list:

$$(v_1, v_2), (v_1, v_3), \ldots, (v_1, v_q)$$

is linearly independent. If $u_1, \ldots, u_q$ are the columns of U, then we clearly have $u_j = \alpha_j v_j$ for some $\alpha_j \in \mathbb{F}$. But for a fixed $j \in \{2, 3, \ldots, q\}$, also $u_1 + zu_j = \alpha(z)(v_1 + zv_j)$, where $z \in \mathbb{F}$ is arbitrary and $\alpha(z) \in \mathbb{F}$ (here we use the condition that the columns v_1 and v_j are linearly independent). It follows that

$$(\alpha_1 - \alpha(z))v_1 + (\alpha_j z - \alpha(z)z)v_j = 0,$$

and hence $\alpha_1 = \alpha(z)$, $\alpha_j = \alpha(z)$ for $z \neq 0$. Thus, all α_j's are equal, and U is a scalar multiple of V. $\square$

Proof of Theorem 2.1. We write $M_{r\times s}$ instead of $M_{r\times s}(\mathbb{F})$. Let k,ℓ be positive integers, and x, y nonzero vectors in $M_{k\times 1}$, $M_{\ell\times 1}$, respectively. Then xy^{t} is a rank-1 matrix and every matrix of rank 1 in $M_{k\times\ell}$ can be written in this form. Consider the sets $L_x=\{xy^{\mathrm{t}} : y\in M_{\ell\times 1}\}$ and $R_y=\{xy^{\mathrm{t}} : x\in M_{k\times 1}\}$. Each of L_x and R_y is a linear subspace of $M_{k\times\ell}$ consisting of matrices having rank 1 or 0; if V is a linear subspace of $M_{k\times\ell}$ whose nonzero members have all rank 1 then V is contained either in some L_x or in some R_y.

Suppose $\phi : M_{m\times n}\to M_{p\times q}$ preserves rank-1 matrices. Then for every nonzero $x\in M_{m\times 1}$ we have either $\phi(L_x)\subseteq L_z$ for some $z\in M_{p\times 1}$ or $\phi(L_x)\subseteq R_y$ for some $y\in M_{q\times 1}$. Of course, an analogue holds true for $\phi(R_y)$ for every nonzero vector y.

If $m=1$ or $n=1$, the above argument shows that ϕ has the form (1) or (2). Assume that $m,n\geqslant 2$. We will prove that we cannot have $\phi(L_x)\subseteq L_u$ and $\phi(L_z)\subseteq R_y$ simultaneously for some nonzero x and z in $M_{m\times 1}$. Assume on the contrary that such vectors x and z exist. Then, clearly, x and z are linearly independent. Because of the injectivity of the restriction of ϕ to L_x we can find linearly independent vectors $a,b\in M_{n\times 1}$ such that $\phi(xa^{t})=uw^{\mathrm{t}}$, $\phi(xb^{\mathrm{t}})=uv^{\mathrm{t}}$, w and y are linearly independent, v and y are linearly independent, and v and w are linearly independent. Now, $\phi(za^{\mathrm{t}})=cy^{\mathrm{t}}$ for some $c\in M_{p\times 1}$, and since $za^{\mathrm{t}}+xa^{\mathrm{t}}$ has rank 1, we have rank $(cy^{\mathrm{t}}+uw^{\mathrm{t}})=1$ which further implies that c and u are linearly dependent. Thus, $\phi(za^{\mathrm{t}})\in\operatorname{span}\{uy^{\mathrm{t}}\}$. Similarly, $\phi(zb^{\mathrm{t}})\in\operatorname{span}\{uy^{\mathrm{t}}\}$, contradicting the fact that the restriction of ϕ to L_z is injective.

So, either for every nonzero $x\in M_{m\times 1}$ there is a vector $y\in M_{p\times 1}$ such that $\phi(L_x)\subseteq L_y$ or for every nonzero $x\in M_{m\times 1}$ there is a vector $y\in M_{q\times 1}$ such that $\phi(L_x)\subseteq R_y$. We will consider only the first possibility since the second one can be reduced to the first one by composing ϕ with the transposition.

If there exists $y\in M_{p\times 1}$ such that $\phi(L_x)\subseteq L_y$ for every nonzero $x\in M_{m\times 1}$, then ϕ has one of the forms described in our statement. So, it remains to consider the case that there are x_0 and z_0 in $M_{m\times 1}$ such that $\phi(L_{x_0})\subseteq L_y$ and $\phi(L_{z_0})\subseteq L_u$ for some linearly independent vectors y and u. In particular, if we choose and fix a nonzero $w\in M_{n\times 1}$, then $\phi(x_0w^{\mathrm{t}})=ya^{\mathrm{t}}$ and $\phi(z_0w^{\mathrm{t}})=ub^{\mathrm{t}}$ for some nonzero vectors a and b. Applying the fact that $x_0w^{\mathrm{t}}+z_0w^{\mathrm{t}}$ has rank 1, we see that a and b are linearly dependent. It follows that $\phi(R_w)\subseteq R_a$. The restriction of ϕ to R_w is injective; consequently, if $x,z\in M_{m\times 1}$ are linearly independent and if $\phi(L_x)\subseteq L_s$ and $\phi(L_z)\subseteq L_t$ for some vectors s and t, then s and t are linearly independent. To verify this conclusion, observe that $\phi(xw^{\mathrm{t}})=\alpha sa^{\mathrm{t}}$ and $\phi(zw^{\mathrm{t}})=\beta ta^{\mathrm{t}}$ for some $\alpha,\beta\in\mathbb{F}$.

So, for every $x\in M_{m\times 1}$ there is $y\in M_{p\times 1}$ such that $\phi(xw^{\mathrm{t}})=yv^{\mathrm{t}}$ for every $w\in M_{n\times 1}$. The map $w\mapsto v$ is linear. Therefore,

$$\phi(xw^{\mathrm{t}})=y(C_xw)^{\mathrm{t}} \tag{2.2}$$

for some linear transformation $C_x : M_{n\times 1}\to M_{q\times 1}$. The linear transformation C_x is clearly injective (otherwise $\phi(A)=0$ for some matrix A of rank 1, a contradiction), and therefore it is not of rank 1.

Assume that x and z are linearly independent. Then $\phi(xw^{\mathrm{t}}) = y(C_x w)^{\mathrm{t}}$ and $\phi(zw^{\mathrm{t}}) = u(C_z w)^{\mathrm{t}}$, $w \in M_{n\times 1}$, and the fact that y and u are linearly independent imply that $C_x w$ and $C_z w$ are linearly dependent for every w. As C_x and C_z are not of rank 1, by Lemma 2.2, C_x and C_z are linearly dependent. If x and z are linearly dependent, then we can find w such that x and w, as well as z and w are linearly independent. We already know that then C_x and C_w, as well as C_z and C_w are linearly dependent. Thus, for every pair of nonzero vectors x and z the linear transformations C_x and C_z are linearly dependent. By absorbing the constant in the first term of the product on the right-hand side in (2.2) we may assume that $C_x = C$ is independent of x. Whence, for every nonzero $x \in M_{m\times 1}$ there exists y such that $\phi(xw^{\mathrm{t}}) = y(Cw)^{\mathrm{t}}$, $w \in M_{n\times 1}$. The map $x \mapsto y$ is linear. Denoting it by D we have $\phi(xw^{\mathrm{t}}) = Dx(Cw)^{\mathrm{t}}$, $w \in M_{n\times 1}$. We already know that both D and C are injective. It follows that ϕ has the form (1). □

There are certain restrictions on the sizes of matrices involved, under which the situations described in (3) and (4) may occur:

Proposition 2.3. *If $m + n - 1 \leqslant q$, then there exists a linear transformation*

$$\phi : M_{m\times n}(\mathbb{F}) \to M_{1\times q}(\mathbb{F})$$

such that

$$A \in M_{m\times n}(\mathbb{F}), \quad \operatorname{rank} A = 1 \Rightarrow \phi(A) \neq 0. \tag{2.3}$$

Conversely, if $\mathbb{F}$ is an algebraically closed field, and there is a linear transformation

$$\phi : M_{m\times n}(\mathbb{F}) \to M_{1\times q}(\mathbb{F})$$

satisfying (2.3), *then $m + n - 1 \leqslant q$.*

Proof. Assume $m + n - 1 \leqslant q$. Define $\phi : M_{m\times n}(\mathbb{F}) \to M_{1\times q}(\mathbb{F})$ by

$$\begin{aligned}\phi\big([a_{j,k}]_{j=1,k=1}^{m,n}\big) = [&a_{m,1}, a_{m-1,1} + a_{m,2}, a_{m-2,1} + a_{m-1,2} + a_{m,3},\\ &\ldots, a_{1,n-2} + a_{2,n-1} + a_{3,n}, a_{1,n-1} + a_{2,n}, a_{1,n}, 0, \ldots, 0].\end{aligned}$$

Then $\phi(A) \neq 0$ if $\operatorname{rank} A = 1$, and therefore ϕ has property (2.3).

To prove the converse, let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbb{F}^m$. If ϕ satisfies (2.3), then for every $j = 1, \ldots, m$, there exists $M_j \in M_{n\times q}(\mathbb{F})$ such that $\phi(e_j x^{\mathrm{t}}) = x^{\mathrm{t}} M_j$. Moreover, for any nonzero $a = (a_1, \ldots, a_m)^{\mathrm{t}} \in \mathbb{F}^m$, $\phi(ax^{\mathrm{t}}) = \sum_{j=1}^m x^{\mathrm{t}}(a_j M_j) \neq 0$ for any nonzero x^{t}. Thus, $\sum_{j=1}^m a_j M_j$ has rank n for any nonzero $a = (a_1, \ldots, a_m) \in \mathbb{F}^m$. So, $\{M_1, \ldots, M_m\}$ is a basis for a subspace in $M_{n\times q}(\mathbb{F})$ whose nonzero elements have rank n. By a result of Meshulam [1, Appendix], we see that $m \leqslant n + q - 2n + 1$, which is our desired inequality after rearrangement. □

Without the additional hypothesis on $\mathbb{F}$, the converse statement of Proposition 2.3 is false, as the following example shows. Let $\phi : M_{2\times 2}(\mathbb{R}) \to M_{1\times 2}(\mathbb{R})$ be a linear transformation such that

$$\operatorname{Ker}\phi = \operatorname{span}\left\{I, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}.$$

Since $\operatorname{Ker}\phi$ does not contain any rank-1 matrix, $\phi(A) \neq 0$ for every rank-1 matrix A.

Corollary 2.4. *Let $\phi : M_{m\times n}(\mathbb{F}) \to M_{p\times q}(\mathbb{F})$ be a linear transformation such that*

$$A \in M_{p\times q}(\mathbb{F}), \quad \operatorname{rank} A = 1 \Leftrightarrow \operatorname{rank}\phi(A) = 1. \tag{2.4}$$

Then there exist invertible matrices $P \in M_{p\times p}(\mathbb{F})$ and $Q \in M_{q\times q}(\mathbb{F})$ such that condition (1) *or* (2) *of Theorem* 2.1 *holds.*

Proof. We need only to show that the situations (3) and (4) of Theorem 2.1 cannot occur under the more restrictive hypothesis (2.4). We may assume $m, n \geqslant 2$. Arguing by contradiction, assume there exists a linear map

$$\psi \ : \ M_{m\times n}(\mathbb{F}) \to M_{p\times 1}(\mathbb{F})$$

such that $\psi(A) \neq 0$ for every $A \in M_{m\times n}(\mathbb{F})$ of rank 1, and $\psi(A) = 0$ for every $A \in M_{m\times n}(\mathbb{F})$ of rank at least 2. Select linearly independent $x, y \in M_{m\times 1}(\mathbb{F})$, and $a, b, c \in M_{1\times n}(\mathbb{F})$ such that a, b are linearly independent, a, c are linearly independent, and $b \neq c$. Then

$$\phi(y(b-c)^{\mathrm{t}}) = \phi(xa^{\mathrm{t}} + yb^{\mathrm{t}}) - \phi(xa^{\mathrm{t}} + yc^{\mathrm{t}}) = 0 - 0 = 0,$$

a contradiction, because $y(b-c)^{\mathrm{t}}$ has rank 1. $\square$

Theorem 2.5. *Let $\mathbb{F}$ be an algebraically closed field of characteristic* 0, *and k be a positive integer. The following conditions are equivalent for a linear transformation $\phi : M_{n\times n}(\mathbb{F}) \to M_{p\times q}(\mathbb{F})$ whose range contains a matrix of rank kn.*

(a) $\operatorname{rank}\phi(A) = k$ *whenever* $\operatorname{rank} A = 1$.

(b) $\operatorname{rank}\phi(A) \leqslant k$ *whenever* $\operatorname{rank} A = 1$.

(c) *ϕ has the form*

$$A \mapsto P \begin{bmatrix} I_r \otimes A & 0 & 0 \\ 0 & I_s \otimes A^{\mathrm{t}} & 0 \\ 0 & 0 & Z \end{bmatrix} Q, \tag{2.5}$$

where Z stands for the $(p - rn - sn) \times (q - rn - sn)$ zero matrix, for some nonnegative integers r and s and some invertible matrices $P \in M_{p\times p}(\mathbb{F})$ and $Q \in M_{q\times q}(\mathbb{F})$.

Proof. The implications (c) $\Rightarrow$ (a) $\Rightarrow$ (b) are clear. We consider (b) $\Rightarrow$ (c). If A has rank r, then it can be written as a sum of r rank-1 linear transformations, and since rank is subadditive, we have $\operatorname{rank}\phi(A) \leqslant rk$. Let B be a matrix with the property that $\operatorname{rank}\phi(B) = nk$. Then B has rank n, and so, we may assume without loss of generality that

$$\phi(I_n) = \begin{bmatrix} I_{kn} & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $P \in M_{n\times n}(\mathbb{F})$ be an idempotent, say of rank r. Then $\phi(I_n) = \phi(P) + \phi(I_n - P)$ and $kn = \operatorname{rank}\phi(I_n) \leqslant \operatorname{rank}\phi(P) + \operatorname{rank}\phi(I_n - P) \leqslant kr + k(n-r) = kn$. So, the inequalities are actually equalities.

Identifying matrices with operators we have the following obvious relation involving range spaces:

$$\mathscr{R}(\phi(I_n)) \subseteq \mathscr{R}(\phi(P)) + \mathscr{R}(\phi(I_n - P)).$$

From

$$\dim \mathscr{R}(\phi(I_n)) = \dim \mathscr{R}(\phi(P)) + \dim \mathscr{R}(\phi(I_n - P))$$

we get

$$\mathscr{R}(\phi(I_n)) = \mathscr{R}(\phi(P)) \dotplus \mathscr{R}(\phi(I_n - P)), \tag{2.6}$$

a direct sum. In particular, $\mathscr{R}(\phi(P)) \subseteq \mathscr{R}(\phi(I_n))$. The same is true for the transposes, so $\phi(P)$ is a matrix having nonzero entries only in the upper left $kn \times kn$ corner. Every $A \in M_{n\times n}(\mathbb{F})$ is a linear combination of idempotents, and so, it is mapped into the upper left $kn \times kn$ corner. Therefore, there is no loss of generality in assuming that $p = q = kn$. For $x \in \mathscr{R}(\phi(P))$ we have $x = \phi(I_n)x = \phi(P)x + \phi(I_n - P)x$, which by (2.6) yields $\phi(P)x = x$ and $\phi(I_n - P)x = 0$. Similarly, $\phi(P)x = 0$ for every $x \in \mathscr{R}(\phi(I_n - P))$. Therefore, $\phi(P)$ is an idempotent. We have thus proved that $\phi : M_{n\times n}(\mathbb{F}) \to M_{nk\times nk}(\mathbb{F})$ maps idempotents into idempotents. By [2, Theorem 2.1], ϕ is a sum of a homomorphism and an antihomomorphism. Now one can complete the proof using the same approach as in [7, p. 77]. $\square$

The assumption that the range of ϕ contains a matrix of rank kn in Theorem 2.5 is essential as shown in the following.

Example 2.6. Let $\eta,\ \mu : M_{n\times n}(\mathbb{F}) \to M_{(n+1)\times(n+1)}(\mathbb{F})$ be linear maps so that for any $A \in M_{n\times n}(\mathbb{F})$, $\eta(A) = A \oplus [0]$ and $\mu(A) = [0] \oplus A$. Then $\phi = \eta + \mu : M_{n\times n}(\mathbb{F}) \to M_{(n+1)\times(n+1)}(\mathbb{F})$ maps every rank-1 matrix into a rank-2 matrix but is not of the form (2.5).

The next example shows that one cannot simply replace in Theorem 2.5 the domain of ϕ by $M_{m\times n}(\mathbb{F})$ and kn by $k \min\{m, n\}$.

Example 2.7. Let $\phi : M_{2\times 3}(\mathbb{F}) \to M_{4\times 4}(\mathbb{F})$ be defined by

$$\phi\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} a & b & c & 0 \\ d & e & f & 0 \\ 0 & a & b & c \\ 0 & d & e & f \end{bmatrix}.$$

Clearly, ϕ maps every rank-1 matrix to a rank-2 matrix, and

$$\phi\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right)$$

has rank 4. However, ϕ is not of the form (2.5).

3. Linear maps on matrices of higher ranks

In view of Theorem 2.1, one may conjecture that if k is fixed, $2 \leqslant k \leqslant \min\{m, n\}$ and $\phi : M_{m\times n}(\mathbb{F}) \to M_{p\times q}(\mathbb{F})$ is a linear mapping having the property that $\operatorname{rank}\phi(A) = k$ whenever $\operatorname{rank} A = k$ then either it is of the form (1) or (2) in Theorem 2.1 or the range of ϕ is a rank-k subspace of $M_{p\times q}(\mathbb{F})$, that is, a subspace whose all nonzero members have rank k. This conjecture is not true as shown in the following examples.

Example 3.1. Assume $k = n < p$ and consider any linear map from $M_{n\times n}(\mathbb{F})$ into $M_{p\times p}(\mathbb{F})$ of the form

$$A \mapsto \begin{bmatrix} A & \psi(A) \\ 0 & 0 \end{bmatrix},$$

where ψ is any linear map. Obviously, such maps need not be of the form (1) or (2) in Theorem 2.1, and their range need not be a rank-n space.

The next example again concerns linear map from $M_{n\times n}(\mathbb{F})$ into $M_{p\times p}(\mathbb{F})$. For simplicity, we describe the construction for $n = 3$, $k = 2$, and $p = 8$. It is easy to construct higher dimensional examples using exactly the same idea.

Example 3.2. Let E_{ij}, $1 \leqslant i, j \leqslant 8$ be the standard matrix units in $M_{8\times 8}(\mathbb{F})$. Define $\phi : M_{3\times 3}(\mathbb{F}) \to M_{8\times 8}(\mathbb{F})$ by

$$\begin{aligned}\phi([a_{ij}]) = {} & a_{11}(E_{11} + E_{22}) + a_{12}(E_{12} + E_{23}) + a_{13}(E_{13} + E_{24}) \\ & + a_{21}(E_{14} + E_{25}) + a_{22}(E_{15} + E_{26}) \\ & + a_{23}(E_{16} + E_{27}) + \varphi([a_{ij}]),\end{aligned}$$

where φ is any linear map from $M_{3\times 3}(\mathbb{F})$ to the linear span of E_{18} and E_{28}. If $A \in M_{3\times 3}$ is any matrix of rank 2, then at least one of the entries $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$

has to be nonzero, and so, the rank $\phi(A)$ is 2. But obviously, ϕ is neither of the form (1) nor of the form (2) in Theorem 2.1, and as we have a complete freedom when choosing φ the range of ϕ is in general not a rank-2 space.

By the above examples, we need some stronger assumptions to get a good description for rank k preservers between matrix spaces. One possibility is to assume preservation of rank k matrices in both directions.

Theorem 3.3. *Assume that $\mathbb{F}$ is infinite. Let m, n, p, q, k be positive integers such that $2 \leqslant k \leqslant \min\{m, n\}$. Suppose $\phi : M_{m\times n}(\mathbb{F}) \to M_{p\times q}(\mathbb{F})$ is a linear transformation such that*

$$\operatorname{rank} \phi(A) = k \iff \operatorname{rank} A = k.$$

Then either $m \leqslant p$ and $n \leqslant q$, or $m \leqslant q$ and $n \leqslant p$, and there exist invertible matrices $P \in M_{p\times p}(\mathbb{F})$ and $Q \in M_{q\times q}(\mathbb{F})$ such that ϕ has the form (1) *or* (2) *in Theorem* 2.1.

Proof. We let $M_{r\times s} = M_{r\times s}(\mathbb{F})$. We start with the special case $k = 2$. Observe that ϕ is continuous in the Zariski topology, i.e., the topology in $M_{m\times n}$ in which closed sets are exactly those that are common zeros of finite sets of polynomials with coefficients in $\mathbb{F}$ of mn independent commuting variables that represent the entries of an element of $M_{m\times n}$, and the analogously defined closed sets in $M_{p\times q}$. It is easy to see that the closure of the set of matrices of rank 2 in the Zariski topology is the set of matrices of rank at most 2. Because of the continuity of ϕ, we see that ϕ maps matrices of rank 1 into matrices of rank at most 2. By the assumption, a rank-1 matrix cannot be mapped into a matrix of rank 2. So, its image has rank at most 1. We will show that rank-1 matrix cannot be mapped into zero matrix. Assume that there is a rank-1 matrix A such that $\phi(A) = 0$. It is easy to find a rank-1 B such that $A + B$ has rank 2. But then $\phi(B)$ must have rank 2, a contradiction. So, ϕ preserves rank-1 matrices and the result follows from Theorem 2.1.

Now let $k \geqslant 3$. We assume (without loss of generality) that $n \leqslant m$.

First consider the case $k = n$. We will prove that in this case ϕ preserves matrices of rank 1 and then the result will follow directly from Theorem 2.1. So, for any rank-1 matrix A we have to show that $\operatorname{rank} \phi(A) = 1$. With no loss of generality we may assume that $A = E_{11}$. There is also no loss of generality in assuming that

$$\phi\left(\begin{bmatrix} I \\ 0 \end{bmatrix}\right) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Our next step will be to prove that

$$\phi\left(\begin{bmatrix} A \\ 0 \end{bmatrix}\right) = \begin{bmatrix} * & * \\ * & 0 \end{bmatrix}$$

for every $A \in M_{n\times n}$. Indeed, it is easy to see that if there is an $A \in M_{n\times n}$ such that $\phi(A)$ has a nonzero entry in the bottom right corner then there is an $\alpha \in \mathbb{F}$ such that $\phi(\alpha I + A)$ has rank larger than n. On the other hand, we know that by continuity (in the Zariski topology) of ϕ every matrix in the range of ϕ has rank at most n, a contradiction.

We define now a new linear map $\psi : M_{n\times n} \to M_{n\times n}$ which associates to each $A \in M_{n\times n}$ the upper left $n \times n$ corner of $\phi(\tilde{A})$, where $\tilde{A} = \left[\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right]$. This linear transformation obviously maps singular matrices into singular matrices. Indeed, if A is singular, then the rank of $\phi(\tilde{A})$ cannot be equal to n, on the other hand, it cannot be larger than n because of the continuity of ϕ; so rank $\phi(A) < n$, and therefore also rank $\psi(A) < n$. Since $\psi(I) = I$, Theorem 1 of [9] implies that $\psi(A) = UAV$ or $\psi(A) = UA^{\mathrm{t}}V$ for some $U, V \in GL(n, \mathbb{F})$ (in fact, since $\psi(I) = I$ we have $V = U^{-1}$). Hence, there is no loss of generality in assuming that ϕ is such that

$$\phi\left(\begin{bmatrix} A \\ 0 \end{bmatrix}\right) = \begin{bmatrix} A & \eta(A) \\ \mu(A) & 0 \end{bmatrix}. \tag{3.1}$$

Here, of course η and μ are linear maps satisfying $\eta(I) = 0$ and $\mu(I) = 0$.

Let $A \in M_{n\times n}$ be any matrix of rank $n - 1$ having the first row equal to zero. Since rank $\phi(\tilde{A}) \leqslant n - 1$ we see using (3.1) that the first row of $\eta(A)$ must be zero. Every matrix from $M_{n\times n}$ having the first row equal to zero can be written as a difference of two such matrices with rank $n - 1$. So, for every such matrix the first row of $\eta(A)$ must be zero. Of course, an analogue holds true for every matrix having the ith row zero. In particular, $\eta(E_{11})$ has nonzero entries only in the first row. Assume that $\eta(E_{11}) \neq 0$. Since $\phi(I) = \phi(E_{11}) + \phi(E_{22} + \cdots + E_{nn})$ the first row of $\eta(E_{22} + \cdots + E_{nn})$ is nonzero, a contradiction. Thus, $\eta(E_{11}) = 0$, and similarly, $\mu(E_{11}) = 0$. Consequently, $\phi(E_{11}) = E_{11}$. Hence, we have proved that ϕ maps rank-1 matrices into rank-1 matrices. This completes the proof in the special case that $k = n$.

Let us now prove the statement for $2 < k < n$. Once again we will prove that ϕ preserves matrices of rank 1 and then the result follows directly from Theorem 2.1. As before it is enough to prove that $\phi(E_{11})$ has rank 1. The linear span V of $\{E_{ij} : 1 \leqslant i, j \leqslant k\}$ is isomorphic to $M_{k\times k}$. We consider the restriction of ϕ to the subalgebra V and applying the previous step we get the desired relation rank $\phi(E_{11}) = 1$. $\square$

A special case of linear maps ϕ such that rank $\phi(A) = s$ for every matrix A of rank k (with k and s fixed) are linear maps that send full rank matrices to full rank matrices. In particular, if $m = n$ and $p = q$, we are studying linear maps preserving invertibility, which is very difficult; see [6]. It was proved in [5] that if a linear transformation $\phi : M_{m\times m}(\mathbb{F}) \to M_{p\times p}(\mathbb{F})$ maps invertible matrices to invertible matrices, then $p = km$ for some positive integer k. An example in [6] shows that without additional assumptions description of all linear transformations (1.1) (where $m = n$ and $p = q$) such that

$$\phi(A) \text{ is invertible} \Leftrightarrow A \text{ is invertible} \tag{3.2}$$

may be intractable. Thus, we need to impose additional assumptions. We have the following result.

Proposition 3.4. *Let $\mathbb{C}$ be the complex field, and suppose $\phi : M_{m\times m}(\mathbb{C}) \to M_{p\times p}(\mathbb{C})$ is linear and maps invertible matrices to invertible matrices. If $\phi(A^*) = \phi(A)^*$ for all $A \in M_{m\times m}(\mathbb{C})$, and $\phi(P)$ is positive or negative definite for some positive definite $P \in M_{m\times m}(\mathbb{C})$, then ϕ is of the from*

$$\phi(A) = \pm T \begin{bmatrix} I_{s_1} \otimes A & 0 \\ 0 & I_{s_2} \otimes A^{\mathrm{t}} \end{bmatrix} T^* \tag{3.3}$$

for some invertible matrix T and some nonnegative integers s_1, s_2 (if $s_j = 0$ for some $j,\ j = 1, 2$, then the corresponding part in the right-hand side of (3.3) *is absent*).

Proof. Suppose $P \in M_{m\times m}(\mathbb{C})$ is positive definite such that $\phi(P) = Q$ is positive or negative definite. Replacing ϕ by a mapping of the form $X \mapsto \pm\phi(P^{1/2}XP^{1/2})$, we may assume that $\phi(I_m)$ is positive definite. We may further replace ϕ by the mapping of the form $X \mapsto \phi(I_m)^{-1/2}\phi(X)\phi(I_m)^{-1/2}$ and assume that $\phi(I_m) = I_p$. Note that the modified transformation still maps Hermitian matrices to Hermitian matrices. Moreover, if $A \in M_{m\times m}(\mathbb{C})$ is a Hermitian idempotent, then $tI_m - A$ is invertible for all $t \in \mathbb{C}\backslash\{0, 1\}$. Thus $\phi(tI_m - A) = tI_p - \phi(A)$ is also invertible for all $t \in \mathbb{C}\backslash\{0, 1\}$. Hence ϕ maps the set of Hermitian idempotents to itself. The proof can now be completed using the arguments from the proofs of Theorem 4.1 and Corollary 4.3 in [7] (see also [2, Theorem 2.1]). □

Note that one cannot remove the hypothesis that $\phi(P)$ is definite for some definite $P \in M_{m\times m}(\mathbb{C})$ in the above proposition.

Example 3.5. Let $\phi : M_{2\times 2}(\mathbb{C}) \to M_{4\times 4}(\mathbb{C})$ be defined by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} 0 & c & a & b \\ b & 0 & c & d \\ a & b & 0 & 0 \\ c & d & 0 & 0 \end{bmatrix}.$$

Then ϕ is linear such that $\phi(A^*) = \phi(A)^*$ for all $A \in M_{2\times 2}(\mathbb{C})$, and maps invertible matrices to invertible matrices. However, ϕ is not of the form (3.3).

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