The boundary layer flow of hyperbolic tangent fluid over a vertical exponentially stretching cylinder

Muhammad Naseer a,*, Muhammad Yousaf Malik a, Sohail Nadeem a, Abdul Rehman b

a Department of Mathematics, Quaid-i-Azam University 45320, Islamabad 44000, Pakistan
b Department of Mathematics, University of Balochistan, Quetta, Pakistan

Received 24 September 2013; revised 4 May 2014; accepted 5 May 2014
Available online 4 June 2014

Abstract The present problem is the steady boundary layer flow and heat transfer of a hyperbolic tangent fluid flowing over a vertical exponentially stretching cylinder in its axial direction. After applying usual boundary layer with a suitable similarity transformation to the given partial differential equations and the boundary conditions, a system of coupled nonlinear ordinary differential equations is obtained. This system of ordinary differential equations subject to the boundary conditions is solved with the help of Runge–Kutta–Fehlberg method. The effects of the involved parameters such as Reynolds numbers, Prandtl numbers, Weissenberg numbers and the natural convection parameter are presented through the graphs. The associated physical properties on the flow and heat transfer characteristics that is the skin friction coefficient and Nusselt numbers are presented for different parameters.

1. Introduction

During the past many years, number of researchers has worked on non-Newtonian fluids. Wang [1] analyzed non-Newtonian fluids for mixed convection heat transfer from a vertical plate. Xu et al. [2] have presented the series solutions of unsteady boundary layer flows of non-Newtonian fluids near a forward stagnation point. Ellahi and Afzal [3] have discussed the effects of variable viscosity on a third grade fluid with porous medium. In another paper Ellahi [4] has presented the effects of MHD (Magneto hydrodynamics) and temperature dependent viscosity on the flow of non-Newtonian nano-fluid in a pipe. Nadeem et al. [5] analyzed the non-orthogonal stagnation point flow of a nano non-Newtonian fluid toward a stretching surface with heat transfer. Labropulu et al. [6] wrote an article on non-orthogonal stagnation-point flow toward a stretching surface in a non-Newtonian fluid with heat transfer.
An important branch of the non-Newtonian fluid models is the hyperbolic tangent fluid model. The hyperbolic tangent fluid is used extensively for different laboratory experiments. Friedman et al. [7] have used the hyperbolic tangent fluid model for large-scale magneto-rheological fluid damper coils. Recently, Nadeem and Akram [8] jointly published a paper on peristaltic transport of a hyperbolic tangent fluid model in an asymmetric channel. In another paper Nadeem and Akram [9] have presented the effects of partial slip on the peristaltic transport of a hyperbolic tangent fluid model in an asymmetric channel. Only few researchers have worked on different non-Newtonian fluid models [10–17]. Ali [18] analyzed the heat transfer characteristics of a continuous stretching surface. Ishak et al. [19] investigated the uniform suction/blowing and the uniform ambient temperature is taken as \( T_\infty \). The expression for coefficient of skin friction and local Nusselt number are computed numerically.

2. Fluid model

For the hyperbolic tangent fluid the continuity and momentum equations are given as

\[
\text{div} \ V = 0,
\]

\[
\rho \frac{dV}{dt} = \text{div} \tau + \rho b,
\]

where \( \rho \) is the density, \( V \) is the velocity vector, \( \tau \) is the Cauchy stress tensor, \( b \) represents the specific body force vector and \( dt/dt \) represents the material time derivative. The constitutive equations for hyperbolic tangent fluid model is given by [8]

\[
S = -pI + \tau,
\]

\[
\tau = -\eta g_0 + (\eta_0 + \eta_{\infty}) \tanh \left( \Gamma g_0 \right) \nabla \nabla \nabla \left( \frac{1}{2} \right)
\]

where \( \Gamma = \sqrt{\frac{1}{2} \sum_j \sum_k \frac{\partial \phi_j}{\partial x_k} \frac{\partial \phi_k}{\partial x_j}} = \sqrt{\frac{1}{2} \Pi} \)

We consider the case for which \( \eta_\infty = 0 \) and \( \Gamma g_0 < 1 \). The component of extra stress tensor, therefore, can be written as

\[
\tau = -\eta g_0 \left[ \Gamma g_0 \right]^n g_0
\]

\[
\tau = -\eta_0 \left[ 1 + \Gamma g_0 - 1 \right] g_0
\]

\[
\tau = -\eta_0 \left[ 1 + n \Gamma g_0 - 1 \right] g_0
\]

3. Formulation

Consider the problem of natural convection boundary layer flow of a hyperbolic tangent fluid flowing over a vertical circular cylinder of radius \( a \). The cylinder is assumed to be stretched exponentially along the axial direction with velocity \( U_w \). The temperature at the surface of the cylinder is assumed to be \( T_w \) and the uniform ambient temperature is taken as \( T_\infty \). Under these assumptions the boundary layer equations of motion and heat transfer are

\[
u_r + u + w_z = 0,
\]

\[
u w_r + w w_z = g \beta (T - T_\infty)
\]

\[
+ v \left[ (1 - n) \left( \frac{w_w + \frac{1}{r} w_r}{2} \right) + \frac{n}{2} \frac{w_r + \frac{1}{r} w_w}{2} \right]
\]

\[
u_T r + wT_z = \frac{z}{T_w + \frac{1}{r} T_r}.
\]

where the velocity components along the \( (r, z) \) axes are \( (u, w) \), \( \rho \) is density, \( v \) is the kinematic viscosity, \( p \) is pressure, \( g \) is the gravitational acceleration along the \( z \)-direction, \( \beta \) is the coefficient of thermal expansion, \( T \) is the temperature, \( \eta_\infty \) is the infinite shear rate viscosity, \( \eta_0 \) is the zero shear rate viscosity, \( \Gamma \) is the time constant, \( n \) is the power law index, and \( \lambda \) is the thermal diffusivity. The corresponding boundary conditions for the problem are

\[
u(a, z) = 0, \quad w(a, z) = U_w \quad w(r, z) \to 0 \quad \text{as} \ r \to \infty,
\]

\[
u(T_w, z) = 0, \quad T(r, z) \to T_\infty \quad \text{as} \ r \to \infty,
\]

where \( U_w = 2 \alpha \kappa^{n/2} \) (\( k \) is dimensional constant) is the fluid velocity at the surface of the cylinder.

4. Solution of the problem

Introduce the following similarity transformations:

\[
u = -\frac{1}{2} U_w \frac{f(\eta)}{\sqrt{\eta}}, \quad w = U_w f'(\eta),
\]

\[
\theta = \frac{T - T_\infty}{T_w - T_\infty}, \quad \eta = \frac{r^2}{a^2},
\]

where the characteristic temperature difference is calculated from the relations \( T_w - T_\infty = T_0 e^{ci \alpha} \). With the help of transformations (6) and (7), Eqs. (1)-(3) take the form

\[
2(1 - n)(\eta f'' + f') + n F_0 \sqrt{\theta} (4 \eta f'' + 3 f') + \text{Re}(f' - f'^2) + \text{Re} \theta^2 = 0,
\]

\[
\eta f'' + f' + \text{RePr}(f' - f'^2) = 0,
\]

in which \( \lambda = g \beta a (T_w - T_\infty) / U_w^2 \) is the natural convection parameter, \( \text{Pr} = v/\kappa \) is the Prandtl number, \( \text{We} = 4 \Gamma U_w / a \) is the Weissenberg number and \( \text{Re} = \alpha U_w / 4v \) is the Reynolds number. The boundary conditions in nondimensional form become

\[
f(1) = 0, \quad f'(1) = 1, \quad f' \to 0, \quad \text{as} \ \eta \to \infty,
\]

\[
\theta(1) = 1, \quad \theta \to 0, \quad \text{as} \ \eta \to \infty.
\]
The important physical quantities such as the shear stress at the surface $\tau_w$, the skin friction coefficient $c_f$, the heat flux at the surface of the cylinder $q_w$ and the local Nusselt number $N_u$ are

$$\tau_w = \tau_{w|_{\eta = 1}}, \quad q_w = -kT_{1|_{\eta = 1}},$$
$$c_f = \frac{\tau_w}{\rho U_w^2}, \quad N_u = \frac{ae^{-\theta}q_w}{k(T_w - T_{\infty})}. \quad (12)$$

The solution of the present problem is obtained numerically by using the Runge–Kutta–Fehlberg method.

5. Results and discussion

In this paper an analysis is carried out for natural convection boundary layer flow of a hyperbolic tangent fluid over an exponentially stretched cylinder. It is assumed that the cylinder is stretching exponentially along its axial direction. Expression $U_w = 2ae^{-\theta}/a$ is the assumed exponential stretching velocity at the surface of the cylinder. For the solution of the problem Runge–Kutta-Fehlberg method is used. The impact of the different parameters such as the Reynolds number $Re$, the Prandtl number $Pr$, the Weissenberg number $We$ and the natural convection parameter $\lambda$ over the non-dimensional velocity and temperature profiles are presented graphically and in the form of tables. Fig. 1 shows the influence of the Weissenberg number $We$ on the velocity function $f'$ when $n = 0.3$. From the graph it is clear that velocity profile decreases by increasing the values of $We$. Fig. 2 shows the effects of Reynolds number $Re$ on velocity profile. Fig. 3 shows the influence of natural convection parameter on velocity profile. Fig. 4 shows the effects of Prandtl number $Pr$ on temperature profile. Fig. 5 shows the influence of Reynolds number $Re$ on temperature profile.
Table 1  Numerical values of local heat flux $-\theta' (1)$ at the surface.

<table>
<thead>
<tr>
<th>$Pr/Re$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\theta' (1)$ For $n = 0.3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.9613</td>
<td>0.9786</td>
<td>0.9963</td>
<td>1.0141</td>
<td>1.0323</td>
<td>1.1263</td>
</tr>
<tr>
<td>3</td>
<td>1.0610</td>
<td>1.0617</td>
<td>1.0624</td>
<td>1.0632</td>
<td>1.0639</td>
<td>1.0667</td>
</tr>
<tr>
<td>5</td>
<td>1.1578</td>
<td>1.1589</td>
<td>1.1601</td>
<td>1.1612</td>
<td>1.1624</td>
<td>1.1684</td>
</tr>
<tr>
<td>7</td>
<td>1.2517</td>
<td>1.2533</td>
<td>1.2548</td>
<td>1.2563</td>
<td>1.2579</td>
<td>1.2658</td>
</tr>
<tr>
<td>15</td>
<td>1.6018</td>
<td>1.6045</td>
<td>1.6071</td>
<td>1.6098</td>
<td>1.6125</td>
<td>1.6264</td>
</tr>
</tbody>
</table>

Table 2  Numerical values of local skin friction $-f' (1)$ at the surface.

<table>
<thead>
<tr>
<th>$We/Re$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-f' (1)$ For $n = 0.3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.9621</td>
<td>1.0041</td>
<td>1.0450</td>
<td>1.0850</td>
<td>1.1240</td>
<td>1.3082</td>
</tr>
<tr>
<td>0.2</td>
<td>0.9767</td>
<td>1.0231</td>
<td>1.0687</td>
<td>1.1136</td>
<td>1.1580</td>
<td>1.4666</td>
</tr>
<tr>
<td>0.3</td>
<td>0.9943</td>
<td>1.0464</td>
<td>1.0982</td>
<td>1.1499</td>
<td>1.2017</td>
<td>1.1684</td>
</tr>
<tr>
<td>0.4</td>
<td>1.0163</td>
<td>1.0761</td>
<td>1.1366</td>
<td>1.1982</td>
<td>1.2612</td>
<td>1.6159</td>
</tr>
</tbody>
</table>

Re over the velocity function $f'$ when $n = 0.3$. The velocity profile increases by increasing the values of Re. Fig. 3 shows the influence of the natural convection parameter $\lambda$ on velocity profile when $n = 0.3$. From the graph it is clear that by increasing the values of $\lambda$ the velocity profile decreases. Fig. 4 describes the impact of Prandtl number Pr over the temperature profile when $n = 0.3$. The temperature profile increases by increasing the values of Pr. Fig. 5 shows the influence of Reynolds number Re on the temperature profile when $n = 0.3$. The temperature profile increases by increasing the values of Re. Table 1 shows the behavior of heat flux at the surface of the stretching cylinder for different values of the Pr and Re when $n = 0.3$, that also corresponds to the local Nusselt numbers. Entries in Table 1 show that the heat flux at the surface increases by increasing both Pr and Re. Table 2 shows that the magnitude of the boundary derivative of the velocity profile when $n = 0.3$. Entries in the Table 2 show that the magnitude of the boundary derivative increases by increasing both the Weissenberg number and the Reynolds number.

References