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Minimum-cost line broadcast in paths

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Abstract

Under the line communication protocol, calls can be placed between pairs of non-adjacent sites over a path of lines connecting them; only one call can utilize a line at any time. This paper addresses questions regarding the cumulative cost, i.e., sum of lengths of calls, of broadcasting under the line protocol in path networks. Let P_n be the path with n vertices, and C_n be the cost of an optimal, line broadcast scheme from a terminal vertex in path P_n. We show that a minimum-cost line broadcast scheme from any source vertex in P_n has cost no more than C_n and no less than $C_n - n + 2$ for any $n \geqslant 2$ and any time $t \geqslant \lceil \log_2 n \rceil$. We derive a closed-form expression for the minimum cost of a minimum-time line broadcast from a terminal vertex in certain paths and relate this to costs from nearby sources.

Keywords: Line broadcast; Cumulative cost

1. Introduction

The synchronous *line communication* protocol in a network is defined as follows: (i) during each time unit, any site can call at most one other site through a path of lines connecting them, (ii) a call succeeds only if it shares no line with other calls during the same time unit, and (iii) if a call succeeds, it takes only one time unit regardless of the distance (i.e., number of lines) between the caller and receiver. Under this protocol, a site can send or receive at most one call during any time unit, but may switch through several calls, depending on connections. The line communication protocol is an approximate model of the *wormhole* and *cut-through* communication protocols [5, 10–12].

We are interested here in the information dissemination task of broadcasting under the line communication model. *Broadcasting* is the one-to-all information dissemination task, where one site, the *source*, originates a message to be transferred to all other sites. Broadcasting under the line communication protocol was first studied by Farley in [3]

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and later examined by Hromkovič et al. in [7, 8]. Several researchers have studied broadcasting under a line communication model wherein sites could *eavesdrop* on calls that pass through them [2, 4]. Bitan and Zaks [1] considered a broadcasting problem in trees under that model, restricting the originator of every call to the given source vertex.

We model a communication network by a connected graph $G=(V,E)$, consisting of a set V of vertices, representing sites of the network, and a set E of edges, representing lines of the network. A graph is *connected* if there exists a path in E between every pair of vertices in V. In [3], Farley shows that, applying the line communication protocol in any connected graph of n vertices, there exists a broadcast scheme that requires $\lceil \log_2 n \rceil$ time units from any vertex of the graph. Since the number of informed vertices can at most double in any time unit, this broadcast scheme is time optimal. We call such a minimum-time scheme a *minimum-time line broadcast scheme*, consisting of sets of calls to be made during successive time units that are sufficient to complete the broadcast process.

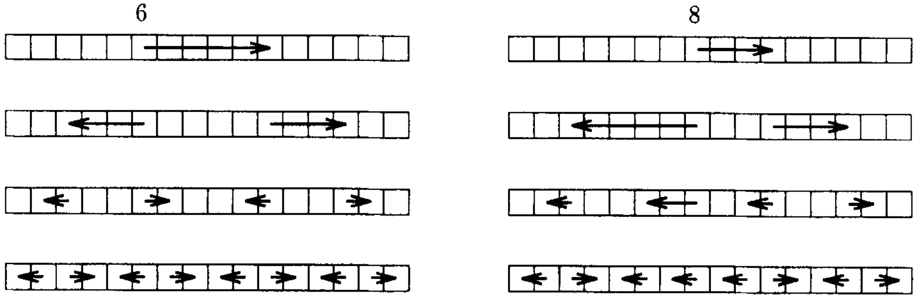
Although we assume every call takes a single time unit, we note that longer-distance calls incur more cost in terms of their use of network resources. Suppose that each call is charged a cost related to its distance; i.e., define the *cost* of a call to be the number of lines on the path between the caller and the receiver. In this paper, we consider the *cumulative cost* of a line broadcast scheme, being the sum of the costs of all calls involved in the given scheme. In particular, we are interested in the *minimum cumulative cost* over all broadcast schemes for a given network graph, source vertex, and broadcast time. The cumulative cost of line broadcast was examined first by Kane, who considered such costs in cycle networks [9].

This paper considers two questions regarding the cumulative cost of line broadcast in path networks. A path network of n sites is modeled by a graph P_n of n vertices labeled 1 through n, such that the set of edges is $(i, i+1)$ for $1 \leqslant i < n$. Vertices 1 and n, at the two ends of the path, are called *terminal vertices*. Our first question addresses the effect of the position of the source in the path on the cumulative cost.

Example 1. Let $P_4=(1,2,3,4)$ be the path of length four. Note that $\lceil \log_2 4 \rceil = 2$, i.e., the minimum-time line broadcast takes 2 time units.

When the source vertex is 1 a minimum-time line broadcast scheme $\mathscr{S}_1$ proceeds as follows. In the first time unit, vertex 1 calls vertex 3; and in the second time unit, vertices 1 and 3 call vertices 2 and 4, respectively. Note that this scheme has the least cumulative cost of 4 among all minimum-time schemes from vertex 1.

When the source vertex is 2, on the other hand, a line broadcast scheme $\mathscr{S}_2$ proceeds as follows. In the first time unit, vertex 2 calls vertex 3; in the second time unit, vertices 2 and 3 call vertices 1 and 4, respectively. This scheme has the least cost of 3 among all minimum-time schemes from vertex 2; moreover, it is less than the cost of scheme $\mathscr{S}_1$.



(a) Broadcast scheme from 6. (b) Broadcast scheme from 8.

Fig. 1. Example 2.

In general, the minimum cumulative cost of a line broadcast for a given number of time units will vary with position of the source vertex. A natural conjecture is that a source vertex closer to a terminal vertex will have greater cumulative cost. This is not the case, however, Example 2 presents a counterexample.

Example 2. Let $P_{16} = (1, 2, \dots, 16)$. A minimum-cost line broadcast scheme from vertex 6 is shown in Fig. 1(a); one from vertex 8 is shown in Fig. 1(b). Although vertex 6 is closer to a terminal vertex than is vertex 8, the former has a lesser minimum cost.

In this paper, we prove that a terminal vertex has the greatest cost of any source vertex. The proof of this claim is not trivial, as cumulative cost does not monotonically increase as we move the source from a central to a terminal vertex. Specifically, we show that a minimum-cost line broadcast scheme from any source vertex in P_n has cost that is not more than that from a terminal vertex and not less than that cost minus $n-2$ for any $n \geqslant 2$ and $t \geqslant \log_2 n$.

We also address issues with respect to the precise value of minimum-time cumulative cost from a terminal vertex and its nearby neighbors in certain paths. In [9], Kane derived a precise value of the cumulative cost for *cycles* of size $n = 2^m$. Note that since a cycle has no terminal vertex, the position of the source vertex in a cycle does not affect the cost. Kane's result represents a lower bound on the cost of a minimum-time line broadcast in the path of length 2^m.

The remainder of this paper is organized as follows. Section 2 introduces preliminary notation and definitions and establishes basic properties of minimum-cost line broadcast schemes in paths. The goodness and the badness of a terminal vertex as source vertex are discussed in Sections 3 and 4, respectively. Section 5 derives a closed form of the cumulative cost of the broadcast from a terminal vertex subject to $n = 2^m$ and relates that cost to the cost from nearby vertices as sources. Section 6 concludes the paper with some future directions for research.

2. Preliminaries

The minimum cumulative cost of a line broadcast varies with the position of the source vertex and the desired broadcast time.

Definition 1. For all $t \geqslant \lceil \log_2 n \rceil$, let $C(n,p,t)$ be the *minimum cumulative cost* of a line broadcast in path P_n from vertex p in t time units.

By symmetry, $C(n,p,t) = C(n,n-p+1,t)$ for any $1 \leqslant p \leqslant n$.

Definition 2. For each $t \geqslant \lceil \log_2 n \rceil$, let

$$\widetilde{C}(n,t) = \min_{1 \leqslant p \leqslant n} C(n,p,t)$$

and

$$D(n,t) = \min_{1 \leqslant p \leqslant n} \{C(n,p,t) + p\}.$$

There are two, immediate results regarding the monotonicity of $\widetilde{C}(n,t)$.

Lemma 1. *For all* $t \geqslant \lceil \log_2 n \rceil$, $\widetilde{C}(n-1,t) < \widetilde{C}(n,t)$ *and* $\widetilde{C}(n,t+1) \leqslant \widetilde{C}(n,t)$.

Definition 3. A line broadcast scheme $\mathscr{S}$ is *nested* iff no call of $\mathscr{S}$ passes through an informed vertex.

Lemma 2 (Kane [9]). *A minimum-cost line broadcast scheme for path P_n is nested.*

Definition 4. Consider a line broadcast scheme $\mathscr{S}$ in path P_n. For each vertex v, the *segment* $Q(v)$ of $\mathscr{S}$ consists of v, those vertices called by v, and those vertices in $Q(x)$ for all x called by v in $\mathscr{S}$.

Definition 5. Given segment $Q(v)$ of scheme $\mathscr{S}$ in P_n, we call v the *leader* of $Q(v)$.

Note that the leader of a segment is the first vertex to become informed in the segment. We also say that a vertex v is leader for subsegments of $Q(v)$ containing v but not all descendants.

Example 3. Consider the broadcast schemes in Example 2, again (see Fig. 1). In the broadcast scheme from vertex 6, $Q(6) = (1,2,\ldots,16)$ and $Q(8) = (8)$. On the other hand, in the broadcast scheme from vertex 8, $Q(6) = (5,6)$ and $Q(8) = (1,2,\ldots,16)$.

Definition 6. A line broadcast scheme $\mathscr{S}$ for path P_n is *contiguous* iff segment $Q(v)$ consists of a (connected) subpath of P_n for all leaders v of $\mathscr{S}$.

Lemma 3. *If a line broadcast scheme $\mathscr{S}$ for path P_n is nested, then $\mathscr{S}$ is contiguous.*

Proof. Assume $\mathscr{S}$ is not contiguous. Then there exists a situation where u calls y and v calls x (as the 'smallest' violation), where $u < x < y < v$. They must place these calls at different times. Suppose u calls y first. Then when v calls x, it calls through y, an informed vertex. Thus, $\mathscr{S}$ is not nested. □

Corollary 1. *A minimum-cost line broadcast scheme for path P_n is contiguous.*

Definition 7. A line broadcast scheme $\mathscr{S}$ for path P_n is *directed* iff the first call made by a vertex after becoming informed of the message is away from the vertex from which it received the message.

Lemma 4. *A minimum-cost line broadcast scheme for a path is directed.*

Proof. Assume $\mathscr{S}$ is not directed. Then there exists a situation where u calls v and v calls w as its first call, where w lies between u and v on the path. We can alter the scheme to produce a scheme with lower cumulative cost by having u call w when it currently calls v and then having w call v when v was calling w. This produces a scheme which has lower cost; thus, a minimum cost scheme must be directed. □

3. Goodness of line broadcast from a terminal vertex

We now consider the cumulative cost of a line broadcast from a terminal vertex of path P_n. We show that for any $n \geqslant 2$, the cost of a minimum-cost line broadcast from a terminal vertex (i.e., $C(n,1,t)$) differs by at most $n-2$ from the minimum cost of any line broadcast on the path P_n with the same broadcast time (i.e., $\widetilde{C}(n,t)$).

Theorem 1 (Goodness). *For any $n \geqslant 2$ and $t \geqslant \lceil \log_2 n \rceil$,*

$$C(n,1,t) - \widetilde{C}(n,t) \leqslant n-2.$$

Proof. We prove it by induction on n. By observation, the proposition holds for all $n \leqslant 4$ for all applicable t.

As our inductive hypothesis, we suppose that for any $i < n$ and $t \geqslant \lceil \log_2 i \rceil$,

$$C(i,1,t) - \widetilde{C}(i,t) \leqslant i-2$$

and consider the case of $i = n$. Let p be a vertex, such that $C(n,p,t) = \widetilde{C}(n,t)$ and $1 \leqslant p \leqslant n/2$.

Let $\mathscr{S}$ be a broadcast scheme which completes the broadcast from vertex p in t time units with minimum cost $\widetilde{C}(n,t)$. Let q be the vertex called in the first time unit by p. By Corollary 1, $Q(q)$ and $P_n \backslash Q(q)$ are both contiguous. Let $R = Q(q)$ and $L = P_n \backslash R$. Note that vertex p is the leader of L in the remainder of the broadcast scheme $\mathscr{S}$ and that broadcast subschemes for segments L and R can be assumed to have minimum cost. Without loss of generality, suppose that R is at least as long as L. Denote $|L|$ by

m ($\leqslant n/2$). Let C_p be the cost of the broadcast from p to L in $t-1$ time units. Since $\widetilde{C}(m,t-1) \leqslant C_p$ and $C_p + D(n-m,t-1) \leqslant \widetilde{C}(n,t)$, we have

$$\widetilde{C}(m,t-1) + D(n-m,t-1) \leqslant \widetilde{C}(n,t). \tag{1}$$

Next, consider the following scheme $\mathscr{S}'$ which completes the broadcast from vertex 1 in t time units as follows: In the first time unit, vertex 1 calls q, and during the next $t-1$ time units, vertices 1 and q are responsible for segments L and R, respectively. Let C' be the cost of the scheme. Then, since $C(n,1,t) \leqslant C'$ and $C' = C(m,1,t-1) + m - 1 + D(n-m,t-1)$, we have

$$C(n,1,t) \leqslant C(m,1,t-1) + m - 1 + D(n-m,t-1). \tag{2}$$

From (1) and (2), we have

$$C(n,1,t) - \widetilde{C}(n,t) \leqslant C(m,1,t-1) - \widetilde{C}(m,t-1) + m - 1.$$

By our inductive hypothesis and the fact that $m \leqslant n/2$, we have

$$C(n,1,t) - \widetilde{C}(n,t) \leqslant 2m - 3 \leqslant n - 2$$

which completes the proof. $\square$

4. Badness of line broadcast from a terminal vertex

This section characterizes the badness of a terminal vertex as the source vertex of a line broadcast in a path. Specifically, we show that it is worst in terms of cumulative cost.

Theorem 2 (Badness). *For all $n \geqslant 1$,*

$$C(n,p,t) \leqslant C(n,1,t)$$

for any $1 \leqslant p \leqslant n$ and $t \geqslant \lceil \log_2 n \rceil$.

We prove it by induction on n. When $n \leqslant 4$, the statement holds by inspection. Suppose that for any $i < n$, $C(i,p,t) \leqslant C(i,1,t)$ holds for any $1 \leqslant p \leqslant i$ and $t \geqslant \lceil \log_2 i \rceil$, and consider the case of $i = n$, for $n > 4$.

Let $\mathscr{A}$ be a scheme which completes a line broadcast from vertex 1 in t time units with minimum cost $C(n,1,t)$. Without loss of generality, we assume that scheme $\mathscr{A}$ is 'eager', i.e., each vertex makes its calls as soon as possible; by our earlier lemmas, this is possible and does not affect broadcast cost. Let p be a vertex in the path. Without loss of generality, we suppose $1 \leqslant p \leqslant n/2$. In the following, we will construct a broadcast scheme $\mathscr{B}$ that completes a broadcast from vertex p with no more cost than $C(n,1,t)$ by modifying scheme $\mathscr{A}$.

Label vertices in P_n from left to right as $1,2,\ldots,n$. For each $0 \leqslant i \leqslant k$, where $k \leqslant t$, define u_i ($\in P_n$) as follows: (1) $u_0 = 1$, and (2) for each $i \geqslant 1$, u_i is the first vertex

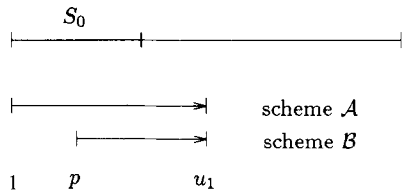


Fig. 2. An explanation of Lemma 5.

called by u_{i-1}. By our 'eager' scheme assumption, u_{i-1} calls u_i during the ith time unit of the scheme. By Lemmas 2–4, scheme $\mathscr{A}$ partitions P_n into contiguous segments that lie consecutively from one end of the path to the other, as

$$S_0, S_1, S_2, \ldots, S_k,$$

where $S_i = Q(u_i) \backslash Q(u_{i+1})$ for each $0 \leqslant i \leqslant k \leqslant t$. Vertex u_i is the *leader* of segment S_i; let s_i be the number of vertices in segment S_i, i.e., $|S_i|$. Note that vertex u_k is the terminal vertex at the other end of path P_n, i.e., vertex n, and that $s_k = 1$. Thus, the cumulative cost of calls between the $k+1$ leaders is $n-1$. The overall cost of a broadcast is equal to $n-1$ plus the sum of the costs to broadcast in the segments S_i. For $1 \leqslant i \leqslant k$, that cost is equal to $\widetilde{C}(s_i, t-(i+1))$, as each such u_i can be selected to be the optimal originator for each segment and time remaining. By Lemma 1, we can assume that $s_i \geqslant s_{i+1}$, for all $1 \leqslant i < k$.

The following lemma eliminates an obvious case.

Lemma 5. *If $p \in S_0$ and $p \neq 1$, then $C(n, p, t) < C(n, 1, t)$.*

Proof. Scheme $\mathscr{B}$ proceeds as follows. In the first time unit, vertex p calls vertex u_1. Note that since p is closer to u_1 than vertex 1 ($= u_0$), the first time unit has less cost than scheme $\mathscr{A}$ (see Fig. 2 for illustration). During the next $t-1$ time units, vertex p is responsible for segment S_0, and vertex u_1 is responsible for $P_n \backslash S_0$. In other words, in scheme $\mathscr{B}$, all vertices in $P_n \backslash S_0$ act as in scheme $\mathscr{A}$.

The broadcast from u_1 to $P_n \backslash S_0$ has the same cost as scheme $\mathscr{A}$. Since $s_0 < n$, by our inductive hypothesis, the broadcast from p to S_0 has no more cost than the broadcast from 1 to S_0, completing the proof. □

Given the above lemma, we need only consider the case of $s_0 < p \leqslant n/2$. We use the following two operations to modify a given line broadcast scheme in a path P_n.

Definition 8 (*FLIP*). Let $Q(v)$ be a (sub)segment of scheme $\mathscr{S}$ on a path P with leader v. *FLIP*($Q(v)$) means the replacement of segment $Q(v)$ by the mirror image of $Q(v)$. It does not change the leader of or transmission order in $Q(v)$.

Definition 9 (*EXCHANGE*). Let $Q(u)$ and $Q(v)$ be two disjoint (sub)segments of scheme $\mathscr{S}$ on a path P with leaders u and v, respectively. The operation

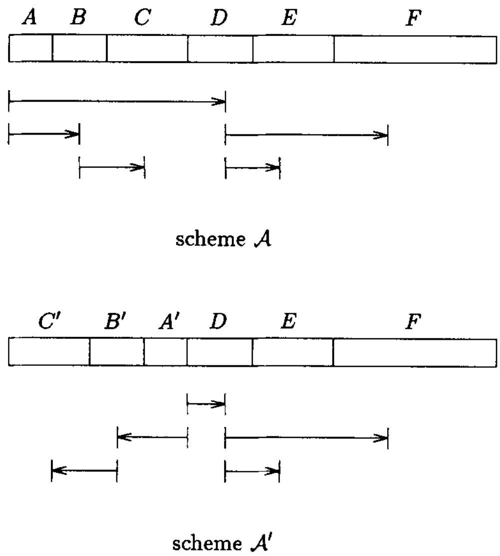


Fig. 3. A partition of P_n according to scheme $\mathscr{A}$ and scheme $\mathscr{A}'$.

$EXCHANGE(Q(u), Q(v))$ exchanges the positions of $Q(u)$ and $Q(v)$ (without flipping) and exchanges the roles of the leaders with respect to external vertices.

The following lemma is crucial to the construction of scheme $\mathscr{B}$.

Lemma 6. *There is a position $q \geqslant s_0$ in path P_n such that*

1. *there is a scheme with cumulative cost less than or equal to that of scheme $\mathscr{A}$ in which the source vertex at the qth position of path P_n calls the vertex at the $(q+1)$th position in the first time unit; and*
2. *for any $s_0 \leqslant p \leqslant q$, there is a broadcast scheme from the vertex at the pth position with cumulative cost less than or equal to that of scheme $\mathscr{A}$.*

Proof. First, according to scheme $\mathscr{A}$, partition P_n into six segments $A, B, \ldots, F$ as follows: the source vertex at the left end of path P_n is the leader of A; the leader of A calls the leader of D in the first time unit; in the second time unit, the leaders of A and D, respectively, call the leaders of B and F; and in the third time unit, the leaders of B and D, respectively, call the leaders of C and E. Some, but not all, of these segments may be empty; given $n > 4$, segments A, D and F are not empty. The catenation of A, B, C coincides with S_0 and the catenation of D and E coincides with S_1, see Fig. 3 for illustration.

Now we begin to modify scheme $\mathscr{A}$ by moving the source interior to the path by applying operation $FLIP(S_0)$ to scheme $\mathscr{A}$. Let $\mathscr{A}'$ be the resulting scheme. If segment B exists and $|B| < |D|$, we apply $EXCHANGE(B, D)$ prior to flipping S_0

without increasing the cumulative cost of $\mathscr{A}'$. Similarly, if B exists and $|C| < |E|$, we apply the operation $EXCHANGE(C, E)$ to scheme $\mathscr{A}$ prior to flipping S_0, as this does not increase the cumulative cost of scheme $\mathscr{A}'$.

In scheme $\mathscr{A}'$, the position of the source vertex has been moved from the left end to the qth position of P_n, where $q = s_0 + \max\{0, |E| - |C|\} + \max\{0, |D| - |B|\}$. These operations reduce the cost of the calls reaching segment F by $q - 1$ and do not increase any other line broadcast costs, as per the semantics of the $FLIP$ and $EXCHANGE$ operations. As such, the cost of scheme $\mathscr{A}'$ is at most

$$C(n, 1, t) - q + 1.$$

The first claim of the lemma is proved as follows. Modify scheme $\mathscr{A}'$ in such a way that the vertex called by the source vertex in the first time unit is at the $(q + 1)$th position. That vertex informs the segment F in the next time unit and then acts as leader for the possibly modified segment S_1 after that. The modification does not affect the cumulative cost of calls of scheme $\mathscr{A}'$ in the first and second time units. The increase in the cost during the next $t - 2$ time units is at most $s_1 - \max\{0, |E| - |C|\} + \max\{0, |D| - |B|\} - 2$ by Theorem 1. This increase cannot be more than the decrease to scheme $\mathscr{A}$ resulting from the earlier flip of S_0. The possible exchanges guarantee that the modified S_1 is not larger than the original S_0. Theorem 1 guarantees the increase is 2 less than the size of the modified S_1, which is less than or equal to q.

The second claim of the lemma is established as follows. By Lemma 5, we need only consider the case of $q > |S_0|$. Modify scheme $\mathscr{A}'$ in such a way that the position of the source vertex is at the pth position, where $|S_0| < p \leqslant q$. This modification increases the cost of the first call by $q - p$, and, by the inductive hypothesis, results in a cost for the first segment of size q that is not greater than from location q. Thus, the cost of this scheme is at most

$$C(n, 1, t) - p + 1. \qquad \square$$

The following lemma examines the remaining case and completes the proof of Theorem 2.

Lemma 7. *Let q be the position determined in Lemma 6. Then, for any $q \leqslant p \leqslant n/2$, there is a broadcast scheme $\mathscr{B}$ from the vertex at the pth position with a cumulative cost no larger than that of scheme $\mathscr{A}$.*

Proof. Let $\mathscr{S}$ be a broadcast scheme from the qth position with minimum cumulative cost in which the source vertex calls the vertex at the $(q + 1)$th position in the first time unit. By Lemma 6, the scheme $\mathscr{S}$ has cost not greater than that of the original scheme $\mathscr{A}$ from an end vertex.

Label vertices in P_n from left to right as $1, 2, \ldots, n$. Let $L = \{1, 2, \ldots, q\}$ and $R = P_n \backslash L$. Since $\mathscr{S}$ is optimal, by Lemma 2 and Theorem 3, segments L and R, respectively, can be partitioned into (sub)segments as $L_1, L_2, \ldots, L_t$ and $R_1, R_2, \ldots, R_t$, in a way equivalent to the initial partition of P_n under scheme $\mathscr{A}$ into subsegments $S_0, S_1, \ldots, S_t$. Subsegments of L are arranged from left to right as $L_t, L_{t-1}, \ldots, L_1$, and subsegments of R are arranged from left to right as $R_1, R_2, \ldots, R_t$. As per our prior lemmas and discussion, we may assume that

- $|L_i| \geqslant |L_{i+1}|$ and $|R_i| \geqslant |R_{i+1}|$ for $i \geqslant 2$, and
- $|L_1 \cup R_1| \geqslant |L_2|$ and $|L_1 \cup R_1| \geqslant |R_2|$.

In the following, we will modify $\mathscr{S}$ in such a way that the source vertex moves to the pth position without increasing the cumulative cost.

First, we apply $EXCHANGE(FLIP(L_i), FLIP(R_i))$ for some $2 \leqslant i \leqslant t$ to scheme $\mathscr{S}$ in such a way that the resultant $L_1 \cup R_1$ contains the pth position of path P_n. An application of one of these exchanges moves the position of the source vertex to the right by at most distance $|R_i| - |L_i|$ and does not alter the cost of the overall scheme. Since $(|R_i| - |L_i|) \leqslant (|L_1 \cup R_1|)$ and $|R| > n/2$, there always exists such a set of $EXCHANGE$ operations.

Let $\mathscr{S}'$ be the resultant scheme, which has the same cost as $\mathscr{S}$. We now modify scheme $\mathscr{S}'$ so that the originator is at location p. Location p calls the leader of the other first segment, either L_1 or R_1, in the first time unit and, in the second time unit, the two informed vertices call the leaders of L_2 and R_2; following this, the broadcast proceeds as in scheme $\mathscr{S}'$, except that p optimizes its broadcast in the first segment, L_1 or R_1, of which it is part. The cost of the first two time units is unchanged. By our inductive hypothesis regarding the cost of broadcast from p to its segment, this results in a broadcast scheme with cost less than or equal to $\mathscr{S}'$, which is less than or equal to the original, optimal scheme from a terminal vertex. □

5. Cumulative cost on paths with length of a power of two

This section derives a simple form of the cumulative cost $C(n, p, \lceil \log_2 n \rceil)$ when $n = 2^m$ for some integer m. In his study of line broadcasting in cycles, Kane proved the following theorem.

Theorem 3 (Kane [9]).

$$\widetilde{C}(2^m, m) = \frac{2^m \cdot m}{3} + \frac{2^m - (-1)^m}{9}.$$

In this section, we derive a simple form of $C(2^m, 1, m)$. First, recall that $D(n,t)$ is defined as

$$D(n,t) = \min_{1 \leqslant p \leqslant n} \{C(n,p,t) + p\}.$$

Then, for any $n \geqslant 1$, $C(n, 1, t)$ is recursively defined as follows:

$$C(n, 1, t) = \min_i \{C(i, 1, t-1) + i - 1 + D(n-i, t-1)\}.$$

If $n=2^m$ and $t=m$, then since $n-2^{t-1}=2^{t-1}=2^{m-1}$ $(=n/2)$, $C(n,1,m)$ is represented as

$$C(n,1,m)=C\left(\frac{n}{2},1,m-1\right)+\frac{n}{2}-1+D\left(\frac{n}{2},m-1\right). \tag{3}$$

By a similar observation, if $n=2^m$ and $t=m$, $D(2^m,m)$ and $\widetilde{C}(2^m,m)$ are represented as follows:

$$D(n,m)=\widetilde{C}\left(\frac{n}{2},m-1\right)+D\left(\frac{n}{2},m-1\right)+\frac{n}{2},$$

and

$$\widetilde{C}(n,m)=\widetilde{C}\left(\frac{n}{4},m-2\right)+D\left(\frac{n}{2},m-1\right)+D\left(\frac{n}{4},m-2\right)+\frac{n}{4}-1.$$

By solving the recurrences, we have the following theorem.

Theorem 4. *If $n=2^m$, then $C(2^m,1,m)$ is given as*

$$C(2^m,1,m)=\frac{2^m\cdot m}{3}+\frac{7(2^m)-5m-6}{9}+\frac{1}{9}\left\lfloor\frac{2^m-1}{2}\right\rfloor.$$

Proof. See [6]. □

Cost $C(2^m,1,m)$ is calculated as in Table 1.

By Theorems 3 and 4, we have the following corollary.

Corollary 2. *Let $\Delta C=C(2^m,1,m)-\widetilde{C}(2^m,m)$.*

$$\Delta C=\frac{2(2^m-1)}{3}-\frac{5m}{9}+\frac{1}{9}\left(\left\lfloor\frac{m-1}{2}\right\rfloor+(-1)^m\right).$$

We now consider the cost of line broadcasting in a path of length 2^m from source vertices that are near to a terminal vertex. We see that cost drops off significantly in the neighborhood of a terminal vertex in such paths.

Theorem 5. *Cost $C(2^m,2,m)=C(2^m,1,m)-(m-1)$ for $m\geqslant 2$.*

Proof. Our proof is by induction on m. We see the proposition is true for P_4, as the cumulative cost from vertex 1 is 4 and from vertex 2 it is 3. We assume it is true for $2\leqslant m\leqslant k$, and consider the case for $m=k+1$. Vertex 2 must call the leader of the other half of the path, just as vertex 1 does in the first time unit. The leader of the other half must be the same; however, the cost of a call from vertex 2 is one less. Then, vertex 2 must complete the line broadcast in its half of the path, now of size 2^k; by our inductive hypothesis, we know it can be done with cost that is $k-1$ less than from vertex 1. Thus, vertex 2 completes the line broadcast in path of length 2^{k+1} with cost reduced by k from the cost of a line broadcast from vertex 1, completing our proof. □

Table 1
Cumulative cost $C(2^m, 1, m)$

m	Cost	2^m
1	1	2
2	4	4
3	12	8
4	31	16
5	75	32
6	174	64
7	394	128
8	877	256
9	1929	512
10	4204	1024
11	9096	2048
12	19563	4096
13	41863	8192
14	89194	16384
15	189318	32768
16	400489	65536

Theorem 6. *Cost* $C(2^m, 3, m) = C(2^m, 2, m) - (m-2)$ *for* $m \geqslant 3$.

Proof. Again, our proof is by induction on m. First, considering P_4, the cumulative cost from vertex 1 is four and from vertex 2 it is three. By symmetry, we know that the cost from vertex 3 is also 3. When looking at P_8, we see that the cost to call the 'other half' is one less from vertex 3 than it is from vertex 2; the costs are equal in their respective segments. This establishes our base case for $m=3$. We now assume our proposition is true for $3 \leqslant m \leqslant k$, and consider the case for $m = k+1$. The argument to complete the proof is as in the previous theorem. □

6. Conclusions

In this paper, we continue the investigation of cumulative cost for line broadcasting begun by Kane in his work on cycle graphs. We establish a set of basic results regarding cumulative cost in paths, determining that indeed terminal vertices have the greatest cost over all vertices as source in a path. By Theorems 1 and 2, we have that for any $n \geqslant 2$,

$$C(n,1,t) - n + 2 \leqslant C(n,p,t) \leqslant C(n,1,t)$$

for any $1 \leqslant p \leqslant n$ and $t \geqslant \lceil \log_2 n \rceil$. Furthermore, we provide closed-form solutions for the cost from a terminal vertex and from its two nearest neighbors in paths of length 2^m, when broadcast time is minimized as m time units.

Before concluding, we consider cases where the time to complete broadcast is on the order of the number of vertices in the path. The least cost possible from any vertex of P_n is of course $n-1$, as every edge must be traversed at least once (i.e., all $n-1$ non-source vertices must be called) if all vertices are to be informed. We have the following, straightforward results regarding cumulative cost when time is near or above $n-1$ time units.

Observation 1. $C(n,1,t) = n-1$ *for* $t \geqslant n-1$.

Lemma 8. $C(n,1,n-2) = n$ *for* $n > 3$.

Proof. One call of length 2 is needed to reach the other end (i.e. vertex n). If there is another time unit left to make a call to 'fill in' behind the length 2 call, then this scheme is optimal. This is the case for all paths of length greater than 3. □

Lemma 9. $C(n,1,n-3) = n+1$ *for* $n > 5$.

Proof. Two calls of length 2 (or one of length 3) with 'fill in' calls to follow are needed in this case. □

We have implemented a recursive procedure that computes minimum cumulative cost $C(n,p,t)$ from any location p in a path of any length n and for any time $t \leqslant n$, catching results for smaller n, p and t to make the process reasonably efficient. The algorithm reflects the following definition, based upon our result as to contiguity of a minimum-cost line broadcast scheme, where $D(n-m,t-1)$ is as defined earlier:

$$C(n,p,t) = \min_{p \leqslant m < n} \{C(m,p,t-1) + m - p + D(n-m,t-1)\}.$$

Further closed-form characterizations of the cumulative cost function $C(2^m,p,m)$ for other position p, as well as for other path lengths for any given time remain as open problems. Looking at classes of graphs other than paths or cycles, e.g., binary, binomial, or general trees, also represents an opportunity for further research in determining minimum-cost line broadcast schemes. Results for such structures do not follow directly from results obtained for cycles and paths as the nested and contiguity properties are likely to no longer hold, i.e, calls will be placed through informed vertices in optimal schemes.

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