



Fix-Mahonian calculus, I: Two transformations

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Abstract

We construct two bijections of the symmetric group \mathfrak{S}_n onto itself that enable us to show that three new three-variable statistics are equidistributed with classical statistics involving the number of fixed points. The first one is equidistributed with the triplet (fix, des, maj), the last two with (fix, exc, maj), where “fix”, “des”, “exc” and “maj” denote the number of fixed points, the number of descents, the number of excedances and the major index, respectively.

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*Tu es Petrus, et super hanc petram,
aedificavisti Lacim Uqam tuam.
To Pierre Leroux, Montreal, Sept. 2006,
on the occasion of the LerouxFest.*

1. Introduction

In this paper *fix-Mahonian calculus* is understood to mean the study of multivariable statistics on the symmetric group \mathfrak{S}_n which involve the number of fixed points “fix” as a marginal component. As for the two transformations mentioned in the title, they make it possible to show that the new statistics defined below are equidistributed with the classical ones.

The *descent set*, $\text{DES } w$, and *rise set*, $\text{RISE } w$, of a word $w = x_1x_2 \cdots x_n$, whose letters are nonnegative integers, are respectively defined as being the *subsets*

$$\text{DES } w := \{i : 1 \leq i \leq n-1, x_i > x_{i+1}\}; \quad (1.1)$$

$$\text{RISE } w := \{i : 1 \leq i \leq n, x_i \leq x_{i+1}\}. \quad (1.2)$$

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By convention, $x_0 = x_{n+1} = +\infty$. If $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ is a permutation of $12\cdots n$, we can then consider $\text{RISE } \sigma$, but also a new statistic $\text{RIZE } \sigma$, which is simply the rise set of the word w derived from σ by replacing each *fixed point* $\sigma(i) = i$ by 0. For instance, with $\sigma = 32541$ having the two fixed points 2, 4, we get $w = 30501$, so that $\text{RISE } \sigma = \{2, 5\}$ and $\text{RIZE } \sigma = \text{RISE } w = \{2, 4, 5\}$. It is quite unexpected that the two set-theoretic statistics “RISE” and “RIZE” are equidistributed on each symmetric group \mathfrak{S}_n . Such an equidistribution property, which is a consequence of our [Theorem 1.4](#), has become a powerful tool, as it has enabled Xin and the second author [6] to give an immediate proof of a conjecture of Stanley [9] on alternating permutations. More importantly, the equidistribution properties are proved by means of new transformations of the symmetric group, the bijections Φ and F_3 introduced in the sequel, whose properties will be fully exploited in our next paper [2]. Those transformations will be described not directly on \mathfrak{S}_n , but on classes of *shuffles*, as now introduced.

Let $0 \leq m \leq n$ and let v be a nonempty word of length m , whose letters are *positive integers* (with possible repetitions). Designate by $\text{Sh}(0^{n-m}v)$ the set of all *shuffles* of the words 0^{n-m} and v , that is, the set of all rearrangements of the juxtaposition product $0^{n-m}v$, whose longest *subword* of positive letters is v . Let $w = x_1x_2\cdots x_n$ be a word from $\text{Sh}(0^{n-m}v)$. It is convenient to write: $\text{Pos } w := v$, $\text{Zero } w := \{i : 1 \leq i \leq n, x_i = 0\}$, $\text{zero } w := \#\text{Zero } w (= n - m)$, so that w is completely characterized by the pair $(\text{Zero } w, \text{Pos } w)$.

The *major index* of w is defined by

$$\text{maj } w := \sum_{i \geq 1} i \quad (i \in \text{DES } w), \tag{1.3}$$

and a new integral-valued statistic “mafz” by

$$\text{mafz } w := \sum_{i \in \text{Zero } w} i - \sum_{i=1}^{\text{zero } w} i + \text{maj Pos } w. \tag{1.4}$$

Note that the first three definitions are also valid for each arbitrary word with nonnegative letters. The link of “mafz” with the statistic “maf” introduced in [1] for permutations will be discussed below.

We shall also be interested in shuffle classes $\text{Sh}(0^{n-m}v)$ when the word v is a *derangement* of the set $[m] := \{1, 2, \dots, m\}$, that is, when the word $v = y_1y_2\cdots y_m$ is a permutation of $12\cdots m$ and $y_i \neq i$ for all i . For short, v is a *derangement of order m*. Let $w = x_1x_2\cdots x_n$ a be word from the shuffle class $\text{Sh}(0^{n-m}v)$. Then $v = y_1y_2\cdots y_m = x_{j_1}x_{j_2}\cdots x_{j_m}$ for a certain sequence $1 \leq j_1 < j_2 < \cdots < j_m \leq n$. Let “red” be the increasing bijection of $\{j_1, j_2, \dots, j_m\}$ onto $[m]$. Say that each positive letter x_k of w is *excedent* (resp. *subexcedent*) if and only if $x_k > \text{red } k$ (resp. $x_k < \text{red } k$). Another kind of rise set, denoted by $\text{RISE}^\bullet w$, can then be introduced as follows.

Say that $i \in \text{RISE}^\bullet w$ if and only if $1 \leq i \leq n$ and if one of the following conditions holds (assuming that $x_{n+1} = +\infty$):

- (1) $0 < x_i < x_{i+1}$;
- (2) $x_i = x_{i+1} = 0$;
- (3) $x_i = 0$ and x_{i+1} is excedent;
- (4) x_i is subexcedent and $x_{i+1} = 0$.

Note that if $x_i = 0$ and x_{i+1} is subexcedent, then $i \in \text{RISE } w \setminus \text{RISE}^\bullet w$, while if x_i is subexcedent and $x_{i+1} = 0$, then $i \in \text{RISE}^\bullet w \setminus \text{RISE } w$.

RISE w	w	$\Phi(w)$	$\text{RISE}^\bullet \Phi(w)$
1, 2, 4, 5	00312	00312	1, 2, 4, 5
1, 3, 4, 5	03012	03120	1, 3, 4, 5
1, 4, 5	03102	03012	1, 4, 5
1, 3, 5	03120	03102	1, 3, 5
2, 3, 4, 5	30012	31200	2, 3, 4, 5
2, 4, 5	30102	30012	2, 4, 5
	31200	31020	
2, 3, 5	30120	30102	2, 3, 5
3, 4, 5	31002	30120	3, 4, 5
3, 5	31020	31002	3, 5

Sh(0^2312)

maj w	w	$\mathbf{F}_3(w)$	mafz $\mathbf{F}_3(w)$
2	12001	00121	2
3	01201	01021	3
4	00121	10021	4
	10201	01201	
5	10021	10201	5
	12100	01210	
6	01021	12001	6
	12010	10210	
7	01210	12010	7
8	10210	12100	8

Sh(0^2121)

Fig. 1.

Example. Let $v = 512364$ be a derangement of order 6. Its excedent letters are 5, 6. Let $w = 501200364 \in \text{Sh}(0^3v)$. Then, $\text{RISE } w = \{2, 3, 5, 6, 7, 9\}$ and $\text{RISE}^\bullet w = \{3, 4, 5, 7, 9\}$. Also $\text{mafz } w = (2 + 5 + 6) - (1 + 2 + 3) + \text{maj}(512364) = 7 + 6 = 13$.

Theorem 1.1. For each derangement v of order m and each integer $n \geq m$ the transformation Φ constructed in Section 2 is a bijection of $\text{Sh}(0^{n-m}v)$ onto itself having the property that

$$\text{RISE } w = \text{RISE}^\bullet \Phi(w) \tag{1.5}$$

holds for every $w \in \text{Sh}(0^{n-m}v)$.

Theorem 1.2. For each arbitrary word v of length m with positive letters and each integer $n \geq m$ the transformation \mathbf{F}_3 constructed in Section 4 is a bijection of $\text{Sh}(0^{n-m}v)$ onto itself having the property that

$$\text{maj } w = \text{mafz } \mathbf{F}_3(w); \tag{1.6}$$

$$Lw = L\mathbf{F}_3(w) \quad (\text{“}L\text{” for “last” or rightmost letter}); \tag{1.7}$$

hold for every $w \in \text{Sh}(0^{n-m}v)$.

We emphasize the fact that **Theorem 1.1** is restricted to the case where v is a derangement, while **Theorem 1.2** holds for an arbitrary word v with possible repetitions. In **Fig. 1** we can see that “RISE” and “RISE[•]” (resp. “maj” and “mafz”) are equidistributed on the shuffle class $\text{Sh}(0^2312)$ (resp. $\text{Sh}(0^2121)$).

Those two transformations are fully exploited once we know how to map those shuffle classes onto the symmetric groups. The permutations from the symmetric group \mathfrak{S}_n will be regarded as linear words $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n)$. If σ is such a permutation, let $\text{FIX } \sigma$ denote the set of its fixed points, i.e., $\text{FIX } \sigma := \{i : 1 \leq i \leq n, \sigma(i) = i\}$ and let $\text{fix } \sigma := \#\text{FIX } \sigma$. Let (j_1, j_2, \dots, j_m) be the increasing sequence of the integers k such that $1 \leq k \leq n$ and $\sigma(k) \neq k$ and “red” be the increasing bijection of $\{j_1, j_2, \dots, j_m\}$ onto $[m]$. The word $w = x_1x_2 \cdots x_n$ derived from $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n)$ by replacing each fixed point by 0 and each other letter $\sigma(j_k)$ by $\text{red } \sigma(j_k)$ will be denoted by $\text{ZDer}(\sigma)$. Also let

$$\text{Der } \sigma := \text{red } \sigma(j_1) \text{red } \sigma(j_2) \cdots \text{red } \sigma(j_m), \tag{1.8}$$

so that $\text{Der } \sigma$ is the word derived from $\text{ZDer}(\sigma)$ by deleting all the zeros. Accordingly, $\text{Der } \sigma = \text{Pos } \text{ZDer}(\sigma)$.

It is important to notice that $\text{Der } \sigma$ is a *derangement* of order m . Also $\sigma(j_k)$ is excedent in σ (i.e. $\sigma(j_k) > j_k$) if and only $\text{red } \sigma(j_k)$ is excedent in $\text{Der } \sigma$ (i.e. $\text{red } \sigma(j_k) > \text{red } j_k$)

Recall that the statistics “DES,” “RISE” and “maj” are also valid for permutations $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ and that the statistics “des” (*number of descents*) and “exc” (*number of excedances*) are defined by

$$\text{des } \sigma := \#\text{DES } \sigma; \tag{1.9}$$

$$\text{exc } \sigma := \#\{i : 1 \leq i \leq n - 1, \sigma(i) > i\}. \tag{1.10}$$

We further define:

$$\text{DEZ } \sigma := \text{DES ZDer}(\sigma); \tag{1.11}$$

$$\text{RIZE } \sigma := \text{RISE ZDer}(\sigma); \tag{1.12}$$

$$\text{dez } \sigma := \#\text{DEZ } \sigma = \text{des ZDer}(\sigma); \tag{1.13}$$

$$\text{maz } \sigma := \text{maj ZDer}(\sigma); \tag{1.14}$$

$$\text{maf } \sigma := \text{mafz ZDer}(\sigma). \tag{1.15}$$

As the zeros of $\text{ZDer}(\sigma)$ correspond to the fixed points of σ , we also have

$$\text{maf } \sigma := \sum_{i \in \text{FIX } \sigma} i - \sum_{i=1}^{\text{fix } \sigma} i + \text{maj Der } \sigma. \tag{1.16}$$

Example. Let $\sigma = 821356497$; then $\text{DES } \sigma = \{1, 2, 6, 8\}$, $\text{des } \sigma = 4$, $\text{maj } \sigma = 17$, $\text{exc } \sigma = 2$. Furthermore, $\text{ZDer}(\sigma) = w = 501200364$ and $\text{Pos } w = \text{Der } \sigma = 512364$ is a derangement of order 6. We have $\text{FIX } \sigma = \{2, 5, 6\}$, $\text{fix } \sigma = 3$, $\text{DEZ } \sigma = \{1, 4, 8\}$, $\text{RIZE } w = \{2, 3, 5, 6, 7, 9\}$, $\text{dez } = 3$, $\text{maz } \sigma = 13$ and $\text{maf } \sigma = (2 + 5 + 6) - (1 + 2 + 3) + \text{maj}(512364) = 7 + 6 = 13$.

For each $n \geq 0$ let D_n be the set of all derangements of order n and $\mathfrak{S}_n^{\text{Der}}$ be the union: $\mathfrak{S}_n^{\text{Der}} := \bigcup_{m,v} \text{Sh}(0^{n-m}v)$ ($0 \leq m \leq n, v \in D_m$).

Proposition 1.3. *The map ZDer is a bijection of \mathfrak{S}_n onto $\mathfrak{S}_n^{\text{Der}}$ having the following properties:*

$$\text{RIZE } \sigma = \text{RISE ZDer}(\sigma) \quad \text{and} \quad \text{RISE } \sigma = \text{RISE}^\bullet \text{ZDer}(\sigma). \tag{1.17}$$

Proof. It is evident to verify that ZDer is bijective and to define its inverse ZDer^{-1} . On the other hand, we have $\text{RIZE} = \text{RISE ZDer}$ by definition. Finally, let $w = x_1x_2\cdots x_n = \text{ZDer}(\sigma)$ and $\sigma(i) < \sigma(i + 1)$ for $1 \leq i \leq n - 1$. Four cases are to be considered:

- (1) both i and $i + 1$ are not fixed by σ and $0 < x_i < x_{i+1}$;
- (2) both i and $i + 1$ are fixed points and $x_i = x_{i+1} = 0$;
- (3) $\sigma(i) = i$ and $\sigma(i + 1)$ is excedent; then $x_i = 0$ and x_{i+1} is also excedent;
- (4) $\sigma(i) < i < i + 1 = \sigma(i + 1)$; then x_i is subexcedent and $x_{i+1} = 0$.

We recover the four cases considered in the definition of RISE^\bullet . The case $i = n$ is banal to study. \square

We next form the two chains

$$\Phi : \sigma \xrightarrow{\text{ZDer}} w \xrightarrow{\Phi} w' \xrightarrow{\text{ZDer}^{-1}} \sigma'; \tag{1.18}$$

$$F_3 : \sigma \xrightarrow{\text{ZDer}} w \xrightarrow{F_3} w'' \xrightarrow{\text{ZDer}^{-1}} \sigma''. \tag{1.19}$$

The next theorem is then a consequence of [Theorems 1.1](#) and [1.2](#) and [Proposition 1.3](#).

Theorem 1.4. *The mappings Φ, F_3 defined by (1.18) and (1.19) are bijections of \mathfrak{S}_n onto itself and have the following properties:*

$$(\text{fix}, \text{RIZE}, \text{Der}) \sigma = (\text{fix}, \text{RISE}, \text{Der}) \Phi(\sigma); \tag{1.20}$$

$$(\text{fix}, \text{maz}, \text{Der}) \sigma = (\text{fix}, \text{maf}, \text{Der}) F_3(\sigma); \tag{1.21}$$

$$(\text{fix}, \text{maj}, \text{Der}) \sigma = (\text{fix}, \text{maf}, \text{Der}) F_3 \circ \Phi^{-1}(\sigma); \tag{1.22}$$

for every σ from \mathfrak{S}_n .

It is evident that if $\text{Der } \sigma = \text{Der } \tau$ holds for a pair of permutations σ, τ of order n , then $\text{exc } \sigma = \text{exc } \tau$. Since $\text{DES } \sigma = [n] \setminus \text{RISE } \sigma$ and $\text{DEZ } \sigma = [n] \setminus \text{RIZE } \sigma$ it follows from (1.20) that

$$(\text{fix}, \text{DEZ}, \text{exc}) \sigma = (\text{fix}, \text{DES}, \text{exc}) \Phi(\sigma); \tag{1.23}$$

$$(\text{fix}, \text{dez}, \text{maz}, \text{exc}) \sigma = (\text{fix}, \text{des}, \text{maj}, \text{exc}) \Phi(\sigma). \tag{1.24}$$

On the other hand, (1.21) implies that

$$(\text{fix}, \text{maz}, \text{exc}) \sigma = (\text{fix}, \text{maf}, \text{exc}) F_3(\sigma). \tag{1.25}$$

As a consequence we obtain the following corollary.

Corollary 1.5. *The two triplets $(\text{fix}, \text{dez}, \text{maz})$ and $(\text{fix}, \text{des}, \text{maj})$ are equidistributed over \mathfrak{S}_n . Moreover, the three triplets $(\text{fix}, \text{exc}, \text{maz})$, $(\text{fix}, \text{exc}, \text{maj})$ and $(\text{fix}, \text{exc}, \text{maf})$ are also equidistributed over \mathfrak{S}_n .*

The distributions of $(\text{fix}, \text{des}, \text{maj})$ and $(\text{fix}, \text{exc}, \text{maj})$ have been calculated by Gessel and Reutenauer ([5], Theorem 8.4) and by Shareshian and Wachs [8], respectively, using the algebra of the q -series (see, e.g., Gasper and Rahman ([4], chap. 1). Let

$$A_n(s, t, q, Y) := \sum_{\sigma \in \mathfrak{S}_n} s^{\text{exc } \sigma} t^{\text{des } \sigma} q^{\text{maj } \sigma} Y^{\text{fix } \sigma} \quad (n \geq 0).$$

Then, they respectively derived the identities

$$\sum_{n \geq 0} A_n(1, t, q, Y) \frac{u^n}{(t; q)_{n+1}} = \sum_{r \geq 0} t^r \left(1 - u \sum_{i=0}^r q^i \right)^{-1} \frac{(u; q)_{r+1}}{(uY; q)_{r+1}}; \tag{1.26}$$

$$\sum_{n \geq 0} A_n(s, 1, q, Y) \frac{u^n}{(q; q)_n} = \frac{(1 - sq)e_q(Yu)}{e_q(squ) - sqe_q(u)}. \tag{1.27}$$

In our third paper [3] we have shown that the factorial generating function for the four-variable polynomials $A_n(s, t, q, Y)$ could be evaluated under the form

$$\sum_{n \geq 0} A_n(s, t, q, Y) \frac{u^n}{(t; q)_{n+1}} = \sum_{r \geq 0} t^r \frac{(1 - sq)(u; q)_r (usq; q)_r}{((u; q)_r - sq(usq; q)_r)(uY; q)_{r+1}}, \tag{1.28}$$

the two identities (1.26) and (1.27) becoming simple specializations. We then know the distributions over \mathfrak{S}_n of the triplets $(\text{fix}, \text{dez}, \text{maz})$, $(\text{fix}, \text{exc}, \text{maz})$ and $(\text{fix}, \text{exc}, \text{maf})$ and also the distribution of the quadruplet $(\text{fix}, \text{dez}, \text{maz}, \text{exc})$. Note that the statistic “maf” was introduced by Clarke et al. [1]. Although it was not explicitly stated, their bijection “CHZ” of \mathfrak{S}_n onto itself satisfies identity (1.22) when $F_3 \circ \Phi^{-1}$ is replaced by “CHZ”.

As is shown in Section 2, the transformation Φ is described as a composition product of bijections ϕ_l . The image $\phi_l(w)$ of each word w from a shuffle class $\text{Sh}(0^{n-m}v)$ is obtained

by moving its l -th zero, to the right or to the left, depending on its preceding and following letters. The description of the inverse bijection Ψ of Φ follows an analogous pattern. The verification of identity (1.5) requires some attention and is done in Section 3. The construction of the transformation F_3 is given in Section 4. Recall that F_3 maps each shuffle class $\text{Sh}(0^{n-m}v)$ onto itself, the word v being an arbitrary word with nonnegative letters. Very much like the *second fundamental transformation* (see, e.g., [7], p. 201, Algorithm 10.6.1) the construction of F_3 is defined by induction on the length of the words and preserves the *rightmost* letter.

2. The bijection Φ

Let v be a *derangement* of order m and $w = x_1x_2 \cdots x_n$ be a word from the shuffle class $\text{Sh}(0^{n-m}v)$ ($0 \leq n \leq m$), so that $v = x_{j_1}x_{j_2} \cdots x_{j_m}$ for $1 \leq j_1 < j_2 < \cdots < j_m \leq n$. Let “red” (“reduction”) be the increasing bijection of $\{j_1, j_2, \dots, j_m\}$ onto the interval $[m]$. Remember that a positive letter x_k of w is said to be *excedent* (resp. *subexcedent*) if and only if $x_k > \text{red } k$ (resp. $x_k < \text{red } k$). Accordingly, a letter is *non-subexcedent* if it is either equal to 0 or excedent.

We define n bijections ϕ_l ($1 \leq l \leq n$) of $\text{Sh}(0^{n-m}v)$ onto itself in the following manner: for each l such that $n - m + 1 \leq l \leq n$ let $\phi_l(w) := w$. When $1 \leq l \leq n - m$, let x_j denote the l -th letter of w , equal to 0, when w is read *from left to right*. Three cases are now considered (by convention, $x_0 = x_{n+1} = +\infty$):

- (1) x_{j-1}, x_{j+1} both non-subexcedent;
- (2) x_{j-1} non-subexcedent, x_{j+1} subexcedent; or x_{j-1}, x_{j+1} both subexcedent with $x_{j-1} > x_{j+1}$;
- (3) x_{j-1} subexcedent, x_{j+1} non-subexcedent; or x_{j-1}, x_{j+1} both subexcedent with $x_{j-1} < x_{j+1}$.

When case (1) holds, let $\phi_l(w) := w$.

When case (2) holds, determine the *greatest* integer $k \geq j + 1$ such that

$$x_{j+1} < x_{j+2} < \cdots < x_k < \text{red}(k),$$

so that

$$w = x_1 \cdots x_{j-1} 0 x_{j+1} \cdots x_k x_{k+1} \cdots x_n$$

and define

$$\phi_l(w) := x_1 \cdots x_{j-1} x_{j+1} \cdots x_k 0 x_{k+1} \cdots x_n.$$

When case (3) holds, determine the *smallest* integer $i \leq j - 1$ such that

$$\text{red}(i) > x_i > x_{i+1} > \cdots > x_{j-1},$$

so that

$$w = x_1 \cdots x_{i-1} x_i \cdots x_{j-1} 0 x_{j+1} \cdots x_n$$

and define

$$\phi_l(w) := x_1 \cdots x_{i-1} 0 x_i \cdots x_{j-1} x_{j+1} \cdots x_n.$$

It is important to note that ϕ_l has no action on the 0’s other than the l -th one. Then the mapping Φ in Theorem 1.1 is defined to be the composition product

$$\Phi := \phi_1\phi_2 \cdots \phi_{n-1}\phi_n.$$

Example. The following word w has four zeros, so $\Phi(w)$ can be reached in four steps:

$$\begin{array}{l}
 \text{Id} = 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \\
 w = 5 \ \mathbf{0} \ 1 \ 2 \ \mathbf{0} \ \mathbf{0} \ 3 \ 6 \ \mathbf{0} \ 7 \ 4 \quad j = 9, \text{ apply } \phi_4, \text{ case (1);} \\
 \quad \quad \quad 5 \ \mathbf{0} \ 1 \ 2 \ \mathbf{0} \ \mathbf{0} \ 3 \ 6 \ \mathbf{0} \ 7 \ 4 \quad j = 6, \text{ apply } \phi_3, \text{ case (2), } k = 7; \\
 \quad \quad \quad 5 \ \mathbf{0} \ 1 \ 2 \ \mathbf{0} \ 3 \ \mathbf{0} \ 6 \ \mathbf{0} \ 7 \ 4 \quad j = 5, \text{ apply } \phi_2, \text{ case (3), } i = 4; \\
 \quad \quad \quad 5 \ \mathbf{0} \ 1 \ \mathbf{0} \ 2 \ 3 \ \mathbf{0} \ 6 \ \mathbf{0} \ 7 \ 4 \quad j = 2, \text{ apply } \phi_1, \text{ case (2), } k = 3; \\
 \Phi(w) = 5 \ 1 \ \mathbf{0} \ \mathbf{0} \ 2 \ 3 \ \mathbf{0} \ 6 \ \mathbf{0} \ 7 \ 4.
 \end{array}$$

We have: $\text{RISE } w = \text{RISE}^\bullet \Phi(w) = \{2, 3, 5, 6, 7, 9, 11\}$, as desired.

To verify that Φ is bijective, we introduce a class of bijections ψ_l , whose definitions are parallel to the definitions of the ϕ_l 's. Let $w = x_1x_2 \cdots x_n \in \text{Sh}(0^{n-m}v)$ ($0 \leq m \leq n$). For each l such that $n - m + 1 \leq l \leq n$ let $\psi_l(w) := w$. When $1 \leq l \leq n - m$, let x_j denote the l -th letter of w , equal to 0, when w is read from left to right. Consider the following three cases (remember that $x_0 = x_{n+1} = +\infty$ by convention):

(1') x_{j-1}, x_{j+1} both non-subexcedent;

(2') x_{j-1} subexcedent, x_{j+1} non-subexcedent; or x_{j-1}, x_{j+1} both subexcedent with $x_{j-1} > x_{j+1}$;

(3') x_{j-1} non-subexcedent, x_{j+1} subexcedent; or x_{j-1}, x_{j+1} both subexcedent with $x_{j-1} < x_{j+1}$.

When case (1') holds, let $\psi_l(w) := w$.

When case (2') holds, determine the *smallest* integer $i \leq j - 1$ such that

$$x_i < x_{i+1} < \cdots < x_{j-1} < \text{red}(j - 1),$$

so that

$$w = x_1 \cdots x_{i-1} x_i \cdots x_{j-1} \mathbf{0} x_{j+1} \cdots x_n$$

and define

$$\psi_l(w) := x_1 \cdots x_{i-1} \mathbf{0} x_i \cdots x_{j-1} x_{j+1} \cdots x_n.$$

When case (3') holds, determine the *greatest* integer $k \geq j + 1$ such that

$$\text{red}(j + 1) > x_{j+1} > x_{j+2} > \cdots > x_k,$$

so that

$$w = x_1 \cdots x_{j-1} \mathbf{0} x_{j+1} \cdots x_k x_{k+1} \cdots x_n$$

and define

$$\psi_l(w) := x_1 \cdots x_{j-1} x_{j+1} \cdots x_k \mathbf{0} x_{k+1} \cdots x_n.$$

We now observe that when case (2) (resp. (3)) holds for w , then case (2') (resp. (3')) holds for $\phi_l(w)$. Also, when case (2') (resp. (3')) holds for w , then case (2) (resp. (3)) holds for $\psi_l(w)$. Therefore

$$\phi_l \psi_l = \psi_l \phi_l = \text{Identity map}$$

and the product $\Psi := \psi_n \psi_{n-1} \cdots \psi_2 \psi_1$ is the inverse bijection of Φ .

3. Verification of $\text{RISE } w = \text{RISE}^\bullet \Phi(w)$

Let us introduce an alternative definition for Φ . Let w belong to $\text{Sh}(0^{n-m}v)$ and w' be a nonempty left factor of w , of length n' . Let w' have p' letters equal to 0. If $p' \geq 1$, write

$$w' = x_1 \cdots x_{j-1} 0^h x_{j+h} x_{j+h+1} \cdots x_{n'},$$

where $1 \leq h \leq p'$, $x_{j-1} \neq 0$ and where the right factor $x_{j+h} x_{j+h+1} \cdots x_{n'}$ contains no 0. By convention $x_{j+h} := +\infty$ if $j+h = n'+1$. If x_{j+h} is subexcedent let k be the *greatest* integer $k \geq j+h$ such that $x_{j+h} < x_{j+h+1} < \cdots < x_k < \text{red}(k)$. If x_{j-1} is subexcedent let i be the *smallest* integer $i \leq j-1$ such that $\text{red}(i) > x_i > x_{i+1} > \cdots > x_{j-1}$. Examine four cases:

(1) If x_{j-1} and x_{j+h} are both excedent, let

$$u := x_1 \cdots x_{j-1}, \quad u' := 0^h x_{j+h} x_{j+h+1} \cdots x_{n'}$$

and define

$$\theta(u') := u'.$$

(2) If x_{j-1} is excedent and x_{j+h} subexcedent, or if x_{j-1}, x_{j+h} are both subexcedent with $x_{j-1} > x_{j+h}$, let

$$u := x_1 \cdots x_{j-1}, \quad u' := 0^h x_{j+h} \cdots x_k x_{k+1} \cdots x_{n'}$$

and define

$$\theta(u') := x_{j+h} \cdots x_k 0^h x_{k+1} \cdots x_{n'}.$$

(3) If x_{j-1}, x_{j+h} are both subexcedent with $x_{j-1} < x_{j+h}$, let

$$u := x_1 \cdots x_{i-1}, \quad u' := x_i \cdots x_{j-1} 0^h x_{j+h} \cdots x_k x_{k+1} \cdots x_{n'}$$

and define

$$\theta(u') := 0 x_i \cdots x_{j-1} x_{j+h} \cdots x_k 0^{h-1} x_{k+1} \cdots x_{n'}.$$

(4) If x_{j-1} is subexcedent and x_{j+h} excedent, let

$$u := x_1 \cdots x_{i-1}, \quad u' := x_i \cdots x_{j-1} 0^h x_{j+h} \cdots x_{n'}$$

and define

$$\theta(u') := 0 x_i \cdots x_{j-1} 0^{h-1} x_{j+h} \cdots x_{n'}.$$

By construction $w' = uu'$. Call it the *canonical factorization* of w' . In the four cases we evidently have

$$\text{RISE } u' = \text{RISE}^\bullet \theta(u'). \tag{3.1}$$

Define $\Theta(w')$ to be the three-term sequence

$$\Theta(w') := (u, u', \theta(u')). \tag{3.2}$$

Let q be the length of u . Then $q = j - 1$ in cases (1) and (2) and $q = i - 1$ in cases (3) and (4).

Lemma 3.1. *We have*

$$\text{RISE } x_q u' = \text{RISE}^\bullet x_q \theta(u'), \tag{3.3}$$

$$\text{RISE } x_q u' = \text{RISE}^\bullet 0\theta(u'), \quad \text{if } x_q \text{ is subexcedent.} \tag{3.4}$$

Proof. Let α (resp. β) be the leftmost letter of u' (resp. of $\theta(u')$). Because of (3.1) we only have to prove that $\text{RISE } x_q \alpha = \text{RISE}^\bullet x_q \beta$ for (3.3) and $\text{RISE } x_q \alpha = \text{RISE}^\bullet 0\beta$ when x_q is subexcedent for (3.4). Let us prove identity (3.3). There is nothing to do in case (1). In case (2) we have to verify $\text{RISE } x_{j-1} 0 = \text{RISE}^\bullet x_{j-1} x_{j+h}$. When x_{j-1} is excedent and x_{j+h} subexcedent, then $1 \notin \text{RISE } x_{j-1} 0$ and $1 \notin \text{RISE}^\bullet x_{j-1} x_{j+h}$. When x_{j-1}, x_{j+h} are both subexcedent with $x_{j-1} > x_{j+h}$, then $1 \notin \text{RISE } x_{j-1} 0$ and $1 \notin \text{RISE}^\bullet x_{j-1} x_{j+h}$.

In cases (3) and (4) we have to verify $\text{RISE } x_{i-1} x_i = \text{RISE}^\bullet x_{i-1} 0$. If $x_{i-1} = 0$ (resp. excedent), then $1 \in \text{RISE } x_{i-1} x_i$ (resp. $1 \notin \text{RISE } x_{i-1} x_i$) and $1 \in \text{RISE}^\bullet x_{i-1} 0$ (resp. $1 \notin \text{RISE}^\bullet x_{i-1} 0$). When x_{i-1} is subexcedent, then $x_{i-1} < x_i$ by definition of i . Hence $1 \in \text{RISE } x_{i-1} x_i$ and $1 \in \text{RISE}^\bullet x_{i-1} 0$.

We next prove identity (3.4). In case (1) x_q is always excedent, so that identity (3.4) need not be considered. In case (2) we have to verify $\text{RISE } x_{j-1} 0 = \text{RISE}^\bullet 0x_{j+h}$. But if x_{j-1} is subexcedent, then x_{j+h} is also subexcedent, so that the above two sets are empty. In cases (3) and (4) we have to verify $\text{RISE } x_{i-1} x_i = \text{RISE}^\bullet 00 = \{1\}$. But if x_{i-1} is subexcedent, then $x_{i-1} < x_i$ by definition of i , so $1 \in \text{RISE } x_{i-1} x_i$. \square

Now, if w has p letters equal to 0 with $p \geq 1$, it may be expressed as the juxtaposition product

$$w = w_1 0^{h_1} w_2 0^{h_2} \dots w_r 0^{h_r} w_{r+1},$$

where $h_1 \geq 1, h_2 \geq 1, \dots, h_r \geq 1$ and where the factors $w_1, w_2, \dots, w_r, w_{r+1}$ contain no 0 and w_2, \dots, w_r are nonempty. We may define $\Theta(w) := (u_r, u'_r, \theta(u'_r))$, where $w = u_r u'_r$ is the canonical factorization of w . As u_r is a left factor of w , we next define $\Theta(u_r) := (u_{r-1}, u'_{r-1}, \theta(u'_{r-1}))$, where $u_r = u_{r-1} u'_{r-1}$ is the canonical factorization of u_r , and successively $\Theta(u_{r-1}) := (u_{r-2}, u'_{r-2}, \theta(u'_{r-2}))$ with $u_{r-1} = u_{r-2} u'_{r-2}, \dots, \Theta(u_2) := (u_1, u'_1, \theta(u'_1))$ with $u_2 = u_1 u'_1$, so that $w = u_1 u'_1 u'_2 \dots u'_r$ and $\Phi(w) = u_1 \theta(u'_1) \theta(u'_2) \dots \theta(u'_r)$.

It can be verified that $u_r \theta(u'_r) = \phi_{p-h_r+1} \dots \phi_{p-1} \phi_p(w)$, $u_{r-1} \theta(u'_{r-1}) \theta(u'_r) = \phi_{p-h_r-h_{r-1}+1} \dots \phi_{p-1} \phi_p(w)$, etc.

With identities (3.1), (3.3) and (3.4) the proof of (1.5) is now completed.

Again, consider the word w of the preceding example:

$$w = 50120036074,$$

so that $r = 3, h_1 = 1, h_2 = 2, h_3 = 1, w_1 = 5, w_2 = 12$ and $w_3 = 36, w_4 = 74$. We have

$$\Theta(w) = (u_3, u'_3, \theta(u'_3)) = (50120036; 074; 074); \quad \text{case (1)}$$

$$\Theta(u_3) = (u_2, u'_2, \theta(u'_2)) = (501; 20036; 02306); \quad \text{case (3)}$$

$$\Theta(u_2) = (u_1, u'_1, \theta(u'_1)) = (5; 01; 10); \quad \text{case (2)}$$

$$\Phi(w) = u_1 \theta(u'_1) \theta(u'_2) \theta(u'_3) = 5 | 10 | 02306 | 074.$$

4. The transformation F_3

The bijection F_3 that we are now defining maps each shuffle class $\text{Sh}(0^{n-m}v)$ with v an arbitrary word of length m ($0 \leq m \leq n$) onto itself. When $n = 1$ the unique element of the shuffle class is sent onto itself. Also let $F_3(w) = w$ when $\text{des}(w) = 0$. Let $n \geq 2$ and assume

that $F_3(w')$ has been defined for all words w' with nonnegative letters, of length $n' \leq n - 1$. Further assume that (1.6) and (1.7) hold for all those words. Let w be a word of length n such that $\text{des}(w) \geq 1$. We may write

$$w = w'a0^r b,$$

where $a \geq 1, b \geq 0$ and $r \geq 0$. Three cases are considered:

(1) $a \leq b$; (2) $a > b, r \geq 1$; (3) $a > b, r = 0$.

In case (1) define $F_3(w) = F_3(w'a0^r b) := (F_3(w'a0^r))b$.

In case (2) we may write $F_3(w'a0^r) = w''0$ by Property (1.7). We then define

$$\gamma F_3(w'a0^r) := 0w'';$$

$$F_3(w) = F_3(w'a0^r b) := (\gamma F_3(w'a0^r))b = 0w''b.$$

In short, add one letter “0” to the left of $F_3(w'a0^r)$, then delete the rightmost letter “0” and add b to the right.

In case (3) remember that $r = 0$. Write

$$F_3(w'a) = 0^{m_1} x_1 v_1 0^{m_2} x_2 v_2 \cdots 0^{m_k} x_k v_k,$$

where $m_1 \geq 0, m_2, \dots, m_k$ are all positive; then x_1, x_2, \dots, x_k are positive letters and v_1, v_2, \dots, v_k are words with positive letters, possibly empty. Then define

$$\delta F_3(w'a) := x_1 0^{m_1} v_1 x_2 0^{m_2} v_2 x_3 \cdots x_k 0^{m_k} v_k;$$

$$F_3(w) = F_3(w'ab) := (\delta F_3(w'a))b.$$

In short, move each positive letter occurring just after a 0-factor of $F_3(w'a)$ to the beginning of that 0-factor and add b to the right.

Example.

$w = 00031220013$	
$F_3(0003) = 0003$	no descent
$F_3(00031) = \delta(0003)1 = 30001$	case (3)
$F_3(0003122) = 3000122$	case (1)
$F_3(00031220) = \delta(3000122)0 = 31000220$	case (3)
$F_3(000312200) = \gamma(31000220)0 = 031000220$	case (2)
$F_3(0003122001) = \gamma(031000220)1 = 0031000221$	case (2)
$F_3(00031220013) = 00310002213.$	

We have: $\text{maj } w = \text{maj}(00031220013) = 4 + 7 = 11$ and $\text{mafz } F_3(w) = \text{mafz}(00310002213) = (1 + 2 + 5 + 6 + 7) - (1 + 2 + 3 + 4 + 5) + (1 + 4) = 11$.

By construction the rightmost letter is preserved by F_3 . To prove (1.6) proceed by induction. Assume that $\text{mafz } w'a0^r = \text{mafz } F_3(w'a0^r)$ holds. In case (1) “maj” and “mafz” remain invariant when b is juxtaposed at the end. In case (2) we have $\text{maj } w = \text{maj}(w'a0^r b) = \text{maj}(w'a0^r)$, but $\text{mafz } \gamma F_3(w'a0^r) = \text{mafz } F_3(w'a0^r) - |w'a0^r|_{\geq 1}$ and $\text{mafz } (\gamma F_3(w'a0^r))b = \text{mafz } \gamma F_3(w'a0^r) + |w'a0^r|_{\geq 1}$, where $|w'a0^r|_{\geq 1}$ denotes the number of positive letters in $w'a0^r$. Hence (1.6) holds. In case (3) remember $r = 0$. We have $\text{maj}(w'ab) = \text{maj}(w'a) + |w'a|$, where $|w'a|$ denotes the length of the word $w'a$. But $\text{mafz } \delta F_3(w'a) = \text{mafz } F_3(w'a) + \text{zero}(w'a)$ and $\text{mafz } (\delta F_3(w'a))b = \text{mafz } \delta F_3(w'a) + |w'a|_{\geq 1}$. The equality holds for $b = 0$ and $b \geq 1$, as easily verified. As $\text{zero}(w'a) + |w'a|_{\geq 1} = |w'a|$, we have

$\text{mafz } \mathbf{F}_3(w) = \text{mafz } (\delta \mathbf{F}_3(w'a))b = \text{mafz } \mathbf{F}_3(w'a) + |w'a| = \text{maj } w'ab = \text{maj } w$. Thus (1.6) holds in the three cases.

To define the inverse bijection \mathbf{F}_3^{-1} of \mathbf{F}_3 we first need the inverses $\gamma^{-1}(w)$ and $\delta^{-1}(w)$ for each word w . Let $w = 0w'$ be a word whose first letter is 0. Define $\gamma^{-1}(w)$ to be the word derived from w by deleting the first letter 0 and adding one letter “0” to the right of w . Clearly, $\gamma^{-1}\gamma = \gamma\gamma^{-1}$ is the identity map.

Next, let w be a word whose first letter is positive. Define $\delta^{-1}(w)$ to be the word derived from w by moving each positive letter occurring just before a 0-factor of w to the end of that 0-factor. Again $\delta^{-1}\delta = \delta\delta^{-1}$ is the identity map.

We may write

$$w = cw'a0^r b,$$

where $a \geq 1, b \geq 0, c \geq 0$ and $r \geq 0$. Three cases are considered:

(1) $a \leq b$; (2) $a > b, c = 0$; (3) $a > b, c \geq 1$.

In case (1) define $\mathbf{F}_3^{-1}(w) := (\mathbf{F}_3^{-1}(cw'a0^r))b$.

In case (2) define $\mathbf{F}_3^{-1}(w) := (\gamma^{-1}(\mathbf{F}_3^{-1}(cw'a0^r)))b$.

In case (3) define $\mathbf{F}_3^{-1}(w) := (\delta^{-1}(\mathbf{F}_3^{-1}(cw'a0^r)))b$.

We end this section by proving a property of the transformation \mathbf{F}_3 , which will be used in our next paper [2].

Proposition 4.1. *Let w, w'' be two words with nonnegative letters, of the same length. If $\text{Zero } w = \text{Zero } w''$ and $\text{DES Pos } w = \text{DES Pos } w''$, then $\text{Zero } \mathbf{F}_3(w) = \text{Zero } \mathbf{F}_3(w'')$.*

Proof. To derive $\mathbf{F}_3(w)$ (resp. $\mathbf{F}_3(w'')$) from w (resp. w'') we have to consider one of the three cases (1), (2) or (3), described above, at each step. Because of the two conditions $\text{Zero } w = \text{Zero } w''$ and $\text{DES Pos } w = \text{DES Pos } w''$, case (i) ($i = 1, 2, 3$) is used at the j -th step in the calculation of $\mathbf{F}_3(w)$ if and only if the same case is used at that j -th step for the calculation of $\mathbf{F}_3(w'')$. Consequently the letters equal to 0 are in the same places in both words $\mathbf{F}_3(w)$ and $\mathbf{F}_3(w'')$. \square

By the very definition of $\Phi : \sigma \xrightarrow{\text{ZDer}} w \xrightarrow{\Phi} w' \xrightarrow{\text{ZDer}^{-1}} \sigma'$, given in (1.18) and of $\mathbf{F}_3 : \sigma \xrightarrow{\text{ZDer}} w \xrightarrow{\mathbf{F}_3} w'' \xrightarrow{\text{ZDer}^{-1}} \sigma''$, given in (1.19) we have $\Phi(\sigma) = \sigma$ and $\mathbf{F}_3(\sigma) = \sigma$ if σ is a derangement. In the next two tables we have calculated $\Phi(\sigma) = \sigma'$ and $\mathbf{F}_3(\sigma) = \sigma''$ for the fifteen non-derangement permutations σ of order 4.

Calculation of $\sigma' = \Phi(\sigma)$.

fix σ	Der σ	RISE σ	σ	w	w'	σ'	RISE σ'	Der σ'	fix σ'
4	e	1,2,3,4	1234	0000	0000	1234	1,2,3,4	e	4
2	21	1,2,4	1243	0021	0021	1243	1,2,4	21	2
		1,4	1324	0210	0201	1432	1,4		
		1,3,4	1432	0201	0210	1324	1,3,4		
		3,4	2134	2100	2010	3214	3,4		
		2,4	3214	2010	2001	4231	2,4		
2,3,4	4231	2001	2100	2134	2,3,4				
1	231	1,2,4	1342	0231	0231	1342	1,2,4	231	1
		1,4	2314	2310	2301	2431	1,4		
		1,3,4	2431	2301	2310	2314	1,3,4		
		2,4	3241	2031	2031	3241	2,4		
1	312	1,3,4	1423	0312	0312	1423	1,3,4	312	1
		2,4	3124	3120	3102	4132	2,4		
		3,4	4132	3102	3012	4213	3,4		
		2,3,4	4213	3012	3120	3124	2,3,4		

Calculation of $\sigma'' = F_3(\sigma)$.

fix σ	Der σ	maz σ	σ	w	w''	σ''	maf σ''	Der σ''	fix σ''
4	e	0	1234	0000	0000	1234	0	e	4
2	21	3	1243	0021	2001	4231	3	21	2
		5	1324	0210	2100	2134	5		
		2	1432	0201	0201	1432	2		
		3	2134	2100	0210	1324	3		
		4	3214	2010	2010	3214	4		
1	231	1	4231	2001	0021	1243	1	231	1
		3	1342	0231	2031	3241	3		
		5	2314	2310	2310	2314	5		
		2	2431	2301	0231	1342	2		
1	312	4	3241	2031	2301	2431	4	312	1
		2	1423	0312	3012	4213	2		
		4	3124	3120	3120	3124	4		
		3	4132	3102	3102	4213	3		
		1	4213	3012	0312	3124	1		

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