ON INTERPOLATING CUBIC SPLINES WITH EQUALLY-SPACED NODES [1]

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In this note we give exact expressions for the norms of some interpolating spline projection operators and establish some new error estimates in the approximation of continuous functions.

We restrict our attention to the simplest case of spline interpolation, namely, the *periodic cubic splines*. Let C denote the Banach space (with supremum norm) of all continuous, periodic, real-valued functions on $[0, 1]$. In this context, "periodic" simply means that $f(0)=f(1)$. To each division of the interval into n subintervals $\{0=x_0<x_1<\ldots<x_n=1\}$ there corresponds an n-dimensional subspace S in C whose members are the periodic cubic spline functions with nodes x_i. Thus, $s \in S$ if and only if

1) s'' exists and belongs to C,
2) on each subinterval $[x_i, x_{i+1}]$, s coincides with a certain cubic polynomial q_i.

For the general theory of splines, the reader should refer to the treatise [1].

To each $f \in C$ there corresponds a uniquely determined element $s \in S$ with the interpolating property: $s(x_i)=f(x_i)$ for $i=0, \ldots, n$. The mapping $L: f \to s$ defined in this manner is a linear and idempotent operator from C onto S. In a previous paper [2], we gave estimates of the operator norm

$$\|L\| = \sup \{\|Lf\| : \|f\|=1, \quad f \in C\}$$

in terms of the spacing numbers $h_i=x_i-x_{i-1}$. In the equally-spaced case, all the numbers h_i are equal to n^{-1}, and it is possible to compute $\|L\|$ exactly. This we do below in Theorem 1. We show, for example, that $\|L\| \leqslant 1.549$ for all n. In other results, we give estimates (which are best possible or nearly so) for the expression $\|Lf-f\|$, in terms of the modulus of continuity of f. Finally, a result of NORD [3] on the derivatives of a spline function at the nodes is improved to a form which is best possible.

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Several basic results are needed for these calculations. First, if $s=Lf$ and $\mu_i=s'(x_i)$ then as in [1, p. 12]

$$\mu_{i-1}+4\mu_i+\mu_{i+1}=3n(f_{i+1}-f_{i-1}) \quad (i=1, \ldots, n). \tag{1}$$

Here f_i denotes $f(x_i)$. Next, if the two numbers μ_{i-1} and μ_i are known then s can be given explicitly on the interval $[x_{i-1}, x_i]$ by the Hermite Interpolation Formula as in [2]:

$$s(x)=f_{i-1}A_i(x)+f_iB_i(x)+\mu_{i-1}C_i(x)+\mu_iD_i(x). \tag{2}$$

Here the functions $A_i, \ldots, D_i$ are certain cubic polynomials defined as follows

$$\begin{cases} A_i(x)=n^3(n^{-1}+2x-2x_{i-1})(x-x_i)^2 \\ B_i(x)=n^3(n^{-1}-2x+2x_i)(x-x_{i-1})^2 \\ C_i(x)=n^2(x-x_{i-1})(x-x_i)^2 \\ D_i(x)=n^2(x-x_i)(x-x_{i-1})^2. \end{cases} \tag{3}$$

We note that A_i, B_i, C_i and $-D_i$ are nonnegative on $[x_{i-1}, x_i]$. Furthermore, $A_i(x)+B_i(x)=1$ and $C_i(x)-D_i(x)=n(x-x_{i-1})(x_i-x)$. Next, we denote by s^i the i-th *cardinal* spline. It is defined by the equation $s^i(x_j)=\delta^i_j$ for i, $j=1, \ldots, n$. In terms of these functions, we have $Lf=\sum_{i=1}^{n} f_is^i$ and consequently $\|L\|=\|\sum_{i=1}^{n}|s^i|\,\|$. The purpose of the first two lemmas is to compute the numbers $(s^i)'(x_j)$ by solving system (1) in an appropriate particular case. If $n=2k$ or $n=2k-1$, we put $\lambda_i=(s^k)'(x_i)$. Since the functions s^i are periodic and the nodes are equally spaced, we have $s^i(x)=s^k(x-x_{i-k})$. Thus it is only necessary to compute one cardinal function, and we chose s^k. Then (with $\mathscr{D}$ denoting differentiation)

$$\mathscr{D}s^i(x_j)=\mathscr{D}s^k(x_j-x_{i-k})=\mathscr{D}s^k(x_{j-i+k})=\lambda_{j-i+k}.$$

Consequently, on the interval $[x_{j-1}, x_j]$ we have

$$\begin{cases} s^i(x)=\delta^i_{j-1}A_j(x)+\delta^i_jB_j(x)+\mathscr{D}s^i(x_{j-1})C_j(x)+\mathscr{D}s^i(x_j)D_j(x)= \\ \quad =\delta^i_{j-1}A_j(x)+\delta^i_jB_j(x)+\lambda_{j-i+k-1}C_j(x)+\lambda_{j-i+k}D_j(x). \end{cases} \tag{4}$$

(Calculations involving the indices are carried out modulo n. Thus, for example, $f_n=f_0$, $\lambda_{n+1}=\lambda_1$, etc.)

Lemma 1. *Define* $n=2k-1$, $a_0=a_1=1$, $a_{i+1}=4a_i-a_{i-1}$ *for* $i\in\{1, 2, \ldots\}$, $\lambda_i=(-1)^{k+i+1}3na_k{}^{-1}a_i$ *for* $i\in\{0, 1, \ldots, k-1\}$, $\lambda_k=0$, *and* $\lambda_i=-\lambda_{2k-i}$ *for*

$i \in \{k+1, \ldots, 2k-1\}$. *Then* $\lambda_{i-1}+4\lambda_i+\lambda_{i+1}=3n(\delta^k_{i+1}-\delta^k_{i-1})$ *for* $i \in \{1, \ldots, n\}$, *and* $\sum_{i=1}^{n} |\lambda_i| = 6na_k^{-1}(a_1+\ldots+a_{k-1})$.

Proof. If $i \in \{1, \ldots k-2\}$ then

$$\lambda_{i-1}+4\lambda_i+\lambda_{i+1}=(-1)^{k+i}3na_k^{-1}(a_{i-1}-4a_i+a_{i+1})=0=3n(\delta^k_{i+1}-\delta^k_{i-1}).$$

For the case $i=k-1$ we have

$$\lambda_{k-2}+4\lambda_{k-1}+\lambda_k=-3na_k^{-1}(a_{k-2}-4a_{k-1})=-3na_k^{-1}(-a_k)=3n=3n(\delta^k_k-\delta^k_{k-2}).$$

For the case $i=k$ we have

$$\lambda_{k-1}+4\lambda_k+\lambda_{k+1}=\lambda_{k-1}+\lambda_{k+1}=0=3n(\delta^k_{k+1}-\delta^k_{k-1}).$$

For the case $i=k+1$ we have

$$\lambda_k+4\lambda_{k+1}+\lambda_{k+2}=-\lambda_{k-2}-4\lambda_{k-1}-\lambda_k=-3n=3n(\delta^k_{k+2}-\delta^k_k).$$

If $i \in \{k+2, \ldots, 2k-1\}$ then

$$\lambda_{i-1}+4\lambda_i+\lambda_{i+1}=-(\lambda_{2k-i+1}+4\lambda_{2k-i}+\lambda_{2k-i-1})=$$
$$-(\lambda_{j-1}+4\lambda_j+\lambda_{j+1})=0=3n(\delta^k_{i+1}-\delta^k_{i-1}),$$

where $j=2k-i$ and $j \in \{1, \ldots, k-2\}$.

In exactly the same way one can prove a similar result when n is even.

Lemma 2. *If* $n=2k$, $b_0=0$, $b_1=1$, $b_{i+1}=4b_i-b_{i-1}$ *for* $i \in \{1, 2, \ldots\}$, $\lambda_i=(-1)^{k+i+1}3nb_k^{-1}b_i$ *for* $i \in \{0, 1, \ldots, k-1\}$, $\lambda_k=0$, *and* $\lambda_i=-\lambda_{n-i}$ *for* $i \in \{k+1, \ldots, n\}$, *then* $\lambda_{i-1}+4\lambda_i+\lambda_{i+1}=3n(\delta^k_{i+1}-\delta^k_{i-1})$ *for* $i \in \{1, \ldots, n\}$, *and* $\sum_{i=1}^{n} |\lambda_i| = 6nb_k^{-1}(b_1+\ldots+b_{k-1})$.

Theorem 1. *In the equally-spaced case, the norm of* L_n *is as follows, with* $\beta=2+\sqrt{3}$,

(a) $\|L_n\|=1+\frac{3}{2}(\beta^k-\beta)(\beta^k+1)^{-1}(\beta-1)^{-1}$, $(n=2k)$

(b) $\|L_n\|=1+\frac{3}{2}(\beta^k-\beta)(\beta^k+\beta)(\beta^{2k}+\beta)^{-1}(\beta-1)^{-1}$, $(n=2k-1)$.

Proof. We have $\|L\|=\| \sum_{i=1}^{n} |s^i| \|$. Select x so that $\|L\|= \sum_{i=1}^{n} |s^i(x)|$ and select j so that $x_{j-1} \leqslant x \leqslant x_j$. By equation (4)

$$\sum_{i=1}^{n} |s^i(x)|=$$
$$=\sum_{i=1}^{n} |\delta^i_{j-1}A_j(x)+\delta^i_jB_j(x)+\lambda_{j-i+k-1}C_j(x)+\lambda_{j-i+k}D_j(x)|=$$
$$=|A_j(x)+\lambda_kC_j(x)+\lambda_{k+1}D_j(x)|+|B_j(x)+\lambda_{k-1}C_j(x)+\lambda_kD_j(x)|+$$
$$+\left(\sum_{i=1}^{k-1}+\sum_{i=k+2}^{n}\right)|\lambda_{i-1}C_j(x)+\lambda_iD_j(x)|.$$

By Lemmas 1 and 2, the coefficients λ_i alternate in sign as i runs through the sets $\{0, \ldots, k-1\}$ and $\{k+1, \ldots, n\}$. Furthermore, $\lambda_k=0$, $\lambda_{k-1}>0$, and $\lambda_{k+1}<0$. These facts together with the properties of $A_j, \ldots, D_j$ imply that

$$\sum_{i=1}^{n} |s^i(x)| = 1 + [C_j(x) - D_j(x)] \sum_{i=1}^{n} |\lambda_i|.$$

Since x was chosen to make $\sum_{i=1}^{n} |s^i(x)|$ a maximum, it also makes $C_j(x)-D_j(x)$ a maximum. Then we obtain

$$\|L_n\| = 1 + \frac{1}{4n} \sum_{i=1}^{n} |\lambda_i|. \tag{5}$$

In the odd case, we use Lemma 1 to obtain

$$\|L_n\| = 1 + \frac{3}{2} \sum_{i=1}^{k-1} \frac{a_i}{a_k}.$$

The numbers a_i satisfy a linear difference equation and can be determined explicitly by the formula $a_i=\beta^{-i}(\beta^{2i}+\beta)(1+\beta)^{-1}$. If we sum the geometric series appearing in the formula for $\|L_n\|$ and perform algebraic manipulations, the result is the formula in the theorem. The analysis for the even case is similar, the formula for b_i being $b_i=(\frac{1}{2})(\beta-2)^{-1}(\beta^i-\beta^{-i})$.

The next theorem is proved by direct calculations based upon Theorem 1.

Theorem 2. *In the equally-spaced case, the norms $\|L_n\|$ are ordered as follows:*

(a) $\|L_3\|<\|L_5\|<\|L_7\|<\ldots<\frac{1}{4}(1+3\sqrt{3})=1.548\ldots$

(b) $\|L_2\|<\|L_4\|<\|L_6\|<\ldots<\frac{1}{4}(1+3\sqrt{3})=1.548\ldots$

(c) $\|L_3\|=\|L_6\|$, $\|L_5\|=\|L_{10}\|$, $\|L_7\|=\|L_{14}\|$, $\ldots$

Theorem 3. *The following error-estimate is valid for all $f \in C$, the nodes being equally spaced:*

$$|(L_nf-f)(x)| \leqslant c_n\omega(f;\delta)$$

where $\delta = \min_i |x-x_i|$ *and* $\|L_n\| \leqslant c_n < 2\|L_n\|$.

Proof. Let $n=2k$ or $n=2k-1$. Let x be any point, and select j so that $x_{j-1}\leqslant x\leqslant x_j$. From equation (4), together with the equation $A_j(x)+B_j(x)=1$, we obtain

$$\left\{\begin{aligned} \varepsilon=(L_nf-f)(x) &= \sum_{i=1}^{n} f_i s^i(x) - f(x)A_j(x) - f(x)B_j(x) = \\ &= [f_{j-1}-f(x)]A_j(x) + [f_j - f(x)]B_j(x) + \\ &+ \sum_{i=1}^{n} f_i[\lambda_{j+k-i-1}C_j(x) + \lambda_{j+k-i}D_j(x)]. \end{aligned}\right. \tag{6}$$

In order to simplify the notation we abbreviate f_{j+k-i} by F_i, $f(x)$ by f_x, $A_j(x)$ by A, etc. We also note from Lemmas 1 and 2 that $\lambda_{k+i}=-\lambda_{k-i}$ for $i=0, \ldots, k-1$. Furthermore, $\lambda_0=0$ when n is even. Hence

$$(7)\quad \begin{cases} \varepsilon=(f_{j-1}-f_x)A+(f_j-f_x)B+\sum_{i=1}^{n} F_i(\lambda_{i-1}C+\lambda_i D)= \\ =(f_{j-1}-f_x)A+(f_j-f_x)B+\sum_{i=1}^{k-1} F_i\lambda_i D+\sum_{i=2}^{k} F_i\lambda_{i-1}C+ \\ \qquad -\sum_{i=1}^{k-1} F_{2k-i}\lambda_i D-\sum_{i=1}^{k-1} F_{2k-i+1}\lambda_i C. \end{cases}$$

Now define $\sigma_0=0$ and $\sigma_i=\lambda_1+\ldots+\lambda_i$ for $i=1, \ldots, k$. We apply to (7) the formula for partial summation, $\sum_1^p \alpha_i\lambda_i=\sum_1^{p-1}(\alpha_i-\alpha_{i+1})\sigma_i+\alpha_p\sigma_p$. The result is

$$(8)\quad \begin{cases} \varepsilon=\sum_{i=1}^{k-1}(F_i-F_{i+1})(\sigma_i D+\sigma_{i-1}C)+(f_j-f_x)(B+\sigma_k C+\sigma_k D)+ \\ \qquad +(f_x-f_{j-1})(-A+\sigma_k C+\sigma_k D)+ \\ \qquad +\sum_{i=1}^{k-1}(F_{k+i}-F_{k+i+1})(\sigma_{k-1-i}D+\sigma_{k-i}C). \end{cases}$$

Now let $\delta=\min_i |x-x_i|=\min\{x-x_{j-1}, x_j-x\}$. If $\delta=0$ then x is a node and the inequality in question is trivial. We assume therefore that $\delta>0$. If $\omega(f;\delta)=0$ then f is constant and $Lf=f$. We assume therefore that $\omega(f;\delta)>0$. Since the inequality in question is homogeneous in f, it is sufficient to give the proof for functions f such that $\omega(f;\delta)=1$. Now let p denote the smallest integer satisfying $p\geqslant(n\delta)^{-1}$. Since each interval of length $1/n$ can be subdivided into p intervals of length at most δ, we have $|F_i-F_{i+1}|\leqslant p$. Assume now that $x-x_{j-1}=\delta$ and that $x_j-x=n^{-1}-\delta$. (The analysis of the other case, when $x_j-x=\delta$, is almost exactly the same.) Then $|f_x-f_{j-1}|\leqslant 1$ and $|f_j-f_x|\leqslant p-1$. Thus

$$(9)\quad \begin{cases} \varepsilon\leqslant p\sum_{i=1}^{k-1}|\sigma_i D+\sigma_{i-1}C|+|B+\sigma_k C+\sigma_k D|(p-1)+|-A+\sigma_k C+\sigma_k D|+ \\ \qquad +p\sum_{i=1}^{k-1}|\sigma_{k-1-i}D+\sigma_{k-i}C|. \end{cases}$$

The sum on the right in inequality (9) is analysed in Lemmas 3 and 5 below. The result is

$$\varepsilon\leqslant p\sum_{i=1}^{k-1}|\lambda_i|(C-D)+2-n\delta.$$

We evaluate $C-D$ at the point $x=x_{j-1}+\delta$ and obtain $\delta(1-n\delta)$. Also we note from the proof of Theorem 1 that $\sum_{i=1}^{k-1} |\lambda_i| = 2n(\|L_n\| - 1)$. Finally, we use the inequality $p \leqslant 1 + (n\delta)^{-1}$. Thus

$$\begin{aligned}\varepsilon &\leqslant [1+(n\delta)^{-1}]2n(\|L_n\|-1)\delta(1-n\delta)+2-n\delta = \\ &= 2(1+n\delta)(1-n\delta)(\|L_n\|-1)+2-n\delta = \\ &= 2\|L_n\| + 2n^2\delta^2(1-\|L_n\|) - n\delta < 2\|L_n\|.\end{aligned}$$

In the special case that $\delta = (2n)^{-1}$, we have $p = 2$, and the bound from Lemma 5 is 1. Hence in this case $\varepsilon \leqslant \|L_n\|$.

In order to see that $c_n \geqslant \|L_n\|$ we construct a particular function f by specifying $F_{k-i} = F_{k+i+1} = p$ for $i = 0, 2, 4, \ldots$ and $F_{k-i} = F_{k+i+1} = 0$ for $i = 1, 3, 5, \ldots$ Also, we let $f_x = p - 1$. The function f varies linearly between the specified values, is periodic, and satisfies $\omega(f; \delta) = 1$. For this function,

$$\begin{aligned}\varepsilon &= p(C-D)\sum_{i=1}^{k-1} |\lambda_i| + (B+\sigma_k C+\sigma_k D) - (-A+\sigma_k C+\sigma_k D) = \\ &= 2n\delta p(1-n\delta)(\|L_n\|-1)+1 \geqslant \|L_n\|.\end{aligned}$$

(This example is satisfactory when n is odd. If n is even, it is modified by defining F_0 to be equal to F_1.)

Corollary. *In the equally-spaced case the estimate*

$$\|f - L_n f\| \leqslant \|L_n\|\,\omega\left(f; \frac{1}{2n}\right)$$

is valid. It is not possible to introduce a constant factor <1 *on the right-hand side.*

Lemma 3. $\sum_{i=1}^{k-1} \{|\sigma_i D + \sigma_{i-1} C| + |\sigma_{k-1-i} D + \sigma_{k-i} C|\} = (C-D)\sum_{i=1}^{k-1} |\lambda_i|.$

Proof. In the notation of Lemmas 1 and 2 we have $a_{i+1} \geqslant 3a_i > 0$ and $b_{i+1} \geqslant 3b_i > 0$ for all $i \in \{1, 2, \ldots\}$. This is readily proved by induction. It then follows that $2|\lambda_i| < |\lambda_{i+1}|$ for all $i \in \{1, \ldots, k-2\}$. Another induction establishes that $|\sigma_i| < |\lambda_{i+1}|$ for $i \in \{0, \ldots, k-2\}$. It follows that $\operatorname{sgn} \sigma_i = \operatorname{sgn}(\lambda_i + \sigma_{i-1}) = \operatorname{sgn} \lambda_i = (-1)^{k+1+i}$ for $i \in \{1, \ldots, k-1\}$. Since $C \geqslant 0$ and $D \leqslant 0$, we have $\operatorname{sgn}(\sigma_i D + \sigma_{i-1} C) = \operatorname{sgn} \sigma_{i-1} = (-1)^{k+i}$.
The sum on the left side in the statement of the lemma therefore is

$$\begin{aligned}&\sum_{i=1}^{k-1} \{|\sigma_i D + \sigma_{i-1} C| + |\sigma_i C + \sigma_{i-1} D|\} = \\ &\quad = \sum_{i=1}^{k-1} (-1)^{k+i+1}\{-\sigma_i D - \sigma_{i-1} C + \sigma_i C + \sigma_{i-1} D\} = \\ &\quad = (C-D)\sum_{i=1}^{k-1} (-1)^{k+i+1}(\sigma_i - \sigma_{i-1}) = \\ &\quad = (C-D)\sum_{i=1}^{k-1} |\lambda_i|.\end{aligned}$$

Lemma 4. *On the interval* $[x_{j-1}, x_j]$ *the functions* $A-\sigma_k(C+D)$ *and* $B+\sigma_k(C+D)$ *are nonnegative.*

Proof. Put $J=A-\sigma_k(C+D)$. It is enough to prove that $0\leqslant J\leqslant 1$ because then $B+\sigma_k(C+D)=1-J\geqslant 0$. Put $x=x_{j-1}+\theta n^{-1}$. From equations (3) we obtain

$$J=(1-\theta)[(1+2\theta)(1-\theta)-\sigma_k n^{-1}\theta(1-2\theta)].$$

If $0\leqslant\theta\leqslant\frac{1}{2}$ then $\sigma_k n^{-1}\theta(1-2\theta)\geqslant 0$. (See the proof of Lemma 3). Hence $J\leqslant(1-\theta)^2(1+2\theta)\leqslant 1$. From Lemma 1 and the proof of Lemma 3,

$$\sigma_k=\sigma_{k-1}=\sigma_{k-2}+\lambda_{k-1}=|\lambda_{k-1}|-|\sigma_{k-2}|\leqslant|\lambda_{k-1}|=3na_{k-1}/a_k\leqslant 3n(1/3)=n.$$

Hence $J\geqslant(1-\theta)[(1+2\theta)(1-\theta)-\theta(1-2\theta)]=1-\theta\geqslant 0$. On the other hand, if $\frac{1}{2}\leqslant\theta\leqslant 1$ then $\sigma_k n^{-1}\theta(1-2\theta)\leqslant 0$ and $J\geqslant 0$. Since $\sigma_k\leqslant n$,

$$J\leqslant(1-\theta)[(1+2\theta)(1-\theta)+\theta(2\theta-1)]=1-\theta\leqslant 1.$$

The analysis when n is even is similar.

Lemma 5. *If* $x=x_{j-1}+\delta$ *and* $0\leqslant\delta\leqslant(2n)^{-1}$ *then*

$$(p-1)|B+\sigma_k C+\sigma_k D|+|A-\sigma_k C-\sigma_k D|<2-n\delta.$$

If $\delta=(2n)^{-1}$, *the bound can be lowered to* 1.

Proof. By Lemma 4, the left side of the asserted inequality is

$$\begin{aligned}J &\equiv (p-1)(B+\sigma_k C+\sigma_k D)+A-\sigma_k C-\sigma_k D=\\ &=A+(p-1)B+\sigma_k(p-2)(C+D).\end{aligned}$$

If we insert the values of A, B, C, and D from equations (3) and simplify, the result is

$$J=1+n\delta(p-2)(3n\delta-2n^2\delta^2)+\sigma_k\delta(p-2)(1-3n\delta+2n^2\delta^2).$$

From the proof of Lemma 4, $\sigma_k<n$. Also, $p\geqslant 2$ and $1-3n\delta+2n^2\delta^2=(1-2n\delta)(1-n\delta)\geqslant 0$. Hence $J\leqslant 1+n\delta(p-2)$. Since $p<1+(n\delta)^{-1}$, we have $pn\delta<n\delta+1$ so that $J<1+(n\delta+1)-2n\delta=2-n\delta$. Note that if $\delta=(2n)^{-1}$ then $p=2$, $J=1$, and the bound $2-n\delta$ can be improved to 1.

Theorem 4. *Every periodic cubic spline function s having n equally-spaced nodes,* $x_i=i/n$, *satisfies the inequality*

$$\max_i |s'(x_i)|\leqslant\sqrt{3}\, n \max |s(x_i)-s(x_{i-1})|.$$

The constant $\sqrt{3}$ *cannot be improved.*

Proof. Let $s=Lf$ so that $s=\sum_{i=1}^{n} f_i s^i$. We assume, without loss of generality, that $\max|f_i - f_{i-1}| = 1$. Since $(s^i)'(x_j) = \lambda_{j-i+k}$, we can proceed as in the proof of Theorem 3, and obtain

$$s'(x_j) = \sum_{i=1}^{n} f_i \lambda_{j-i+k} = \sum_{i=1}^{n} F_i \lambda_i = $$
$$= \sum_{i=1}^{k} (F_i - F_{i+1})\sigma_i + \sum_{i=1}^{k-1} (F_{k+i} - F_{k+i+1})\sigma_{k-i-1}.$$

We obtain immediately the upper bound

$$|s'(x_j)| \leqslant 2 \sum_{i=1}^{k-1} |\sigma_i| =$$
$$= 2(\lambda_{k-1} + \lambda_{k-3} + \ldots + \lambda_p), \text{ where } p=1 \text{ or } p=2.$$

If $n = 2k-1$ and k is odd, this upper bound is attained by the following extremal function: $F_{k-i} = F_{k+i} = 0$ if i is even and $-F_{k-i} = F_{k+i} = 1$ if $i \in \{1, 3, \ldots, k-2\}$. The proof of Theorem 4 is completed by establishing that

$$\frac{1}{n}(\lambda_{k-1} + \lambda_{k-3} + \ldots + \lambda_p) < \tfrac{1}{2}\sqrt{3}$$

and that the limit of the left hand side, as $n = 2k-1$ and $k = 1, 3, 5, \ldots$, is $\frac{1}{2}\sqrt{3}$. These calculations, based upon the explicit formulas of λ_i given in Lemma 1, are straightforward and hence omitted.

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REFERENCES

1. Ahlberg, J. H., E. N. Nilson and J. L. Walsh, *The Theory of Splines and Their Applications*, Academic Press, New York, 1967.
2. Cheney, E. W. and F. Schurer, *A note on the operators arising in spline approximation*, Journal of Approximation Theory, 1, 94–102 (1968).
3. Nord, S., *Approximation properties of the spline fit*, Nordisk Tidskr. Informat.-Behand. (BIT) 7, 132–144 (1967).
4. Sharma, A. and A. Meir, *Degree of approximation of spline interpolation*, J. Math. Mech. 15, 759–767 (1966).
5. Schurer, F., *A note on interpolating periodic quintic splines with equally-spaced nodes*. (To appear in Journal of Approximation Theory).