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An inventory model with backorders with fuzzy parameters and decision variables

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ABSTRACT

The paper considers an inventory model with backorders in a fuzzy situation by employing two types of fuzzy numbers, which are trapezoidal and triangular. A full-fuzzy model is developed where the input parameters and the decision variables are fuzzified. The optimal policy for the developed model is determined using the Kuhn–Tucker conditions after the defuzzification of the cost function with the graded mean integration (GMI) method. Numerical examples and a sensitivity analysis study are provided to highlight the differences between crisp and the fuzzy cases.

1. Introduction

The two basic questions an inventory manager usually answers are when and how much to order. Over the past decades, many papers and books on inventory theory and management have been published presenting numerous models that describe various conditions and assumptions [22]. Most of these models are based on the economic order/production quantity (EOQ/EPQ) model developed by Harris [15].

EOQ-based inventory models minimize the sum of mainly two costs, which are the holding and the ordering costs. These models assume that the input parameters and the decision variables are described as crisp values or having crisp statistical distributions where their total inventory cost functions are minimized without ambiguity in the results. Although these models provide some general understating of the behavior of inventory under different assumptions, they are not capable of representing real-life situations. So, applying these models as they are, generally, leads to erroneous decisions. Further, using these models require inventory managers to have some flexibility when deciding on the sizes of the order quantities to reduce the cost of uncertainty. Hence, using fuzzy set theory to solve inventory problems, instead of the traditional probability theory, produces more accurate results (see for instance Guiffrida [13]).

Fuzzy set theory, introduced by Zadeh [29], has been receiving considerable attention from researchers in production and inventory management (see for instance Guiffrida and Nagi, [14]), as well as in other fields. Here is a brief review of the literature. Sommer [24] applied a fuzzy dynamic programming approach to solve a production-inventory scheduling problem with capacity constraints. Kacprzyk and Staniewski [17] investigated long-term inventory policy-making using fuzzy decision models for a multi-stage inventory planning problem. Park [21] proposed an EOQ model with the ordering and inventory holding costs being trapezoidal fuzzy numbers. In his model, the mode and median rules were applied after scaling the fuzzy cost inputs in conformance with the EOQ model. Roy and Maiti [23] transformed an EOQ model into a nonlinear

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programming problem after fuzzifying the objective function and the available storage space. Mandal and Maiti [20] proposed a nonlinear fuzzy modeling for a multi-item EOQ model with 'imprecise storage space' and 'number of production run' constraints where few input parameters were fuzzified. Yao and Chiang [28] developed an EOQ model with the total demand and the unit carrying cost being triangular fuzzy numbers. They used the signed distance and the centroid as defuzzification methods. Vijayan and Kumaran [27] considered inventory models with partial backorders and fuzzy stock-out periods. Chang [4] modified an EOQ model with imperfect quality items by fuzzyfying the defect and demand rates with no shortages. Chang et al. [5] considered an EOQ model with triangular fuzzy backorder. Björk and Carlsson [3] investigated the effect of flexibility in lead times on the distributors' performance in a supply chain. They handled the imprecision in lead time using fuzzy sets. Björk [2] fuzzified the decision variable, cycle time, in an EPQ model with no shortages. In a recent paper, Björk [1] investigated the EOQ model for demand and the lead time being triangular fuzzy numbers. A fairly extensive review of the application of fuzzy sets to inventory management is provided in Guiffrida [13].

The literature review reveals that there is no EOQ model with an analytical solution that has both its input parameters and decision variable(s) fuzzified, which is a limitation that this paper addresses. This paper chose a classic EOQ model that has two decision variable; namely, the batch size and the maximum inventory level. It is unlike the work of Björk [1] who fuzzified one input parameters (demand) and one decision variable (maximum inventory level). Similar problems to that of Björk [1] and to the one in the paper are found in Chen and Wang [11], Chen et al. [12], Chen and Chang [7] and Vijayan and Kumaran [26].

The next section, Section 2, provides a brief introduction and background to fuzzy set theory. This section is followed by a brief introduction to the Kuhn–Tucker conditions and optimization method. Section 4 presents the mathematics for the crisp case of an EOQ model with backorders. Section 5 presents the mathematics of the full-fuzzy version of the model described in Section 4. Section 6 is for numerical examples and Section 7 is for summary and conclusions.

2. Fuzzy preliminaries

Fuzzy set theory has emerged as a powerful tool to quantitatively represent and manipulate the imprecision that sometimes governs the decision-making process. Fuzzy sets or fuzzy numbers can be used to encounter the imprecision by setting the values of the input parameters to be functions of triangular or trapezoidal shapes [18]. Some basic definitions, taken from [19,30], that are related to fuzzy set theory are briefly reviewed below for the interest of the readers.

Definition 1. $\widetilde{A}$ is a fuzzy set in a universe of discourse X. It is characterized by a membership function $\mu_{\widetilde{A}}(x)$, which is associated with each element x, where x is a real number in the interval $[0,1]$. The function value $\mu_{\widetilde{A}}(x)$ is termed as the grade of membership of x in $\widetilde{A}$.

Definition 2. The fuzzy set $\widetilde{A}$ of the universe of discourse X is convex, where $\mu_{\widetilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geqslant min\Big(\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_2)\Big)$ for all $x_1, x_2 \in X$ and for $\lambda \in [0,1]$.

Definition 3. The fuzzy set $\widetilde{A}$ of the universe of discourse X is called a normal fuzzy set when $\exists x_i \in X, \mu_{\widetilde{A}}(x_i) = 1$.

Definition 4. A fuzzy number is a fuzzy subset in the universe of discourse X that is both convex and normal.

$\widetilde{A}$ is said to be a trapezoidal fuzzy number represented by the crisp numbers (a_1, a_2, a_3, a_4), where $a_1 < a_2 < a_3 < a_4$, when its membership function is denoted as:

$$\mu_{\widetilde{A}}(x) = \begin{cases} m(x) = \frac{x-a_1}{a_2-a_1} & a_1 \leqslant x \leqslant a_2, \\ 1 & a_2 \leqslant x \leqslant a_3, \\ n(x) = \frac{x-a_4}{a_3-a_4} & a_3 \leqslant x \leqslant a_4, \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

When $a_2 = a_3$, the trapezoidal fuzzy number described in Eq. (1) becomes a triangular, which is a special case of the first.

In this paper, the Function Principle Method, from [6], is used to simplify the model calculations. Now, define $\widetilde{A} = (a_1, a_2, a_3, a_4)$ and $\widetilde{B} = (b_1, b_2, b_3, b_4)$ as two trapezoidal fuzzy numbers with the following properties:

1. $\widetilde{A} + \widetilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$.
   If a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 are all positive real numbers, then
2. $\widetilde{A} \cdot \widetilde{B} = (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4)$.
   Let λ be a real number, then for $\lambda \geqslant 0,\ \lambda\widetilde{A} = (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4)$ and $\lambda < 0 \lambda\widetilde{A} = (\lambda a_4, \lambda a_3, \lambda a_2, \lambda a_1)$
3. $\widetilde{B} = (-b_4, -b_3, -b_2, -b_1),\ \widetilde{A} - \widetilde{B} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$.
   If a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 are all positive real numbers, then
4. $\frac{1}{B} = \left(\frac{1}{b_4}, \frac{1}{b_3}, \frac{1}{b_2}, \frac{1}{b_1}\right), \frac{A}{B} = \left(\frac{a_4}{b_4}, \frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1}\right)$.

The above principle also holds for trapezoidal fuzzy number.

2.1. Graded mean integration (GMI) representation method

While most decision processes have some fuzzy properties, decision makers are more comfortable using crisp values instead of fuzzy ones. Hence, a modeler should attempt converting fuzzy values into crisp. In order to defuzzify the fuzzy cost function presented in a later section, the GMI representation method introduced by Chen and Hseih [8–10] is applied. Assume that $\widetilde{A} = (a_1, a_2, a_3, a_4)$ is a trapezoidal fuzzy number as defined earlier and that m^{-1}, n^{-1} are respectively the inverse functions of m and n. Also, define the graded α-level value of $\widetilde{A}$ as $\frac{\alpha(m^{-1}(\alpha)+n^{-1}(\alpha))}{2}$. Then, the GMI representation of fuzzy number $\widetilde{A}$ can be computed as

$$\vartheta(\widetilde{A}) = \frac{\int_0^1 \frac{\alpha(m^{-1}(\alpha)+n^{-1}(\alpha))}{2} d\alpha}{\int_0^1 \alpha d\alpha} = \int_0^1 \alpha(m^{-1}(\alpha) + n^{-1}(\alpha))d\alpha, \tag{2}$$

Substituting $m^{-1}(\alpha) = a_1 + (a_2 - a_1)\alpha$ and $n^{-1}(\alpha) = a_4 - (a_4 - a_3)\alpha$, which are trapezoidal fuzzy numbers, into Eq. (2), reduces Eq. (2) to

$$\vartheta(\widetilde{A}) = \frac{\int_0^1 \frac{\alpha[(a_1+a_4)+((a_2+a_3)-(a_1+a_4))\alpha]}{2} d\alpha}{\int_0^1 \alpha d\alpha} = \frac{1}{6}(a_1 + 2a_2 + 2a_3 + a_4). \tag{3}$$

Since the triangular fuzzy number is a special case of the generalized trapezoidal fuzzy number when $a_2 = a_3 = a$, then from Eq. (1), the GMI representation of the triangular fuzzy number $\widetilde{A}(a_1, a_2, a_3)$ becomes

$$\vartheta(\widetilde{A}) = \frac{1}{6}(a_1 + 4a + a_4). \tag{4}$$

3. The Kuhn–Tucker conditions

Kuhn–Tucker (KKT) conditions is a method of finding optimal solutions for nonlinear programming problems (with differentiable functions). Readers may refer to Taha [25] and Hillier and Liberman [16] for details. The basic result of the KKT conditions is embodied in the following theorem [16]:

Theorem 1. *Assume that an objective function f(X) and the constraints $g_1(X)$, $g_2(X)$, ..., $g_r(X)$ are differentiable satisfying certain regularity conditions. Then $\mathbf{X}^* = (x_1^*, x_2^*, \ldots, x_s^*)$ is the optimal solution for the nonlinear programming problem only if there exists r numbers λ_1, λ_2, ..., λ_r such that all the following conditions are satisfied: $\frac{\partial f}{\partial x_j} - \sum_{i=1}^m \lambda_i \frac{\partial g_i}{\partial x_j} \geqslant 0$ at $x = x^*$ for $j = 1, 2, \ldots, s$.*

1. $x_j^*\left(\frac{\partial f}{\partial x_j} - \sum_{i=1}^m \lambda_i \frac{\partial g_i}{\partial x_j}\right) = 0,$	at $x = x^*$ for $j = 1, 2, \ldots, s$.
2. $g_i(x)^* - b_i \leqslant 0$	for $i = 1, 2, \ldots, r$.
3. $\lambda_i(g_i(x)^* - b_i) = 0$	for $i = 1, 2, \ldots, r$.
4. $x_j^* \geqslant 0$	for $j = 1, 2, \ldots, s$.
$\lambda_i \geqslant 0$	for $i = 1, 2, \ldots, r$.

4. The EOQ model with backorders

The EOQ model with planned shortages (EOQ-S), or backorders, is perhaps the first extension of the model of Harris (EOQ). The behavior of inventory for this model is depicted in Fig. 1.

The cost function for the EOQ-S is given as

$$TCU(y, M) = \frac{KD}{y} + \frac{M^2 h}{2y} + \frac{(y - M)^2 p}{2y}, \tag{5}$$

where y is the batch size (in units), M is the maximum inventory level (just after replenishment and measured in units), K is the fixed cost per order, D is the demand rate (units per unit of time), h is the unit holding cost per unit per unit of time, and p is the penalty cost due to shortages per unit per unit of time. The optimal values of y and M are computed using differential calculus as

$$M^* = \sqrt{\frac{2KD}{h}} \cdot \sqrt{\frac{p}{h + p}}, \tag{6}$$

$$y^* = M^*\left(\frac{h + p}{p}\right) = \sqrt{\frac{2KD(h + p)}{hp}} = \sqrt{\frac{2KD}{p} + \frac{2KD}{h}}. \tag{7}$$

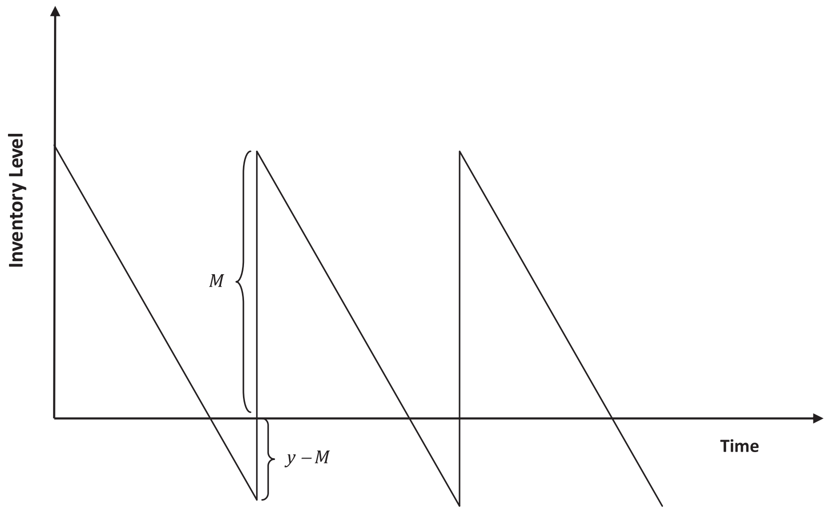

Fig. 1. The behavior of the inventory model.

Eq. (7) reduces to the EOQ formula in [15] as p approaches a very large value; i.e.,

$$y^* = \lim_{p\to\infty}\sqrt{\frac{2KD(h+p)}{hp}} = \sqrt{\frac{2KD}{h}}. \tag{8}$$

5. Fuzzy Modeling of the EOQ model with backorders

In this section, the model presented in Section 4 is fully fuzzified; i.e., by fuzzifying the input parameters (K, D, h and p) and the decision variables (y and M).

Assume now that each input parameter is a trapezoidal fuzzy number consisting of four components as:

$$\begin{aligned}
&\text{Order cost}: \quad \widetilde{K} = (K-\delta_1, K-\delta_2, K+\delta_3, K+\delta_4),\\
&\text{Demand rate}: \quad \widetilde{D} = (D-\delta_5, D-\delta_6, D+\delta_7, D+\delta_8),\\
&\text{Holding cost}: \quad \tilde{h} = (h-\delta_9, h-\delta_{10}, h+\delta_{11}, h+\delta_{12}),\\
&\text{Penalty cost}: \quad \tilde{p} = (p-\delta_{13}, p-\delta_{14}, p+\delta_{15}, p+\delta_{16}),
\end{aligned}$$

where δ_i, $i = 1, 2, \ldots, 16$, is the amount by which a parameter arbitrarily deviates (above or below) from its base value such that $\delta_1 > \delta_2$, $\delta_3 < \delta_4$, $\delta_5 > \delta_6$, $\delta_7 < \delta_8$, $\delta_9 > \delta_{10}$, $\delta_{11} < \delta_{12}$, $\delta_{13} < \delta_{14}$, $\delta_{15} < \delta_{16}$. Also assume that the decision variables are too fuzzified according to the trapezoidal rule as

$$\begin{aligned}
&\text{Maximum inventory level}: \quad \widetilde{M} = (M-\gamma_1, M-\gamma_2, M+\gamma_3, M+\gamma_4),\\
&\text{Batch size}: \quad \tilde{y} = (y-\Delta_1, y-\Delta_2, y+\Delta_3, y+\Delta_4),
\end{aligned}$$

where $\gamma_i > 0$, $i = 1, 2, 3, 4$, $\Delta_i > 0$, $i = 1, 2, 3, 4$, where $\gamma_1 > \gamma_2$ $\gamma_3 < \gamma_4$, $\Delta_1 > \Delta_2$, $\Delta_3 < \Delta_4$ The values of δ_i, γ_i and Δ_i are determined by the inventory system's decision maker. The full-fuzzy form of the annual inventory cost function in Eq. (5) is given as

$$T\widetilde{C}U(\tilde{y},\widetilde{M}) = \frac{\widetilde{K}\widetilde{D}}{\tilde{y}} + \frac{\tilde{h}\widetilde{M}^2}{2\tilde{y}} + \frac{(\tilde{y}-\widetilde{M})^2\tilde{p}}{2\tilde{y}} = \frac{\widetilde{K}\widetilde{D}}{\tilde{y}} + \frac{(\tilde{h}+\tilde{p})\widetilde{M}^2}{2\tilde{y}} + \frac{\tilde{y}}{2}\tilde{p} - \widetilde{M}\tilde{p}, \tag{9}$$

where

$$\frac{\widetilde{K}\widetilde{D}}{\tilde{y}} = \left(\frac{(K-\delta_1)(D-\delta_5)}{y+\Delta_4}, \frac{(K-\delta_2)(D-\delta_6)}{y+\Delta_3}, \frac{(K+\delta_3)(D+\delta_7)}{y-\Delta_2}, \frac{(K+\delta_4)(D+\delta_8)}{y-\Delta_1}\right), \tag{10}$$

$$\begin{aligned}
&(\tilde{h}+\tilde{p})\frac{\widetilde{M}^2}{2\tilde{y}}\\
&= \left(\frac{[(h-\delta_9)+(p-\delta_{13})](M-\gamma_1)^2}{2(y+\Delta_4)}, \frac{[(h-\delta_{10})+(p-\delta_{14})](M-\gamma_2)^2}{2(y+\Delta_3)}, \frac{[(h+\delta_{11})+(p+\delta_{15})](M+\gamma_3)^2}{2(y-\Delta_2)}, \frac{[(h+\delta_{12})+(p+\delta_{16})](M+\gamma_4)^2}{2(y-\Delta_1)}\right),
\end{aligned} \tag{11}$$

$$\frac{\tilde{y}}{2}\tilde{p} = \left(\frac{1}{2}(y-\Delta_1)(p-\delta_{13}),\ \frac{1}{2}(y-\Delta_2)(p-\delta_{14}),\ \frac{1}{2}(y+\Delta_3)(p+\delta_{15}),\ \frac{1}{2}(y+\Delta_4)(p+\delta_{16})\right), \tag{12}$$

$$-\widetilde{M}\tilde{p} = (-(M+\gamma_4)(p-\delta_{13}),\ -(M+\gamma_3)(p-\delta_{14}),\ -(M-\gamma_2)(p+\delta_{15}),\ -(M-\gamma_1)(p+\delta_{16})). \tag{13}$$

Substituting Eqs. (10)–(13) in Eq. (9), the fuzzy cost function is represented as

$$T\widetilde{C}U(y,M) = (C_1, C_2, C_3, C_4),$$

where

$$C_1 = \frac{(K-\delta_1)(D-\delta_5)}{y+\Delta_4} + \frac{[(h-\delta_9)+(p-\delta_{13})](M-\gamma_1)^2}{2(y+\Delta_4)} + \frac{1}{2}(y-\Delta_1)(p-\delta_{13}) - (M+\gamma_4)(p-\delta_{13})$$
$$C_2 = \frac{(K-\delta_2)(D-\delta_6)}{y+\Delta_3} + \frac{[(h-\delta_{10})+(p-\delta_{14})](M-\gamma_2)^2}{2(y+\Delta_3)} + \frac{1}{2}(y-\Delta_2)(p-\delta_{14}) - (M+\gamma_3)(p-\delta_{14})$$
$$C_3 = \frac{(K+\delta_3)(D+\delta_7)}{y-\Delta_2} + \frac{[(h+\delta_{11})+(p+\delta_{15})](M+\gamma_3)^2}{2(y-\Delta_2)} + \frac{1}{2}(y+\Delta_3)(p+\delta_{15}) - (M-\gamma_2)(p+\delta_{15})$$
$$C_4 = \frac{(K+\delta_4)(D+\delta_8)}{y-\Delta_1} + \frac{[(h+\delta_{12})+(p+\delta_{16})](M+\gamma_4)^2}{2(y-\Delta_1)} + \frac{1}{2}(y+\Delta_4)(p+\delta_{16}) - (M-\gamma_1)(p+\delta_{16}),$$

where C_1, C_2, C_3 and C_4 are components of trapezoidal fuzzy numbers.

To defuzzify the all-fuzzy annual inventory cost function, the GMI method is applied as:

$$\begin{aligned}\vartheta(T\widetilde{C}U(\tilde{y},\widetilde{M})) = &\frac{1}{6}\left[\frac{(K-\delta_1)(D-\delta_5)}{y_4} + \frac{[(h-\delta_9)+(p-\delta_{13})]M_1^2}{2y_4} + \frac{1}{2}y_1(p-\delta_{13}) - M_4(p-\delta_{13})\right]\\ &+\frac{2}{6}\left[\frac{(K-\delta_2)(D-\delta_6)}{y_3} + \frac{[(h-\delta_{10})+(p-\delta_{14})]M_2^2}{2y_3} + \frac{1}{2}y_2(p-\delta_{14}) - M_3(p-\delta_{14})\right]\\ &+\frac{2}{6}\left[\frac{(K+\delta_3)(D+\delta_7)}{y_2} + \frac{[(h+\delta_{11})+(p+\delta_{15})]M_3^2}{2y_2} + \frac{1}{2}y_3(p+\delta_{15}) - M_2(p+\delta_{15})\right]\\ &+\frac{1}{6}\left[\frac{(K+\delta_4)(D+\delta_8)}{y_1} + \frac{[(h+\delta_{12})+(p+\delta_{16})]M_4^2}{2y_1} + \frac{1}{2}y_4(p+\delta_{16}) - M_1(p+\delta_{16})\right],\end{aligned} \tag{14}$$

where $0 < M_1 \leqslant M_2 \leqslant M_3 \leqslant M_4$ and $0 < y_1 \leqslant y_2 \leqslant y_3 \leqslant y_4$

Thus, the optimal solution of $\vartheta(T\widetilde{C}U(\tilde{y},\widetilde{M}))$ given in Eq. (14), subject to the following inequality constraints:

$$M_1 - M_2 \leqslant 0, \quad M_2 - M_3 \leqslant 0, \quad M_3 - M_4 \leqslant 0, \quad -M_1 < 0$$

and

$$y_1 - y_2 \leqslant 0, \quad y_2 - y_3 \leqslant 0, \quad y_3 - y_4 \leqslant 0, \quad -y_1 < 0.$$

The Kuhn–Tucker conditions were used to find the optimal solution of $\vartheta(T\widetilde{C}U(\tilde{y},\widetilde{M}))$ subject to eight inequalities as imposed conditions. Kuhn–Tucker conditions based on Theorem 1 are:

$$\frac{1}{6}\left(\frac{[(h-\delta_9)+(p-\delta_{13})]M_1}{y_4} - (p+\delta_{16})\right) - u_1 + u_4 \leqslant 0, \tag{15.1}$$
$$\frac{2}{6}\left(\frac{[(h-\delta_{10})+(p-\delta_{14})]M_2}{y_3} - (p+\delta_{15})\right) + u_1 - u_2 \leqslant 0, \tag{15.2}$$
$$\frac{2}{6}\left(\frac{[(h+\delta_{11})+(p+\delta_{15})]M_3}{y_2} - (p-\delta_{14})\right) + u_2 - u_3 \leqslant 0, \tag{15.3}$$
$$\frac{1}{6}\left(\frac{[(h+\delta_{12})+(p+\delta_{16})]M_4}{y_1} - (p-\delta_{13})\right) + u_3 \leqslant 0, \tag{15.4}$$
$$\frac{1}{12}(p-\delta_{13}) + \frac{1}{6}\left[-\frac{(K+\delta_4)(D+\delta_8)}{y_1^2} - \frac{[(h+\delta_{12})+(p+\delta_{16})]M_4^2}{2y_1^2}\right] - u_5 + u_8 \leqslant 0, \tag{15.5}$$

$$\frac{1}{6}(p-\delta_{14})+\frac{2}{6}\left[-\frac{(K+\delta_3)(D+\delta_7)}{y_2^2}-\frac{[(h+\delta_{11})+(p+\delta_{15})]M_3^2}{2y_2^2}\right]+u_5-u_6\leqslant 0, \quad (15.6)$$

$$\frac{1}{6}(p+\delta_{15})+\frac{2}{6}\left[-\frac{(K-\delta_2)(D-\delta_6)}{y_3^2}-\frac{[(h-\delta_{10})+(p-\delta_{14})]M_2^2}{2y_3^2}\right]+u_6-u_7\leqslant 0, \quad (15.7)$$

$$\frac{1}{12}(p+\delta_{16})+\frac{1}{6}\left[-\frac{(K-\delta_1)(D-\delta_5)}{y_4^2}-\frac{[(h-\delta_9)+(p-\delta_{13})]M_1^2}{2y_4^2}\right]+u_7\leqslant 0, \quad (15.8)$$

$$M_1\times\left[\frac{1}{6}\left(\frac{[(h-\delta_9)+(p-\delta_{13})]M_1}{y_4}-(p-\delta_{13})\right)-u_1+u_4\right]=0, \quad (15.9)$$

$$M_2\times\left[\frac{2}{6}\left(\frac{[(h-\delta_{10})+(p-\delta_{14})]M_2}{y_3}-(p-\delta_{14})\right)+u_1-u_2\right]=0, \quad (15.10)$$

$$M_3\times\left[\frac{2}{6}\left(\frac{[(h-\delta_{11})+(p+\delta_{15})]M_3}{y_2}-(p+\delta_{15})\right)+u_2-u_3\right]=0, \quad (15.11)$$

$$M_4\times\left[\frac{1}{6}\left(\frac{[(h-\delta_{12})+(p+\delta_{16})]M_4}{y_1}-(p+\delta_{16})\right)+u_2-u_3\right]=0, \quad (15.12)$$

$$y_1\times\left[\frac{1}{12}(p-\delta_{13})+\frac{1}{6}\left(-\frac{(K+\delta_4)(D+\delta_8)}{y_1^2}-\frac{[(h+\delta_{12})+(p+\delta_{16})]M_4^2}{2y_1^2}\right)-u_5+u_8\right]=0, \quad (15.13)$$

$$y_2\times\left[\frac{1}{6}(p-\delta_{14})+\frac{2}{6}\left(-\frac{(K+\delta_3)(D+\delta_7)}{y_2^2}-\frac{[(h+\delta_{11})+(p+\delta_{15})]M_3^2}{2y_2^2}\right)+u_5-u_6\right]=0, \quad (15.14)$$

$$y_3\times\left[\frac{1}{6}(p+\delta_{15})+\frac{2}{6}\left(-\frac{(K-\delta_2)(D-\delta_6)}{y_3^2}-\frac{[(h-\delta_{10})+(p-\delta_{14})]M_2^2}{2y_3^2}\right)+u_6-u_7\right]=0, \quad (15.15)$$

$$y_4\times\left[\frac{1}{12}(p+\delta_{16})+\frac{2}{6}\left(-\frac{(K-\delta_1)(D-\delta_5)}{y_4^2}-\frac{[(h-\delta_9)+(p-\delta_{13})]M_1^2}{2y_4^2}\right)+u_6-u_7\right]=0, \quad (15.16)$$

$$M_i-M_{i+1}\leqslant 0,\quad i=1,2,3, \quad (15.17)$$

$$-M_1<0, \quad (15.18)$$

$$y_i-y_{i+1}\leqslant 0,\quad i=1,2,3, \quad (15.19)$$

$$-y_1<0, \quad (15.20)$$

$$u_{i'}(M_i-M_{i+1})=0,\quad i=1,2,3, \quad (15.21)$$

$$u_{4'}-M_1=0, \quad (15.22)$$

$$u_{4+i'}(y_i-y_{i+1})0,\quad i=1,2,3, \quad (15.23)$$

$$u_{8'}-y_1=0, \quad (15.24)$$

$$M_i\geqslant 0,\quad y_i\geqslant 0,\quad i=1,2,3,4\quad \text{and } u_j\geqslant 0,\quad j=1,\ldots,8. \quad (15.25)$$

From constraints (15-18) and (15-22), we have $M_1 > 0$ and $u_4 - M_1 = 0$, implying that $u_4 = 0$. If $u_i = u_{4+1} = 0$ in (15-21) and (15-23), respectively, then $0 < M_1 < M_2 \leqslant M_3 \leqslant M_4$. Hence, $M_1 = M_2$, $M_2 = M_3$ and $M_3 = M_4$ in Eq. (15-19) suggesting $M_1 = M_2 = M_3 = M_4 = M^*$. Since $y_1 > 0$ then from constraint (15-24), $u_8 = 0$. If $u_{4+i} = 0$, then $0 < y_1 \leqslant y_2 \leqslant y_3 \leqslant y_4$ Therefore, $y_1 = y_2$, $y_2 = y_3$ and $y_3 = y_4$ that $y_1 = y_2 = y_3 = y_4 = y^*$ With these interpretations, the solution of the model is determined by solving Eqs. (15-1)–(15-25) as:

$$y^*=\frac{(h-\delta_9)+2(h-\delta_{10})+2(h+\delta_{11})+(h+\delta_{12})+(p-\delta_{13})+2(p-\delta_{14})+2(p+\delta_{15})+(p+\delta_{16})}{(p-\delta_{13})+2(p-\delta_{14})+2(p+\delta_{15})+(p+\delta_{16})}M^*,$$

$$M^*=\sqrt{\frac{[2(K-\delta_1)(D-\delta_5)+4(K-\delta_2)(D-\delta_6)+4(K+\delta_3)(D+\delta_7)+2(K+\delta_4)(D+\delta_3)]\times[(p-\delta_{13})+2(p-\delta_{14})+2(p+\delta_{15})+(p+\delta_{16})]}{[(h-\delta_9)+2(h-\delta_{10})+2(h+\delta_{11})+(h+\delta_{12})+(p-\delta_{13})+2(p-\delta_{14})+2(p+\delta_{15})+(p+\delta_{16})]'\times[(h-\delta_9)+2(h-\delta_{10})+2(h+\delta_{11})+(h+\delta_{12})]}}, \quad (16)$$

$$y^*=\sqrt{\frac{[2(K-\delta_1)(D-\delta_5)+4(K-\delta_2)(D-\delta_6)+4(K+\delta_3)(D+\delta_7)+2(K+\delta_4)(D+\delta_3)]\times[(h-\delta_9)+(p-\delta_{13})+2(h-\delta_{10})+2(p-\delta_{14})+2(h+\delta_{11})+2(p+\delta_{15})+(h+\delta_{12})+(p+\delta_{16})]}{[(h-\delta_9)+2(h-\delta_{10})+2(h+\delta_{11})+(h+\delta_{12})]\times[(p-\delta_{13})+2(p-\delta_{14})+2(p+\delta_{15})+(p+\delta_{16})]}}. \quad (17)$$

Note that when substituting for $\gamma_i = 0$, $i = 1, 2, 3, 4$,$\Delta_i = 0$, $i = 1, 2, 3, 4$, and $\delta_i = 0$,$i = 1, 2, \ldots, 16$, in Eqs. (16) and (17) they reduce to Eqs. (6) and (7), respectively. With some modifications, the above model can be used for the case when the input parameters and decision variables are triangular fuzzy numbers as: $\widetilde{K}=(K-\delta_1,K,K+\delta_4)$, $\widetilde{D}=(D-\delta_5,D,D+\delta_8)\tilde{h}=(h-\delta_9,h,h+\delta_{12})\tilde{p}=(p-\delta_{13},p,p+\delta_{16})$, $\widetilde{M}=(M-\gamma_1,M,M+\gamma_4)$ and $\tilde{y}=(y-\Delta_1,y,y+\Delta_4)$ where $k > \delta_1$, $D > \delta_5$, $h > \delta_9$, $p > \delta_{13}$, $M > \gamma_1$, and $y > \Delta_1$. The triangular fuzzy forms of Eqs. (16) and (17) are then given as:

$$y^*=\sqrt{\frac{[2(K-\delta_1)(D-\delta_5)+2(K-\delta_4)(D+\delta_9)+8KD]\times[(h-\delta_9)+(h+\delta_{12})+(p-\delta_{13})+(p+\delta_{16})+8(h+p)]}{[(h-\delta_9)+(h+\delta_{12})+4h]\times[(p-\delta_{13})+(p+\delta_{16})+4p]}}, \quad (18)$$

$$M^*=\sqrt{\frac{[2(K-\delta_1)(D-\delta_5)+2(K+\delta_4)(D+\delta_9)+8KD]\times[(p-\delta_{13})+(p+\delta_{16})+4p]}{[(h-\delta_9)+(h+\delta_{12})+(p-\delta_{13})+(p+\delta_{16})+4(h+p)]\times[(h-\delta_9)+(h+\delta_{12})+4h]}}. \quad (19)$$

6. Numerical examples

In this section, numerical examples are presented to illustrate the behavior of the model developed in Section 5 with the results compared to those of the crisp case using the parameters in Björk [1].

Consider an inventory situation with crisp parameters having the following values (from Björk [1]): D=50,000 kg/year, C = 1 Euro/kg (purchase price per each unit), h = % 25 of the purchase price, K = 200 Euros in each purchase, p=5 Euros/kg in a year. According to Björk [1, Table 2], the optimal order quantity, the optimal maximum inventory and the optimal total cost for this inventory system are y^* = 9165.15, M^* = 8728.72 and TCU^* = 2128.18 respectively. In Tables 1 and 2, we set some trapezoidal fuzzy numbers of the input parameters (K,D,h,p) only, a special case of the model developed in Section 5, to test the model. For each of these parameters, the variations in the values are arranged arbitrary and their defuzzified values are determined by applying the GMI method are shown in the second and fifth columns of Tables 1 and 2. Furthermore, the third and sixth columns in these tables display the percentage difference between the optimal crisp and fuzzy cost values. Based on these values from Eqs. (16) and (17), we can ascertain these two optimal policies for each set of trapezoidal fuzzy numbers. The results are summarized in Table 3.

In Table 3, columns 2 and 4 present the difference in the optimal values of the decision variables, y^* and M^*, between the fuzzy and the crisp cases, which were found to be identical. The range of change for the input parameters was set to be −30 to + 30% corresponding to a change in the TCU^* value (computed from (15)) ranging from −34.5 to 58.69%. This shows that assuming crisp values can lead to erroneous inventory policies that may be a cause in business failure. This also suggests that using fuzzy theory may help in reducing the uncertainty that governs setting some of the input parameters (e.g., holding and stock-out costs). The advantage of the non-classical approach of using fuzzy sets to model and analyze inventory systems have been discussed in Guiffrida [13]. The results in Table 3 show that the model is more sensitive to negative levels of fuzziness. For example, a −30% in the values of the input parameters results in a reduction in the y^* and M^* values of −3.01%, corresponding to a reduction in the TCU^* value of −34.5%, whereas an increase of 20.8% in the value of $y^*(M^*)$ increases the TCU^* value by 58.69%; i.e., −34.5/−3.01 = 11.46 > 58.69/20.8 = 2.82. Furthermore, a simple linear regression analysis was performed and we found that the change in TCU^* (ΔTCU^*) increased linearly; i.e., $\Delta TCU^* = -0.1614 + 0.3994 \times \Delta y^*$ where $R^2 = 97.7\%$. Similarly, the Δy^* increased linearly as the fuzziness of the input parameter $(K,D h,p)$ increased; $\Delta y^* = 0.06775 + 0.4125 \times \Delta\,(K,D h,p)$ where $R^2 = 94.2\%$. These linear relationships may help develop simpler fuzzy inventory models that managers can implement and interpret their results easily.

The results in Table 4 were reproduced form Tables 1 and 2 where triangular fuzzy numbers were used instead of the trapezoidal ones. Similar to the previous analysis, a simple linear regression analysis was performed and we found $\Delta TCU^* = -0.0249 + 1.877^*\,\Delta M^*$ where $R^2 = 81.4\%$, and a poor linear relationship ($R^2 < 40\%$) between ΔTCU^* and Δy^*. Comparing the results in Tables 3 and 5, shows that ΔTCU^* is more sensitive (ΔTCU^* in Table 5; from −34.5 to 58.69%) to $\Delta(K,D h,p)$ (from −30 to +30) for triangular than it is for trapezoidal fuzzy number (ΔTCU^* in Table 5; from −9.28 to 26.59%). Fuzzifying the decision variables produced almost identical results to those presented above.

Table 1
Fuzzy trapezoidal values for the input parameters K and D.

K	$\vartheta(K)$	Change (%)	D	$\vartheta(D)$	Change (%)
(20, 90, 210, 220)	140	−30	(5000, 18000, 50500, 68000)	35000	−30
(60, 110, 220, 240)	160	−20	(12000, 22000, 53000, 78000)	40000	−20
(120, 130, 230, 260)	180	−10	(20000, 35000, 55000, 70000)	45000	−10
(100, 140, 240, 280)	190	−5	(32000, 34000, 60000, 65000)	47500	−5
(145, 170, 250, 275)	210	5	(30000, 35000, 70000, 75000)	52500	5
(150, 180, 255, 300)	220	10	(29000, 41000, 63000, 93000)	55000	10
(150, 185, 295, 330)	240	20	(42000, 47000, 75000, 94000)	60000	20
(185, 195, 300, 385)	260	30	(33000, 42000, 81000, 111000)	65000	30

Table 2
Fuzzy trapezoidal values for the input parameters h and p.

h	$\vartheta(h)$	Change (%)	p	$\vartheta(p)$	Change (%)
(0.05, 0.07, 0.26, 0.34)	0.1750	−30	(0.5, 1.5, 5.6, 6.5)	3.500	−30
(0.06, 0.12, 0.27, 0.36)	0.2000	−20	(1, 2, 5.7, 7.6)	4.000	−20
(0.11, 0.13, 0.3, 0.38)	0.2250	−10	(1.5, 3, 6, 7.5)	4.500	−10
(0.09, 0.2, 0.28, 0.375)	0.2375	−5	(2.6, 3.4, 6.3, 6.5)	4.750	−5
(0.1, 0.15, 0.35, 0.475)	0.2625	5	(2.4, 4.1, 6.2, 8.5)	5.250	5
(0.13, 0.17, 0.33, 0.52)	0.2750	10	(3.6, 4.2, 6.3, 8.4)	5.500	10
(0.16, 0.21, 0.4, 0.42)	0.3000	20	(3.7, 4.1, 7.1, 9.9)	6.000	20
(0.18, 0.22, 0.43, 0.47)	0.3550	30	(4, 4.6, 8.1, 9.6)	6.500	30

Table 3
The change in optimal policy from the crisp case using the trapezoidal fuzzy number in Tables 1 and 2.

y^*	Change in y^* (%)	M^*	Change in M^* (%)	TCU^*	Change in TCU^* (%)
8889.32	−3.01	8466.02	−3.01	756.28	−34.5
9126.88	−0.42	8692.27	−0.42	1738.46	−20.33
9249.62	0.92	88091.17	0.92	1982.79	−9.14
9381.28	2.36	8934.55	2.36	2121.98	−2.76
9789.45	6.81	9323.29	6.81	2472.75	13.31
10036.12	9.5	9558.21	9.5	2619.46	20.03
10744.92	17.24	10233.26	17.24	3069.89	40.68
11071.38	20.8	10544.17	20.8	3426.84	58.69

Table 4
Triangular fuzzy numbers corresponding to trapezoidal fuzzy number.

K	D	h	p
(20,200,220)	(5000,50,000,68,000)	(0.05,0.25,0.34)	(0.5,5,6.5)
(60,200,240)	(12,000,50,000,78000)	(0.06,0.25,0.36)	(1,5,7.6)
(120,200,260)	(20,000,50,000,70,000)	(0.11,0.25,0.38)	(1.5,5,7.5)
(100,200,280)	(32,000,50,000,65,000)	(0.09,0.25,0.375)	(2.6,5,6.5)
(145,200,275)	(30,000,50,000,75000)	(0.1,0.25,0.475)	(2.4,5,8.5)
(150,200,300)	(29,000,50,000,93,000)	(0.13,0.25,0.52)	(3.6,5,8.4)
(150,200,330)	(42,000,50,000,94,000)	(0.16,0.25,0.42)	(3.7,5,9.9)
(185,200,385)	(33,000,50,000,111,000)	(0.18,0.25,0.47)	(4,5,9.6)

Table 5
Optimal policy by using triangular fuzzy number.

y^*	Change in Y^* (%)	M^*	Change in M^* (%)	TCU^*	Change in TCU^* (%)
12038.25	31.35	8680.1	−0.56	1979.75	−9.28
12220.74	33.34	8930.75	2.26	2114.36	3.11
12017.68	31.12	8795.86	0.71	2184.55	0.11
12197.45	33.09	8933.36	2.34	2180.84	−0.99
11949.34	30.38	8860.38	1.51	2325.62	6.57
12229.59	33.44	9125.87	4.55	2506.17	14.85
12792.45	39.58	9668.70	10.77	2532.30	16.04
13426.69	46.50	10131.35	16.07	2762.41	26.59

7. Summary and conclusions

In this paper, an inventory model with planned backorder with fuzzy parameters and decision variables was developed. The model was solved for triangular and trapezoidal fuzzy numbers using Kuhn–Tucker conditions. The results showed that the changes in the values of the decision variables (the maximum inventory level and the batch size) to changes in the costs between the crisp (deterministic) and fuzzy cases demonstrated a linear relationship. That is, increasing the values of the decision variables increases the difference in the cost between the crisp and the fuzzy case linearly. This may be an enticement to benefit from this relationship and develop simpler fuzzy inventory models that can be easily utilized by managers. The results also showed that the cost and the values of the decision variables were more sensitive to changes in the input parameters when triangular fuzzy numbers were used. The full-fuzzy approach presented in this paper could be applied to other inventory models with crisp and/or fuzzy conditions. For example, applying the full-fuzzy approach to the works of Chang et al. [5], Björk [2] and Björk and Carlsson [3] would be interesting immediate extensions.

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