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An Explicit Finite Differences Scheme over Hexagonal Tessellation

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Abstract—We introduce a new numerical method inspired in the cellular automata methodology to study the transmission of waves in two-dimensional solids The stability of the second-order method is investigated and compared with that of a classical finite differences method for both wave and elastic equations

Keywords—Finite differences methods, Elastic waves, Hexagonal tessellation

1. INTRODUCTION

In recent years, the interest in understanding the inner phenomena that occur when stress waves propagate along materials has grown dramatically This is mainly due to the rise of applications in civil and military engineering and the investigations in new materials [1–3] Moreover, geophysicists need to understand how seismic signals travel through different paths to investigate the inner structure of the Earth [4,5] This is an example of what is known as an inverse problem

For many years, it has been well known how acoustic waves propagate along a homogeneous plate Although the acoustic wave equation is a simplification of the phenomena concerned in real materials, it actually is a good workbench to probe the suitability of a numerical method

In Section 2, the stability of the proposed hexagonal method will be compared with the stability of the standard explicit square-cell method in the simulation of the 2D wave equation In Section 3, we propose a hexagonal discretisation for the elastic equation Finally, some conclusions are summarized in Section 4

2. SIMULATION OF THE WAVE EQUATION

The solutions of the 2D wave equation can be analytically derived The solutions obtained with our scheme will be compared with those obtained by means of a conventional explicit square scheme

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2.1. The Wave Equation

The basic governing equation for a taut membrane under in-plane tension that describes the motion of small deflections normal to the membrane is the 2D wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi = c^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right), \tag{1}$$

where c represents the propagation velocity along the membrane and ϕ is the displacement along the normal to the membrane

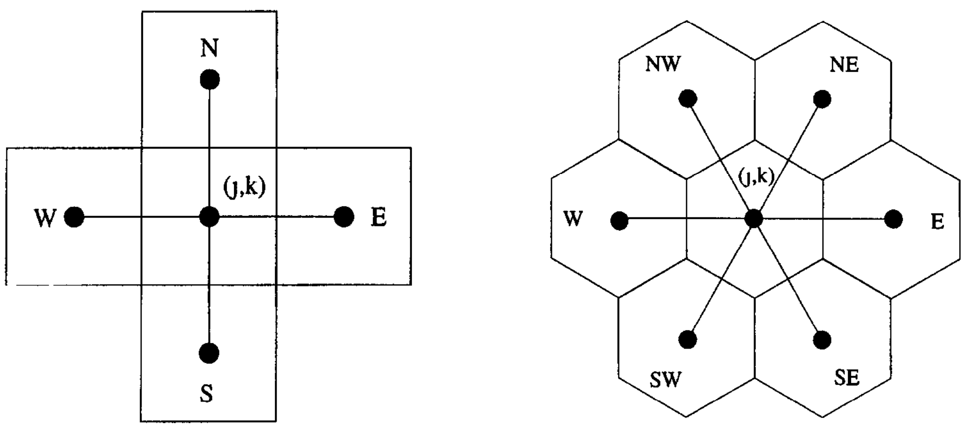
A plane wave propagating along the plane in the positive x direction takes the form

$$\phi(x, y, t) = f(x) \quad e^{i(\xi x - \omega t)} \tag{2}$$

It is easily shown that the propagation velocity c is independent of the frequency ω This means that for a given disturbance, it propagates without distortion along the membrane

2.2. Square Discretisation

First, the conventional explicit square discretisation of a 2D wave equation is studied It corresponds to the tessellation shown in Figure 1a



(a) Square computational molecules (b) Hexagonal computational molecules

Figure 1

The time derivative is discretised in the usual explicit form

$$\frac{\partial^2 \phi}{\partial t^2} \approx \frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2}, \tag{3}$$

where τ is the time stepsize and u^n is the estimated value of the function ϕ in the instant $n \ \tau$

In order to approximate the Laplacian operator, a centered discrete scheme will be used, with h being the spatial stepsize

$$\nabla^2 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi \approx \frac{u_{\mathrm{E}} + u_{\mathrm{N}} + u_{\mathrm{W}} + u_{\mathrm{S}} - 4u_{j,k}}{h^2}, \tag{4}$$

where u_{jk} represents the discrete value of the function ϕ at coordinates $(j \ h, k \ h)$ Then the whole discretization rule of the wave equation is written as

$$u_{j,k}^{n+1} = 2u_{j,k}^n - u_{j,k}^{n-1} + c^2 \frac{\tau^2}{h^2} \left(u_{\mathrm{E}}^n + u_{\mathrm{N}}^n + u_{\mathrm{W}}^n + u_{\mathrm{S}}^n - 4u_{j,k}^n \right) \tag{5}$$

From this iteration rule, by applying the Von Neumann method for stability analysis, we find the well-known stability condition

$$\frac{\tau}{h} \le c^{-1} \frac{\sqrt{2}}{2} \tag{6}$$

2.3. Hexagonal Scheme

Most two-dimensional numerical methods use square grids for simplicity reasons However, the main disadvantage of these methods is their intrinsic numerical anisotropy [6] This anisotropy appears, for example, in the simulation of a KdV equation, is inherent to the square scheme, and cannot be avoided with a finer grid Hexagonal schemes, on the other hand, show a higher isotropy

2.3.1. Discretisation

The discretisation of the wave equation can be obtained from a Taylor expansion of the Laplacian operator

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\approx\frac{(2/3)\left(u_{\mathrm{E}}+u_{\mathrm{NE}}+u_{\mathrm{NW}}+u_{\mathrm{W}}+u_{\mathrm{SW}}+u_{\mathrm{SE}}\right)-4u_{j,k}}{h^2} \tag{7}$$

The computational molecule is shown in Figure 1b The space of simulation is divided into regular hexagons and the magnitude ϕ is measured at the center of each element

The discretisation of the temporal derivative is similar to (3), so the iteration formula we are using is

$$u_{j,k}^{n+1}=2u_{j,k}^{n}-u_{j,k}^{n-1}+c^2\frac{\tau^2}{h^2}\left[\frac{2}{3}\left(u_{\mathrm{E}}^{n}+u_{\mathrm{NE}}^{n}+u_{\mathrm{NW}}^{n}+u_{\mathrm{W}}^{n}+u_{\mathrm{SW}}^{n}+u_{\mathrm{SE}}^{n}\right)-4u_{j,k}n^{n}\right] \tag{8}$$

2.3.2. Stability of the numerical scheme

It is of interest to evaluate the stability condition of the previous scheme as a function of the discretisation parameters τ and h

Assuming that the solution of (1) can be expressed by a Fourier expansion with separated time and space variables, a general term of the series can be written as

$$\begin{aligned}\phi(x,y,t)&=e^{\imath\alpha x}\quad e^{\imath\beta y}\quad e^{\imath\omega t}\\&=e^{\imath\alpha jh}\quad e^{\imath\beta kh}\quad e^{\imath\omega n\tau}\end{aligned} \tag{9}$$

We substitute this solution in (8), and taking into account the following relations

$$\begin{aligned}&u_{\mathrm{E}}^{n}=e^{\imath\alpha h}u_{j,k}^{n}, && u_{\mathrm{NE}}^{n}=e^{\imath(1/2)\alpha h}e^{\imath(\sqrt{3}/2)\beta h}u_{j,k}^{n}, && u_{\mathrm{NW}}^{n}=e^{-\imath(1/2)\alpha h}e^{\imath(\sqrt{3}/2)\beta h}u_{j,k}^{n},\\&u_{\mathrm{W}}^{n}=e^{-\imath\alpha h}u_{j,k}^{n}, && u_{\mathrm{SW}}^{n}=e^{-\imath(1/2)\alpha h}e^{-\imath(\sqrt{3}/2)\beta h}u_{j,k}^{n}, && u_{\mathrm{SE}}^{n}=e^{\imath(1/2)\alpha h}e^{-\imath(\sqrt{3}/2)\beta h}u_{j,k}^{n},\end{aligned} \tag{10}$$

and rearranging terms, we obtain

$$\begin{aligned}u_{j,k}^{n+1}+u_{j,k}^{n-1}=&\left\{2+c^2\frac{\tau^2}{h^2}\left[\frac{2}{3}\left(e^{\imath\alpha h}+e^{-\imath\alpha h}+e^{\imath(1/2)\alpha h}e^{\imath(\sqrt{3}/2)\beta h}\right.\right.\right.\\&+e^{-\imath(1/2)\alpha h}e^{\imath(\sqrt{3}/2)\beta h}+e^{-\imath(1/2)\alpha h}e^{-\imath(\sqrt{3}/2)\beta h}\\&\left.\left.\left.+e^{\imath(1/2)\alpha h}e^{-\imath(\sqrt{3}/2)\beta h}\right)-4\right]\right\}u_{j,k}^{n}\\=&\left(e^{\imath\omega\tau}+e^{-\imath\omega\tau}\right)u_{j,k}^{n}\end{aligned} \tag{11}$$

or

$$2\cos(\omega\tau)=2+c^2\frac{\tau^2}{h^2}\left[\frac{2}{3}\left(2\cos(\alpha h)+4\cos\left(\frac{1}{2}\alpha h\right)\cos\left(\frac{\sqrt{3}}{2}\beta h\right)\right)-4\right] \tag{12}$$

Since $-1\le\cos(\omega\tau)\le1$, then

$$-1\le1+c^2\frac{\tau^2}{h^2}\left[\frac{1}{3}\left(2\cos(\alpha h)+4\cos\left(\frac{1}{2}\alpha h\right)\cos\left(\frac{\sqrt{3}}{2}\beta h\right)\right)-2\right]\le1, \tag{13}$$

and we get

$$-6 \leq c^2 \frac{\tau^2}{h^2} (s - 6) \leq 0, \tag{14}$$

where

$$s = 2\cos(\alpha h) + 4\cos\left(\frac{1}{2}\alpha h\right)\cos\left(\frac{\sqrt{3}}{2}\beta h\right) \tag{15}$$

The extrema of the function $s(\alpha, \beta)$ are

$$\max(s) = 6, \tag{16}$$

$$\min(s) = -3 \tag{17}$$

From (16), it is found that the right-hand inequality in (14) is always satisfied From the left-hand inequity and (17),

$$-6 \leq c^2 \frac{\tau^2}{h^2} (-3 - 6), \tag{18}$$

and therefore,

$$c^2 \frac{\tau^2}{h^2} \leq \frac{2}{3} \tag{19}$$

Finally, the hexagonal scheme is stable if

$$\frac{\tau}{h} \leq c^{-1} \sqrt{\frac{2}{3}}, \tag{20}$$

which compares favourably with (6)

3. ELASTIC 2D EQUATION

The wave equation studied in the previous section describes wave propagation in a medium that cannot sustain finite shear stress, such as fluids or membranes vibrating normally to their surface In general 2D solids, however, two kinds of waves can exist propagating simultaneously dilatational (primary) waves and shear (secondary) waves These waves propagate with different velocities and it is known that when a wave of either type reaches a boundary or discontinuity in the solid, waves of both types are generated

In Cartesian coordinates, the 2D elastic equation can be written as [7]

$$\begin{aligned} \rho \frac{\partial^2 \phi}{\partial t^2} &= (\lambda + 2\mu) \frac{\partial^2 \phi}{\partial x^2} + \mu \frac{\partial^2 \phi}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 \psi}{\partial x \partial y}, \\ \rho \frac{\partial^2 \psi}{\partial t^2} &= \mu \frac{\partial^2 \psi}{\partial x^2} + (\lambda + 2\mu) \frac{\partial^2 \psi}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 \phi}{\partial x \partial y}, \end{aligned} \tag{21}$$

where ϕ, ψ are the displacements along x and y directions, ρ is the material density, and λ, μ are the Lamé parameters

3.1. Hexagonal Scheme

Equation (21) is approximated using a hexagonal tessellation

3.1.1. Hexagonal discretisation

The temporal derivative is again approximated as (3) The spatial derivatives are approximated by the following formulae

$$
\begin{aligned}
&(\lambda+2\mu)\frac{\partial^2\phi}{\partial x^2}+\mu\frac{\partial^2\phi}{\partial y^2}+(\lambda+\mu)\frac{\partial^2\psi}{\partial x\partial y}\approx\frac{F}{h^2}\\
&=\frac{1}{h^2}\left[-A_0u_{j,k}+A_1\left(u_{\mathrm{NE}}+u_{\mathrm{NW}}+u_{\mathrm{SW}}+u_{\mathrm{SE}}\right)\right.\\
&\left.+A_2\left(u_{\mathrm{E}}+u_{\mathrm{W}}\right)+C_1\left(v_{\mathrm{NE}}-v_{\mathrm{NW}}+v_{\mathrm{SW}}-v_{\mathrm{SE}}\right)\right],
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
&\mu\frac{\partial^2\psi}{\partial x^2}+(\lambda+2\mu)\frac{\partial^2\psi}{\partial y^2}+(\lambda+\mu)\frac{\partial^2\phi}{\partial x\partial y}\approx\frac{G}{h^2}\\
&=\frac{1}{h^2}\left[-A_0v_{j,k}+B_1\left(v_{\mathrm{NE}}+v_{\mathrm{NW}}+v_{\mathrm{SW}}+v_{\mathrm{SE}}\right)\right.\\
&\left.+B_2\left(v_{\mathrm{E}}+v_{\mathrm{W}}\right)+C_1\left(u_{\mathrm{NE}}-u_{\mathrm{NW}}+u_{\mathrm{SW}}-u_{\mathrm{SE}}\right)\right],
\end{aligned}
\tag{23}
$$

with the coefficients

$$
\begin{aligned}
A_0&=2\lambda+6\mu, & A_1&=\frac{2\mu}{3}, & A_2&=\frac{3\lambda+5\mu}{3},\\
B_1&=\frac{2\lambda+4\mu}{3}, & B_2&=\frac{\mu-\lambda}{3}, & C_1&=\frac{\lambda+\mu}{\sqrt{3}}
\end{aligned}
\tag{24}
$$

These coefficients have been obtained after a Taylor expansion and ensure a truncation error of $O(h^2)$

Hence, the iteration formulae used in the scheme are written as

$$
\begin{aligned}
u_{j,k}^{n+1}&=2u_{j,k}^{n}-u_{j,k}^{n+1}+\frac{1}{\rho}\frac{\tau^2}{h^2}F,\\
v_{j,k}^{n+1}&=2v_{j,k}^{n}-v_{j,k}^{n+1}+\frac{1}{\rho}\frac{\tau^2}{h^2}G
\end{aligned}
\tag{25}
$$

3.1.2. Numerical stability

By means of a method of Von Neumann similar to the method used in the previous section, the following stability condition is obtained

$$\frac{\tau}{h}\le\sqrt{\frac{3}{4}}\sqrt{\frac{\rho}{\lambda+2\mu}} \tag{26}$$

This condition has been numerically tested, and when compared with stability conditions obtained by other methods [8], it is seen that (26) represents a less restrictive condition In particular, using the same number of nodes in the (2,2)-order method of [8] and in our scheme, the last one allows us to increase the time stepsize by a factor of $\sqrt[4]{3}\simeq 1\ 32$

4. CONCLUSIONS

An explicit numerical scheme has been proposed for the approximation of the 2D elastic equation over a hexagonal tessellation

Numerical stability and isotropy are enhanced with a similar computational work when compared with the standard numerical methods In the simulation of heterogeneous and random media, the proposed method shows excellent behaviour with regard to stability and isotropy To the best of our knowledge, equation (26) is the least restrictive condition for an explicit finite difference method for the 2D elastic equation

Moreover, the same techniques which are applied to other usual numerical schemes (automatic refining, multigrid, etc) can likely be used in this new family of numerical schemes based on hexagonal tessellation

Other equations, such as the nonlinear 2D+1 Sine-Gordon equation, are currently being investigated by means of similar methods based on hexagonal schemes

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