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Colocalization on Grothendieck Categories with Applications to Coalgebras

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INTRODUCTION

Rings and modules of quotients with respect to an additive topology $\mathscr{F}$ or a localizing subcategory of $R-\text{Mod}$ were introduced by Gabriel in his thesis [3], and have been an important tool in ring theory for more than 20 years. If $\mathscr{C}_{\mathscr{F}}$ is the localizing subcategory of $R-\text{Mod}$ associated to $\mathscr{F}$, then we can consider the quotient category $R-\text{Mod}/\mathscr{C}_{\mathscr{F}}$ and the canonical functor $T_{\mathscr{F}}$: $R-\text{Mod} \to R-\text{Mod}/\mathscr{C}_{\mathscr{F}}$ (see [3] and [7]). It is well known that $T_{\mathscr{F}}$ has a right adjoint $S_{\mathscr{F}}$: $R-\text{Mod}/\mathscr{C}_{\mathscr{F}} \to R-\text{Mod}$ (in fact the existence of such a right adjoint functor is equivalent to the fact that the subcategory we factor by is localizing; see [3]). The quotient ring $R_{\mathscr{F}}$ associated to $\mathscr{F}$ is an approach to the quotient category in the sense that under certain conditions $R_{\mathscr{F}}-\text{Mod}$ is equivalent to the quotient category $R-\text{Mod}/\mathscr{C}_{\mathscr{F}}$. This situation represents the perfect localization of rings (or the flat epimorphism of rings; see [7, p. 225]). Starting from the localization for rings, we develop a theory of ‘‘localization’’ for coalgebras,

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giving reasonable answers in this paper. More exactly, if C is a coalgebra over the field k, and $\mathscr{T}$ is a dense subcategory (or a Serre class) of the category M^C of right C-comodules, we can consider the quotient category $M^C/\mathscr{T}$ and the canonical functor $T: M^C \to M^C/\mathscr{T}$. The key step comes now: instead of considering that $\mathscr{T}$ is a localizing subcategory (i.e., $\mathscr{T}$ is closed under arbitrary direct sums or, equivalently, T has a right adjoint), we will ask $\mathscr{T}$ to be a colocalizing subcategory (i.e., the functor T has a left adjoint H). We will see later that a colocalizing subcategory is also localizing. The colocalizing subcategory $\mathscr{T}$ is called perfect if H is an exact functor. This situation is dual to the perfect localization of rings (or the flat epimorphism of rings).

This paper is divided into five sections. In Section 1 we give some properties of left coflat monomorphisms of coalgebras. A study of colocalization in an abelian category is made in Section 2. We apply these results, in Section 3, to the category M^C of right C-comodules, introducing the quotient coalgebra with respect to a colocalization as an analogue of the quotient ring. We also give the main properties of the quotient coalgebra. In Section 4 we study perfect colocalization on M^C, proving that a perfect colocalization is given by a (left) coflat monomorphism. It follows from Section 1 that we can associate a perfect colocalization to any coflat monomorphism of coalgebras. The last section is concerned with applications. A relevant example is the Goldie torsion theory, to which case we apply our theory.

NOTATION AND PRELIMINARIES

Let k be a field. By k-space (k-map) we mean a k-vector space (k-linear map). All unadorned tensor products, Hom, etc., will be over k. The reader is referred to the books [1] and [8] for notions and notations concerning coalgebras and comodules. The category of k-coalgebras is denoted by $\mathbf{Cog}_k$. If C is a coalgebra, the categories of right (resp. left) C-comodules is denoted by $\mathbf{M}^C$ (resp. ${}^C\mathbf{M}$). The fact that a k-space M is an object of such a category is denoted by M_C (resp. ${}_CM$). If $M, N \in \mathbf{M}^C$, the k-space of C-comodules maps between M and N is denoted by $\mathrm{Com}_C(M, N)$. If C, D are coalgebras, the category of (C, D)-bicomodules (i.e., left C-comodules, right D-comodules with compatible structures) is denoted by ${}^C\mathbf{M}^D$; an object in this category is represented by ${}_CM_D$.

We will freely use "sigma notation": $\Delta(c) = \sum c_1 \otimes c_2$ for the comultiplication of a coalgebra C and $\rho(m) = \sum m_0 \otimes m_1$ for the structure map of a right C-comodule M.

For any abelian category $\mathscr{A}$ we denote $Z(\mathscr{A})$ its centre, i.e., the commutative ring of all natural morphisms of the identity functor $1_{\mathscr{A}}: \mathscr{A} \to \mathscr{A}$.

$\mathscr{A}$ is called a k-abelian category whenever there exists a (preserving unit) ring morphism $\sigma: k \to Z(\mathscr{A})$. Giving such a σ is equivalent to defining on any $\operatorname{Hom}_{\mathscr{A}}(M, N)$ a k-space structure such that the composition maps $\operatorname{Hom}_{\mathscr{A}}(M, N) \times \operatorname{Hom}_{\mathscr{A}}(N, P) \to \operatorname{Hom}_{\mathscr{A}}(M, P)$ are k-bilinear. $\mathbf{M}^C$, ${}^C\mathbf{M}$, and ${}^C\mathbf{M}^D$ are instances of k-abelian categories.

An abelian category $\mathscr{A}$ is called locally finite if it has a family of generators of finite length (see [7]). Following Takeuchi [9], a k-abelian category $\mathscr{A}$ is of finite type if $\mathscr{A}$ is locally finite and $\operatorname{Hom}_{\mathscr{A}}(M, N)$ is finite dimensional (over k) for any objects M, N of finite length. (It is easily seen that this last property is equivalent to $\operatorname{End}_{\mathscr{A}}(M)$ is finite dimensional for any simple object M of $\mathscr{A}$.) We recall from [9, Theorem 5.1] that a k-abelian category is of finite type if and only if it is k-equivalent to a category $\mathbf{M}^C$ for some coalgebra C. If $\mathscr{A}$ is of finite type, an object $M \in \mathscr{A}$ is quasifinite if $\operatorname{Hom}_{\mathscr{A}}(S, M)$ is finite dimensional for all simple objects S of $\mathscr{A}$ (or equivalently $\operatorname{Hom}_{\mathscr{A}}(X, M)$ is finite dimensional for all objects $X \in \mathscr{A}$ of finite length).

Let C be an arbitrary coalgebra, M be a right C-comodule, and N be a left C-comodule. The *cotensor product* $M \square_C N$ is the kernel of the k-map $\rho_M \otimes 1 - 1 \otimes \rho_N$: $M \otimes N \to M \otimes C \otimes N$. Following [2], the cotensor product is a left exact functor and preserves inductive limits $\mathbf{M}^C \times {}^C\mathbf{M} \to \mathbf{M}_k$ ($\mathbf{M}_k$ is the category of k-spaces). Moreover the mappings $m \otimes c \to \epsilon(c)m$ and $c \otimes n \to \epsilon(c)n$ yield natural isomorphisms $M \square_C C \simeq M$ and $C \square_C N \simeq N$. The cotensor product is associative. If $N \in \mathbf{M}^C$ has finite dimension, then $N^* = \operatorname{Hom}(N, k)$ has a natural structure of left C-comodule and $M \square_C N^* \simeq \operatorname{Com}_C(N, M)$. A left C-comodule M is called left coflat if the functor $- \square_C M$: $\mathbf{M}^C \to k - \operatorname{Mod}$ is exact. It is proved by Takeuchi that M is left coflat if and only if M is an injective object of ${}^C\mathbf{M}$ (cf. [2]).

Let now $M \in {}^C\mathbf{M}^D$. Then M_D is quasifinite if and only if the functor $- \square_C M$: $\mathbf{M}^C \to \mathbf{M}^D$ has a left adjoint denoted by $h_{-D}(M, —)$ ([9]). The functor $h_{-D}(M, —)$ is called the co-hom functor. The following description is given in [9]: if $Y \in \mathbf{M}^D$, then

$$h_{-D}(M, Y) = \varinjlim_{i} \operatorname{Com}_D(Y_i, M)^* = \varinjlim_{i} (M \square_D Y_i^*)^*,$$

where $(Y_i)_i$ is the family of finite dimensional subcomodules of Y_D. We have in particular that $h_{-D}(D, Y) \simeq Y$ for any $Y \in \mathbf{M}^D$. The functor $h_{-D}(M, —)$ is right exact and commutes with inductive limits; it is an exact functor if and only if M_D is injective. If M_D is quasifinite, then $e_{-D}(M) = h_{-D}(M, M)$ has a natural structure of coalgebra, called the co-endomorphism coalgebra of M (see [9]); M becomes then an $(e_{-D}(M), D)$-bicomodule. For any (bi)comodules M_C, N_D, and ${}_DX_C$, with quasifinite M_C, there exists a canonical map δ: $h_{-C}(M, N \square_D X) \to N \square_D h_{-C}(M, X)$, which is an isomorphism if either N_D is injective or M_C is quasifinite and injective (see [9, 1.13]).

1. COFLAT MONOMORPHISMS

Let $\varphi: C \to D$ be a coalgebra morphism. Then any $M \in \mathbf{M}^C$ (with the structure map $\rho_M: M \to M \otimes C$) becomes a right D-comodule by $(1 \otimes \varphi)\rho_M: M \to M \otimes D$. This defines an exact functor $(—)_\varphi: \mathbf{M}^C \to \mathbf{M}^D$. In particular, C itself may be regarded as a left and right D-comodule. Since $C \in^D \mathbf{M}^C$, we can consider $N^\varphi = N \square_D C \in \mathbf{M}^C$ for any $N \in \mathbf{M}^D$, which is a right adjoint of $(—)_\varphi$ by [2, Proposition 6]. We observe that $(—)_\varphi = - \square_C C$, where C is considered a (C, D)-bicomodule. C is a (C, C)-bicomodule, thus it is also a (D, D)-bicomodule via φ. Let now $M \in \mathbf{M}^C$. Since $(1_M \otimes \varphi \otimes 1_C)(\rho_M \otimes 1_C)\rho_M = (1_M \otimes \varphi \otimes 1_C)(1_M \otimes \Delta)\rho_M$, we have that $\rho_M(M) \subseteq M_\varphi \square_D C$. Denote by $\bar{\rho}_M: M \to M_\varphi \square_D C$ the co-restriction of ρ_M. Clearly the maps $\{\bar{\rho}_M, M \in \mathbf{M}^C\}$ define a natural morphism $\bar{\rho}: 1_{\mathbf{M}^C} \to (—)^\varphi \circ (—)_\varphi$.

For $M = C$ we have $\rho_C = \Delta_C$; this yields a canonical morphism $\overline{\Delta}: C \to C \square_D C$, which is a (C, C)-bicomodule morphism. Let

$$C \square_D C \underset{p_2}{\overset{p_1}{\rightrightarrows}} C$$

be the restriction of the canonical maps $C \otimes C \to C$ defined by $c_1 \otimes c_2 \to \epsilon(c_1)c_2$ (resp. $c_1 \otimes c_2 \to c_1\epsilon(c_2)$). Clearly p_1 is a (D, C)-bicomodule morphism and p_2 is a (C, D)-bicomodule morphism. Also $p_1\overline{\Delta} = p_2\overline{\Delta} = 1_C$.

A characterization of monomorphism in the category of $\mathbf{Cog}_k$ was given in [5, Theorem 3.5]. Assume now that φ is left coflat morphism, i.e., the functor $- \square_D C$ is exact. In this case $\mathscr{T}_\varphi = \{M \in \mathbf{M}^D \mid M \square_D C = 0\}$ is a localizing subcategory of $\mathbf{M}^D$. Let $T: \mathbf{M}^D \to \mathbf{M}^D/\mathscr{T}_\varphi$ be the canonical functor. By [5, Theorem 4.2], there exists a subcoalgebra A of D such that $\mathscr{T}_\varphi = \{M \in \mathbf{M}^D \mid \rho_M(M) \subseteq M \otimes A\}$. Moreover, A is a co-idempotent coalgebra, i.e., $A = A \wedge A$.

PROPOSITION 1.1. *If $\varphi: C \to D$ is a left coflat monomorphism, then*

(1) Ker φ *and* Coker φ *belong to* $\mathscr{T}_\varphi$;

(2) *D/A is quasifinite in* $\mathbf{M}^D$;

(3) *If $E(D/A)$ is the injective envelope of D/A in $\mathbf{M}^D$ and $j: D/A \to E(D/A)$ is the inclusion map, then* Coker j *belongs to* $\mathscr{T}_\varphi$;

(4) *$T(D_D)$ is a quasifinite injective cogenerator of the quotient category* $\mathbf{M}^D/\mathscr{T}_\varphi$.

Proof. (1) We consider the functors

$$\mathbf{M}^C \underset{(-)^{\varphi}}{\overset{(-)_{\varphi}}{\rightleftarrows}} \mathbf{M}^D \underset{S}{\overset{T}{\rightleftarrows}} \mathbf{M}^D/\mathscr{T}_{\varphi},$$

where S is the right adjoint of T. If $F = T \circ (-)_{\varphi}$ and $G = (-)^{\varphi} \circ S$, then by [5, Theorem 6.1], F and G give an equivalence of categories between $\mathbf{M}^C$ and $\mathbf{M}^D/\mathscr{T}_{\varphi}$. Since φ is a monomorphism, we obtain by [5, Theorem 3.5] that $\operatorname{Ker}\varphi \,\square_D C = 0$, hence $\operatorname{Ker}\varphi \in \mathbf{M}^D/\mathscr{T}_{\varphi}$. Next consider the exact sequence in $\mathbf{M}^D$, $C \to_{\varphi} D \to \operatorname{Coker}\varphi \to 0$. Applying $- \square_D C$ we get $C \square_D C \to_{\varphi \square_D} D \square_D C \to \operatorname{Coker}\varphi \,\square_D C \to 0$. Identifying $D \square_D C$ with C, $\varphi \square_D C$ is replaced by $p_1 \colon C \square_D C \to C$. Since $p_1 \circ \overline{\Delta} = 1_C$ and $\overline{\Delta}$ is an isomorphism, we obtain that p_1 is an isomorphism; therefore $\operatorname{Coker}\varphi \,\square_D C = 0$ and $\operatorname{Coker}\varphi \in \mathscr{T}_{\varphi}$.

(4) Since $A \in \mathscr{T}_{\varphi}$, then $T(D_D) = T(D/A)$. Since $\operatorname{Ker}\varphi, \operatorname{Coker}\varphi \in \mathscr{T}_{\varphi}$, we obtain that $T(C_D) = T(D_D)$. Therefore $F(C_C) = T(C_D) = T(D/A) = T(D_D)$. However, F is an equivalence; hence $F(C_C)$ is a quasifinite injective cogenerator. Thus $T(D_D)$ has the same properties.

(2) Let M be a simple object in $\mathbf{M}^D$. Then $\operatorname{Com}_D(M, D/A) = 0$ if $M \in \mathscr{T}_{\varphi}$. Assume that M is $\mathscr{T}_{\varphi}$-torsion-free. Then the canonical morphism

$$\operatorname{Com}_D(M, D/A) \to \operatorname{Hom}_{\mathbf{M}^D/\mathscr{T}_{\varphi}}(T(M), T(D/A)),$$

$f \to Tf$, is injective. Since $T(M)$ is simple in $\mathbf{M}^D/\mathscr{T}_{\varphi}$, $\operatorname{Hom}_{\mathbf{M}^D/\mathscr{T}_{\varphi}}(T(M), T(D/A))$ is finite dimensional. It follows that D/A is quasifinite.

(3) Applying T to the exact sequence $0 \to D/A \to_j E(D/A) \to \operatorname{Coker} j \to 0$, we obtain the exact sequence $0 \to T(D/A) \to_{T(j)} T(E(D/A)) \to T(\operatorname{Coker} j) \to 0$. Since $T(D/A) \simeq T(D_D)$ is injective, the essential monomorphism $T(j)$ is an isomorphism; therefore $T(\operatorname{Coker} j) = 0$, i.e., $\operatorname{Coker} j \in \mathscr{T}_{\varphi}$.

PROPOSITION 1.2. *Let C be a coalgebra and $\mathscr{T}$ a localizing subcategory of $\mathbf{M}^C$. Then the quotient category $\mathbf{M}^C/\mathscr{T}$ is a k-abelian category of finite type.*

Proof. Let $T \colon \mathbf{M}^C \to \mathbf{M}^C/\mathscr{T}$ be the canonical functor and S its right adjoint. Following [9], it is sufficient to prove that $\operatorname{End}_{\mathbf{M}^C/\mathscr{T}}(X)$ is finite dimensional for any simple object $X \in \mathbf{M}^C/\mathscr{T}$. The functor S is fully faithful; thus $\operatorname{Hom}_{\mathbf{M}^C/\mathscr{T}}(X, X) \simeq \operatorname{Com}_C(S(X), S(X))$. However, $S(X)$ contains a nonzero simple object M. Since $T(S(X)) \simeq X$, we have that $S(X)/M \in \mathscr{T}$ and $T(M) \simeq X$. Finally $\operatorname{Hom}_{\mathbf{M}^C/\mathscr{T}}(X, X) \simeq \operatorname{Hom}_{\mathbf{M}^C/\mathscr{T}}(T(M), X) \simeq \operatorname{Com}_C(M, S(X)) \simeq \operatorname{Com}_C(M, M)$, which is finite dimensional.

2. COLOCALIZATION IN ABELIAN CATEGORIES

Let $\mathscr{A}$ be an abelian category and $\mathscr{C}$ a dense subcategory (or a Serre class) of $\mathscr{A}$, i.e., for any exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathscr{A}$, $M \in \mathscr{C}$ if and only if $M', M'' \in \mathscr{C}$ (see [3, p. 365]). We can construct the quotient category $\mathscr{A}/\mathscr{C}$ and the canonical exact functor $T: \mathscr{A} \to \mathscr{A}/\mathscr{C}$. We recall from [3] that the objects of $\mathscr{A}/\mathscr{C}$ are just the objects of $\mathscr{A}$, and if $M, N \in \mathscr{A}$, then

$$\operatorname{Hom}_{\mathscr{A}/\mathscr{C}}(M, N) = \varinjlim_{M', N'} \left\{\operatorname{Hom}_{\mathscr{A}}(M', N/N') \,\middle|\, M/M' \in \mathscr{C}, N' \in \mathscr{C}\right\}.$$

The quotient category $\mathscr{A}/\mathscr{C}$ is abelian. The dense subcategory $\mathscr{C}$ is called localizing if T has a right adjoint. In the case where $\mathscr{A}$ is a Grothendieck category, the dense subcategory $\mathscr{C}$ is localizing if and only if $\mathscr{C}$ is closed under arbitrary direct sums.

We are interested in the situation where $\mathscr{C}$ is a colocalizing subcategory, i.e., the canonical functor T has a left adjoint $H: \mathscr{A}/\mathscr{C} \to \mathscr{A}$. The following result is not new; it is an exercise in [3, p. 369].

Proposition 2.1. *Let $\mathscr{C}$ be a colocalizing subcategory of the abelian category $\mathscr{A}$ and let X be an object of $\mathscr{A}/\mathscr{C}$. The following assertions hold*:

(a) *If $Y \in \mathscr{C}$ is a quotient object of $H(X)$, then $Y = 0$.*

(b) *If $0 \to M \to_f P \to H(X) \to 0$ is an exact sequence in $\mathscr{A}$ and $M \in \mathscr{C}$, then f splits.*

(c) *The canonical morphism $\operatorname{Hom}_{\mathscr{A}}(H(X), M) \to \operatorname{Hom}_{\mathscr{A}/\mathscr{C}}(TH(X), T(M))$ sending f to $T(f)$ is an isomorphism for any $M \in \mathscr{A}$.*

(d) *If $\Psi: 1_{\mathscr{A}/\mathscr{C}} \to T \circ H$ and $\Phi: H \circ T \to 1_{\mathscr{A}}$ are the natural morphisms defined by the adjunction, then Ψ is an isomorphism and, for any $M \in \mathscr{A}$, there exists an exact sequence*

$$0 \to \operatorname{Ker} \Phi(M) \to HT(M) \xrightarrow{\Phi(M)} M \to \operatorname{Coker} \Phi(M) \to 0$$

with $\operatorname{Ker} \Phi(M), \operatorname{Coker} \Phi(M) \in \mathscr{C}$.

(e) *The functor H is fully faithful.*

Proof. Everything follows by [3, Lemmas 1 and 2 and Proposition 3, pp. 370, 371] using the dual category $\mathscr{A}^o$. In this case $\mathscr{C}$ remains a Serre class in $\mathscr{A}^o$, but H becomes a right adjoint of the canonical functor $T: \mathscr{A}^o \to \mathscr{A}^o/\mathscr{C}$.

PROPOSITION 2.2. *Let $\mathscr{C}$ be a colocalizing subcategory of $\mathscr{A}$. Then the following statements hold*:

(1) *$M \in \mathscr{C}$ if and only if* $\mathrm{Hom}_{\mathscr{A}}(H(X), M) = 0$ *for any $X \in \mathscr{A}/\mathscr{C}$.*

(2) *If $\mathscr{A}$ satisfies $AB3^*$ (i.e., $\mathscr{A}$ has arbitrary direct products), then $\mathscr{C}$ is closed to direct products.*

(3) *If moreover $\mathscr{A}$ is a Grothendieck category, then $\mathscr{C}$ is also a localizing subcategory.*

Proof. (1) If $M \in \mathscr{C}$, the desired relation follows by Proposition 2.1(a). Conversely, if $M \notin \mathscr{C}$, then $T(M) \neq 0$. Hence $\mathrm{Hom}_{\mathscr{A}/\mathscr{C}}(T(M), T(M)) \neq 0$. However, $\mathrm{Hom}_{\mathscr{A}/\mathscr{C}}(T(M), T(M)) \simeq \mathrm{Hom}_{\mathscr{A}}(HT(M), M) = 0$, providing a contradiction.

(2) Directly from (1).

(3) Let $(M_i)_{i \in I}$ be a family of objects of $\mathscr{C}$. We have by (2) that $\prod_{i \in I} M_i \in \mathscr{C}$, and since $\bigoplus_{i \in I} M_i$ is a subobject of $\prod_{i \in I} M_i$, we obtain that $\bigoplus_{i \in I} M_i \in \mathscr{C}$, i.e., $\mathscr{C}$ is a localizing subcategory.

From now on, we assume in this section that $\mathscr{A}$ is an abelian category with $AB3^*$, and $\mathscr{C}$ is a colocalization subcategory of $\mathscr{A}$. If $M \in \mathscr{A}$ we can define $s(M) = \bigcap_i M_i$, where M_i are the subobjects of M with the property that $M/M_i \in \mathscr{C}$. Since $\mathscr{C}$ is closed to direct product, we obtain from the exact sequence $0 \to M/s(M) \to \prod_i M/M_i$ that $M/s(M) \in \mathscr{C}$. The correspondence $M \to s(M)$ defines a subfunctor of the identity functor. Indeed, if $u\colon M \to N$ is a morphism in $\mathscr{A}$, we obtain from $s(M) \subseteq M \to_u N \to_\pi N/s(N)$ that $\mathrm{Im}(\pi \circ u) \subseteq N/s(N)$, hence $\mathrm{Im}(\pi \circ u) \in \mathscr{C}$. Therefore $\mathrm{Ker}(\pi \circ u) \supseteq s(M)$. Thus $(\pi \circ u)(s(M)) = 0$, showing that $u(s(M)) \subseteq s(N)$.

PROPOSITION 2.3. *Let $M \in \mathscr{A}$. Then there exists $X \in \mathscr{A}/\mathscr{C}$ such that $M \simeq H(X)$ if and only if $s(M) = M$ and M is $\mathscr{C}$-projective, i.e., for any diagram*

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow{\scriptstyle f} & & \\ P & \xrightarrow[u]{} & P' & \longrightarrow & 0 \end{array}$$

with $\mathrm{Ker}\, u \in \mathscr{C}$, *there exists $g\colon M \to P$ such that $f = ug$.*

Proof. If $M \simeq H(X)$, Proposition 2.1(a) shows that $s(M) = M$. That M is $\mathscr{C}$-projective follows from the fact that H is a left adjoint of T.

Conversely, $\operatorname{Coker}\phi(M) \in \mathscr{C}$ and $s(M) = M$ imply that $\operatorname{Coker}\phi(M) = 0$. The diagram

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow{\scriptstyle 1_M} & & \\ HT(M) & \xrightarrow[\phi(M)]{} & M & \longrightarrow & 0, \end{array}$$

where $\operatorname{Ker}\phi(M) \in \mathscr{C}$ shows that there is $g: M \to HT(M)$ such that $\phi(M)g = 1_M$. Hence $HT(M) \simeq M \oplus \operatorname{Ker}\phi(M)$; that is, $\operatorname{Ker}\phi(M)$ is a quotient object of $HT(M)$. Therefore $\operatorname{Ker}\phi(M) = 0$ and $HT(M) \simeq M$.

PROPOSITION 2.4. *Let $\mathscr{C}$ be a colocalizing subcategory of $\mathscr{A}$ and $M \in \mathscr{A}$. The following properties hold*:

(1) $\operatorname{Im}\phi(M) = s(M)$ *for any* $M \in \mathscr{A}$.

(2) *If $\mathscr{C}$ is closed under taking injective envelopes, then $\phi(M)$ is a monomorphism for any M.*

Proof. (1) Let $u: M \to Y$ be a morphism, where $Y \in \mathscr{C}$. Since $\operatorname{Im}(u \circ \phi(M)) \subseteq Y$, we obtain that $\operatorname{Im}(u \circ \phi(M)) = 0$. Hence $\operatorname{Im}\phi(M) \subseteq \operatorname{Ker} u$. Thus we have that $\operatorname{Im}\phi(M) \subseteq s(M)$. On the other hand $s(M) \subseteq \operatorname{Im}\phi(M)$, since $\operatorname{Coker}\phi(M) \in \mathscr{C}$.

(2) We consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Ker}\phi(M) & \xrightarrow{i} & HT(M) & \xrightarrow{\phi(M)} & M, \\ & & \downarrow{\scriptstyle j} & & & & \\ & & E(\operatorname{Ker}\phi(M)) & & & & \end{array}$$

where $E(\operatorname{Ker}\phi(M)) \in \mathscr{C}$ is the injective envelope of $\operatorname{Ker}\phi(M)$. There is some $u: HT(M) \to E(\operatorname{Ker}\phi(M))$ such that $ui = j$. Proposition 2.1(a) shows that $u = 0$. Hence $j = 0$ and $\operatorname{Ker}\phi(M) = 0$.

3. COLOCALIZATION IN THE CATEGORY $\mathbf{M}^C$

Let C be a k-coalgebra and $\mathscr{T}$ a localizing subcategory of $\mathbf{M}^C$. We know [5] that there exists a subcoalgebra A of C such that $\mathscr{T}_A = \{M \in \mathbf{M}^C \mid \rho_M(M) \subseteq M \otimes A\} = \{M \in \mathbf{M}^C | A^{\perp} M = 0\}$, where $A^{\perp} = \{f \in C^* \mid f(A) = 0\}$ is a two-sided ideal of the dual algebra C^*. Moreover, A is a co-idempotent subcoalgebra, i.e., $A = A \wedge A$.

We denote by $\mathcal{T}_A$ the localizing subcategory associated to a co-idempotent subcoalgebra A. It is also proved in [5] that $\mathcal{T}_A$ is a T.T.F. class, i.e., $\mathcal{T}_A$ is closed under direct products (cf. [7, VI.8]). Let us consider the canonical functor $T_A\colon \mathbf{M}^C \to \mathbf{M}^C/\mathcal{T}_A$, which has a right adjoint $S_A\colon \mathbf{M}^C/\mathcal{T}_A \to \mathbf{M}^C$, since $\mathcal{T}_A$ is localizing.

PROPOSITION 3.1. *With the above notation, the following assertions are equivalent*:

(1) *T_A has left adjoint H_A (i.e., $\mathcal{T}_A$ is a colocalizing subcategory).*

(2) *C/A is a quasifinite right C-comodule.*

Proof. By Proposition 1.2. Hence $\mathbf{M}^C/\mathcal{T}_A$ is a k-abelian category of finite type, then [9, Theorem 5.1] there exists a coalgebra D such that $\mathbf{M}^C/\mathcal{T}_A$ is equivalent via the pair of functors

$$\mathbf{M}^C/\mathcal{T}_A \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{M}^D .$$

Since T_A is exact and commutes with direct sums, the functor $F \circ T_A\colon \mathbf{M}^C \to \mathbf{M}^D$ also has these properties. By [9, Proposition 2.1] we obtain that $F \circ T_A \simeq - \square_D P$, where $P_D = F(T_A(C_C))$. Now by [9, Proposition 1.9] $- \square_D P$ has a left adjoint if and only if P_D is quasifinite in $\mathbf{M}^D$. Since F is an equivalence, P_D is quasifinite if and only if $T_A(C_C)$ is quasifinite in $\mathbf{M}^C/\mathcal{T}_A$. Clearly $T_A(C_C) = T_A(C/A)$. If X is a simple object in $\mathbf{M}^C/\mathcal{T}_A$, then there exists a simple object M in $\mathbf{M}^C$ such that $X = T_A(M)$. Then $\mathrm{Hom}_{\mathbf{M}^C/\mathcal{T}_A}(X, T_A(C/A)) = \mathrm{Hom}_{\mathbf{M}^C/\mathcal{T}_A}(T_A(M), T_A(C/A)) \simeq \mathrm{Com}_C(M, S_A T_A(C/A)) \simeq \mathrm{Com}_A(M, C/A)$, since C/A is an essential subobject of $S_A T_A(C/A)$. We conclude that $T_A(C_C)$ is quasifinite in $\mathbf{M}^C/\mathcal{T}_A$ if and only if C/A is so.

Remark 3.2. With the above notations $T_A(C_C)$ is a cogenerator of the category $\mathbf{M}^C/\mathcal{T}_A$. Indeed, if $X \in \mathbf{M}^C/\mathcal{T}_A$, then $X = T_A(M)$ for some $M \in \mathbf{M}^C$. We have an exact sequence $0 \to M \to (C_C)^{(I)}$ in $\mathbf{M}^C$ for some set I. Since T_A is exact and commutes with direct sums, we obtain an exact sequence $0 \to T_A(M) \to T_A((C_C))^{(I)}$ in $\mathbf{M}^C/\mathcal{T}_A$. Thus $T_A(C_C)$ is a cogenerator in $\mathbf{M}^C/\mathcal{T}_A$.

For the rest of the section we assume that C/A is quasifinite in $\mathbf{M}^C$. With the above notations we have two pairs of adjoint functors:

$$\mathbf{M}^C \underset{H_A}{\overset{T_A}{\rightleftarrows}} \mathbf{M}^C/\mathcal{T}_A \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{M}^D$$

and the isomorphisms $H_A \circ T_A \simeq (H_A \circ G) \circ (F \circ T_A)$ and $T_A \circ H_A \simeq (F \circ T_A) \circ (H_A \circ G) \simeq 1_{\mathbf{M}^C/\mathcal{T}_A}$. Moreover $H_A \circ G$ is a left adjoint of the functor $F \circ T_A$. Therefore the object $(H_A \circ T_A)(C_C)$ does not depend on the choice of the coalgebra D and the functors F, G. Let us denote $P_D =$

$(F \circ T_A)(C_C)$. Since C/A is quasifinite, P_D is a quasifinite D-comodule; also P_D is a cogenerator of $\mathbf{M}^D$. We obtain by [9, Proposition 1.10] that $H_A \circ G$ is the co-hom functor sending $Y_D \in \mathbf{M}^D$ to $h_{-D}(P_D, Y_D)$. In particular $(H_A \circ T_A)(C_C) \simeq h_{-D}(P_D, P_D) = e_{-D}(P_D)$, which has a natural coalgebra structure. This coalgebra structure does not depend on the choice of D. Indeed, D' is another coalgebra such that there exists an equivalence U: $\mathbf{M}^D \to \mathbf{M}^{D'}$ such that $F' = U \circ F$. If $P'_{D'} = (F' \circ T_A)(C_C)$, then $P'_{D'} = U(P_D)$. Since U is an equivalence we clearly have $e_{-D}(P_D) \simeq e_{-D'}(P'_{D'})$.

As a consequence we can assume that $\mathbf{M}^C/\mathscr{T}_A = \mathbf{M}^D$. Since T_A is exact and commutes with inductive limits, we have that $T_A = - \square_C P_D$, where $P_D = T_A(C_C)$ and $H_A = h_{-D}(P_D, —)$. If $X_D \in \mathbf{M}^D$ the natural morphism $\psi(X)$: $X_D \to h_{-D}(P_D, X_D) \square_C P_D$ is an isomorphism (from Section 2). In fact $\psi(X)$ is just the natural morphism defined in [9, 1.7]. Let $M \in \mathbf{M}^C$. From Section 2 we have the canonical map $\phi(M)$: $h_{-d}(P_D, M \square_C P_D) \to M$, with $\operatorname{Ker} \phi(M), \operatorname{Coker} \phi(M) \in \mathscr{T}$. We can write $M = \bigcup_\lambda M_\lambda$, where M_λs are the finite dimensional subcomodules of M. Then $M \square_C P_D = \lim_{\to \lambda}(M_\lambda \square_C P_D)$. Since the functor $h_{-D}(P_d, —)$ commutes with inductive limits, we obtain that $h_{-D}(P_D, M \square_C P_D) = \lim_{\to \lambda} h_{-D}(P_D, M_\lambda \square_C P_D)$. On the other hand the functor T_A sends any simple object S of $\mathbf{M}^C$ either to 0 or a simple object of $\mathbf{M}^C/\mathscr{T}_A$. Therefore $T_A(M)$ has finite dimension for any finite dimensional $M \in \mathbf{M}^C$. However, $T_A = - \square_C P_D$; thus $M_\lambda \square_C P_D$ is a finite dimensional D-comodule for any λ. Hence $h_{-D}(P_D, M \square_C P_D) = \lim_{\to \lambda} \operatorname{Com}_D(M_\lambda \square_C P_D, P_D)^*$.

On the other hand $M = h_{-C}(C, M) = \lim_{\to \lambda} \operatorname{Com}_C(M_\lambda, C)^*$. Since $P_D = C \square_C P_D$, then we have for each λ a natural morphism $\operatorname{Com}_C(M_\lambda, C) \to \operatorname{Com}_D(M_\lambda \square_C P_D, P_D)$, $u \to u \square_C 1_{P_D}$. By dualizing we obtain a natural morphism $\operatorname{Com}_D(M_\lambda \square_C P_D, P_D)^* \to \operatorname{Com}_C(M_\lambda, C)^*$. Taking inductive limits we just obtain the morphism $\phi(M)$: $h_{-D}(P_D, M \square_C P_D) \to M$. In particular, for $M = C_C$ we obtain the morphism $\phi(C)$: $e_{-D}(P_D) \to C$. Since $T_A = - \square_C P_D$, it follows from the above facts that $\phi(C)$ is a coalgebra morphism. By [9, 1.13] we have a natural morphism δ: $h_{-D}(P_D, M \square_C P_D) \to M \square_C h_{-D}(P_D, P_D) = M \square_C e_{-D}(P_D)$. Then for any $M \in \mathbf{M}^C$, the morphism $\phi(M)$ can be obtained as follows: $h_{-D}(P_D, M \square_C P_D) \to_\delta M \square_C h_{-D}(P_D, P_D) = M \square_C e_{-D}(P_D) \to_{1 \square_C \phi(C)} M \square_C C \simeq M$. Hence $\phi(M) = (1 \square_C \phi(C)) \circ \delta$. Moreover $HT(M) = h_{-D}(P_D, M \square_C P_D)$ is a right $e_{-D}(P_D)$-comodule, and the right C-comodule structure that $HT(M)$ has via $\phi(C)$ is just the original C-comodule structure. Summarizing we have the following result:

Proposition 3.3. *Assume that C/A is quasifinite in $\mathbf{M}^C$. With the preceding notation we have*:

(1) *$E = (H_A \circ T_A)(C)$ has a natural coalgebra structure and $\phi(C)$: $E \to C$ is a coalgebra morphism. Moreover* $\operatorname{Ker} \phi(C), \operatorname{Coker} \phi(C) \in \mathscr{T}_A$.

(2) $\mathrm{Im}\,\phi(C) = s(C) = \bigcap_{f \in \mathrm{Com}_C(C,A)} \mathrm{Ker}\, f = A^{\perp}C$, *where* A *is regarded as a right* C*-comodule*.

(3) $(H_A \circ T_A)(M)$ *has a natural right* E*-comodule structure for any* $M \in \mathbf{M}^C$.

Proof. It remains to show only the second assertion. Clearly $s(C) \subseteq \bigcap_{f \in \mathrm{Com}_C(C,A)} \mathrm{Ker}\, f$.

Conversely, let us take some $M \in \mathscr{T}_A$ and $g\colon C \to M$ a comodule morphism. We have a monomorphism $u\colon M \to A^{(I)} \subseteq A^I$ for some set I. Then $\mathrm{Ker}\, g = \bigcap_{i \in I} \mathrm{Ker}(\pi_i u g)$, where π_i is the canonical projection. Hence $\bigcap_{f \in \mathrm{Com}_C(C,A)} \mathrm{Ker}\, f \subseteq \bigcap_{g \in \mathrm{Com}_C(C,M),\, M \in \mathscr{T}_A} \mathrm{Ker}\, g = s(C)$. Next $A^{\perp}(C/A^{\perp}C) = 0$ implies that $C/A^{\perp}C \in \mathscr{T}_A$. Thus $s(C) \subseteq A^{\perp}C$.

On the other hand if $f\colon C \to A$ is a comodule morphism then $f(A^{\perp}C) \subseteq A^{\perp}f(C) \subseteq A^{\perp}A = 0$. Hence $A^{\perp}C \subseteq \mathrm{Ker}\, f$. This shows that $A^{\perp}C \subseteq s(C)$, proving the desired equality.

The coalgebra E is called the *quotient coalgebra* with respect to the localizing subcategory $\mathscr{T}_A$.

We recall that a coalgebra C is called *right semiperfect* if the category $\mathbf{M}^C$ has enough projective objects (see [4]). The next result characterizes the localizing subcategories for these coalgebras.

PROPOSITION 3.4. *If* C *is a right semiperfect coalgebra, then* C/A *is a right quasifinite comodule, i.e.,* $\mathscr{T}_A$ *is a colocalizing subcategory.*

Proof. Let M be a simple object of $\mathbf{M}^C$. Since C is right semiperfect, M has a projective cover $P \to M \to 0$ [4]. Moreover, since M is finite dimensional, P is also finite dimensional. The fact that C is quasifinite yields that $\mathrm{Com}_C(P,C)$ is finite dimensional. The exact sequence $C \to C/A \to 0$ produces the exact sequence $\mathrm{Com}_C(P,C) \to \mathrm{Com}_C(P,C/A) \to 0$ (since P is projective). Hence $\mathrm{Com}_C(P,C/A)$ is finite dimensional. From the exact sequence $0 \to \mathrm{Com}_C(M,C/A) \to \mathrm{Com}_C(P,C/A)$, we conclude that $\mathrm{Com}_C(M,C/A)$ is finite dimensional. Thus C/A is quasifinite.

4. PERFECT (EXACT) COLOCALIZATION

Through this section we keep the notations of Section 3.

PROPOSITION 4.1. *The following assertions are equivalent*:

(1) *The functor* $T_A\colon \mathbf{M}^C \to \mathbf{M}^C/\mathscr{T}_A$ *has an exact left adjoint* $H_A\colon \mathbf{M}^C/\mathscr{T}_A \to \mathbf{M}^C$.

(2) (i) C/A *is quasifinite right* C*-comodule and* (ii) $\mathrm{Coker}\, j \in \mathscr{T}_A$, *where* $j\colon C/A \to E(C/A)$ *is the inclusion morphism of* C/A *into its injective envelope*.

Proof. (1) ⇒ (2). Clearly (i) follows from Proposition 3.1. Since H_A is exact, then T_A carries injective objects to injective objects. In particular $T_A(C_C)$ is injective. However, $T_A(C_C) = T_A(C/A)$; therefore $T_A(A/C)$ is injective. The morphism j is essential, so $T_A(j)$ is essential. Since $T_A(E(C/A))$ is injective, we obtain that $T_A(j)$ is an isomorphism. Thus $\operatorname{Coker} j \in \mathscr{T}_A$.

(2) ⇒ (1). By Proposition 3.1, T_A has a left adjoint H_A. Let $Q \in \mathbf{M}^C$ be an injective object. Then Q is a direct summand of $(C_C)^{(I)}$ for some nonempty set (I). Since T_A commutes with direct sums, we obtain that $T_A(Q)$ is a direct summand of $(T_A(C_C))^{(I)}$. Now $T_A(C_C) = T_A(C/A)$ and by (ii) they are equal to $T_A(E(C/A))$. Since $E(C/A)$ is $\mathscr{T}$-torsion-free, we obtain that $T_A(E(C/A))$ is injective in $\mathscr{T}_A$ (see [3, Chapter III]). We have seen that T_A carries injectives to injectives; therefore H_A is an exact functor.

The colocalizing subcategory $\mathscr{T}$ of $\mathbf{M}^C$ will be called *perfect* (or *exact*) if any of the equivalent conditions in the last proposition holds.

PROPOSITION 4.2. *The condition* $\operatorname{Coker} j \in \mathscr{T}$ *is satisfied in the following cases*:

(1) *C is a hereditary coalgebra* (*i.e.*, $\operatorname{gl.dim} C \leq 1$, [6]).

(2) *$\mathscr{T}_A$ is closed under injective envelopes*.

(3) *The quotient category $\mathbf{M}^C/\mathscr{T}_A$ is a semisimple category*.

Proof. (1) C/A is a right (and left) injective C-comodule since C is a hereditary coalgebra.

(2) Since $\mathscr{T}$ is closed under injective envelopes, we obtain that A is an injective right C-comodule and $C_C = A \oplus C/A$. Therefore C/A is also an injective right C-comodule.

(3) $\mathscr{T}_A$ semisimple implies that $T_A(j)$ is an isomorphism; therefore $\operatorname{Coker} j \in \mathscr{T}$.

We can state now the main result of this section.

THEOREM 4.3. *Let C be a coalgebra and A be a co-idempotent subcoalgebra such that $\mathscr{T}_A$ is a colocalizing subcategory* (*i.e.*, *C/A is a quasifinite right C-comodule*). *Then the following assertions are equivalent*:

(1) *$\mathscr{T}_A$ is a perfect colocalizing subcategory.*

(2) *If E is the quotient coalgebra associated to $\mathscr{T}_A$, then the natural coalgebra morphism $\varphi\colon E \to C$ is a left coflat monomorphism.*

Moreover, if (1) *and* (2) *hold, the localizing subcategory $\mathscr{T}_\varphi$ associated to φ is $\mathscr{T}_A$.*

Proof. (2) ⇒ (1). Follows from Propositions 1.1 and 4.1.

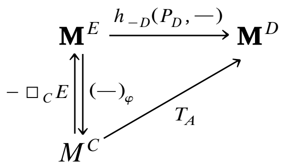
(1) ⇒ (2). With the notations from Section 3, we have $T_A = - \square_C P_D$. Since $\operatorname{Ker} \varphi, \operatorname{Coker} \varphi \in \mathcal{T}_A$, then $\operatorname{Ker} \varphi \square_C P_D = 0$. Since P_D is injective and quasifinite, [9, Proposition 1.14] shows that the natural morphism (*)

$$\delta \colon h_{-D}(P_D, M \square_C P_D) \to M \square_C e_{-D}(P_D)$$

is an isomorphism for any $M_C \in \mathbf{M}^C$. Then P_D is an injective cogenerator. Hence $M \square_C P_D = 0$ if and only if $M \square_C E = 0$. Indeed, $M \square_C P_D = 0$ implies $M \square_C E = 0$. The converse follows from the fact that $h_{-D}(P_D, X) = 0$ if and only if $X = 0$. This is clear since $h_{-D}(P_D, X) = \lim_{\to i \in I} \operatorname{Com}_C(X_i, P_D)^*$, where $X = \bigcup_{i \in I} X_i$ and $(X_i)_{i \in I}$ is the family of all finite dimensional subcomodules of X (ordered by inclusion). If $X \neq 0$, since P_D is an injective cogenerator we have $\operatorname{Com}_c(X_i, P_D)^* \neq 0$ for some $i \in I$. However, this implies that $h_{-D}(P_D, X) \neq 0$.

Since $\operatorname{Ker} \varphi \square_C P_D = 0$, then $\operatorname{Ker} \varphi \square_C E = 0$. By [5, Proposition 3.5] we obtain that φ is a monomorphism in the category $\mathbf{Cog}_k$.

Let us consider the diagram



The isomorphism (*) shows that $- \square_C E \simeq h_{-D}(P_D, -) \circ T_A$. Since $h_{-D}(P_D, -)$ is an equivalence [9, Theorem 3.5], then the functor $- \square_C E$ is exact, i.e., E is a left coflat C-comodule.

The isomorphism (*) also shows that $\mathcal{T}_\varphi = \{M \in \mathbf{M}^C \mid M \square_C E = 0\} = \{M \in \mathbf{M}^C \mid M \square_C P_D = 0\} = \operatorname{Ker} T_A = \mathcal{T}_A$.

COROLLARY 4.4. *Let C be a right semiperfect coalgebra and suppose that $\mathcal{T}_A$ is a localizing subcategory of $\mathbf{M}^C$. If $\mathbf{M}^C/\mathcal{T}_A$ is a semisimple category, then $\mathcal{T}_A$ is a perfect colocalizing subcategory.*

Proof. By Propositions 4.2 and 3.4.

COROLLARY 4.5. *Let C be a hereditary coalgebra. If $\mathcal{T}_A$ is a colocalizing subcategory in $\mathbf{M}^C$, then $\mathcal{T}_A$ is a perfect localizing subcategory.*

COROLLARY 4.6. *Let $\mathcal{T}_A$ be a localizing subcategory closed under injective envelopes. Then $\mathcal{T}_A$ is a perfect colocalizing subcategory. Moreover the associated quotient coalgebra is a subcoalgebra E of C such that E is a left coflat C-comodule and $E \cap A = 0$.*

Proof. Since $\mathcal{T}_A$ is closed under injective envelopes, then $A = E(A)$, where A is regarded as a right C-comodule. Then $C = A \oplus C/A$ in $\mathbf{M}^C$. Hence C/A is an injective and quasifinite right C-comodule. This means that $\mathcal{T}_A$ is a perfect colocalizing subcategory. The second part follows from Proposition 2.4 and Theorem 4.3. Also $E \cap A = 0$ since A and E are subcoalgebras with $A \square_C E = 0$.

5. GOLDIE TORSION THEORY

Let $\mathcal{A}$ be a Grothendieck category and $\mathcal{P}$ be the class of all objects of $\mathcal{A}$ of the form M/L, where $M, L \in \mathcal{A}$ and L is an essential subobject of M. Then $\mathcal{P}$ is closed under homomorphic images. Let $\mathcal{G}$ be the smallest localizing subcategory of $\mathcal{A}$ containing $\mathcal{P}$. It is easily seen that $\mathcal{G}$ is the class of all $M \in \mathcal{A}$ with the property that M/M' has a subobject belonging to $\mathcal{P}$ for any $M' \subset M$. The localizing subcategory $\mathcal{G}$ is called the *Goldie torsion theory* of $\mathcal{A}$. If $M \in \mathcal{G}$, since M is essential in $E(M)$ i.e., $E(M)/M \in \mathcal{P}$, the exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$ yields that $E(M) \in \mathcal{G}$. Thus $\mathcal{G}$ is closed under injective envelopes.

We focus our attention on the case $\mathcal{A} = \mathbf{M}^C$, C a coalgebra. Let us denote by Spec(C) the class of all types of simple right C-comodules, i.e., Spec$(C) = \{[S] \mid S \in \mathbf{M}^C,\ S$ is a simple right comodule$\}$, where $[S] = \{S' \in \mathbf{M}^C \mid S' \simeq S\}$. Clearly Spec$(C)$ is a set. If X is a subset of Spec(C), we denote by $\mathcal{C}_X$ the smallest localizing subcategory containing X. It is easily seen that $\mathcal{C}_X = \{M \in \mathbf{M}^C \mid$for any strict subobject M' of M, M/M' has a subobject isomorphic with an object in X for any $X' \subseteq M\}$. We denote by P the set of all types of simple projective objects of $\mathbf{M}^C$. The following example shows that it is possible to have $P = \varnothing$.

EXAMPLE 5.1. Let $S = \{c_0, c_1, \dots\}$ be a countable set and $C = kS$, the free k-module of basis S. Then C is a coalgebra with the comultiplication $\Delta(c_n) = \sum_{i+j=n} c_i \otimes c_j$ and co-unit $\epsilon(c_n) = \delta_{n,0}$ for $n \in \mathcal{N}$. The dual algebra C^* is isomorphic to the algebra $k[[x]]$ of formal series in the indeterminate X. Also $\mathbf{M}^C$ is isomorphic to the category Tors of all torsion modules over the algebra $k[[x]]$, but the only projective object of Tors is zero.

PROPOSITION 5.2. *The Goldie torsion theory $\mathcal{G}$ of $\mathbf{M}^C$ is generated by the simple objects of $\mathbf{M}^C$, which are not projective, i.e., $\mathcal{G} = \mathcal{C}_{\mathrm{Spec}(C)-P}$. In particular if $P = \varnothing$, then $\mathcal{G} = \mathbf{M}^C$.*

Proof. Let $M \in \mathbf{M}^C$ be a simple projective object such that $M \in \mathcal{G}$. Then $M \simeq X/Y$, where Y is essential in X. The exact sequence $0 \to Y \to X \to X/Y \to 0$ shows that Y is a direct summand of X. Hence $X = Y$ and

$M = 0$, which is a contradiction. It remains to show that any simple M with $M \notin \mathscr{G}$ is projective. Let us consider the following diagram in $\mathbf{M}^C$:

$$\begin{array}{ccccccccc} & & & & & & M & & \\ & & & & & & \Big\downarrow f & & \\ 0 & \longrightarrow & \operatorname{Ker} u & \longrightarrow & X & \xrightarrow{u} & X'' & \longrightarrow & 0. \end{array}$$

Consider the pullback diagram of $M \prod_{X''} X$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker} q_M & \longrightarrow & M \prod_{X''} X & \xrightarrow{q_M} & M & \longrightarrow & 0 \\ & & \Big\downarrow \rho & & \Big\downarrow q_X & & \Big\downarrow f & & \\ 0 & \longrightarrow & \operatorname{Ker} u & \longrightarrow & X & \xrightarrow{u} & X'' & \longrightarrow & 0. \end{array}$$

This shows that $\operatorname{Ker} q_M$ is not essential in $M \prod_{X''} X$. However, M is simple. Then there exits a morphism v: $M \to M \prod_{X''} X$ such that $q_M v = 1_M$. If $g = q_X v$, then $ug = uq_X v = fq_M v = f$. Thus M is projective.

Since $\mathscr{G}$ is closed under injective envelopes, Corollary 4.6 shows that $\mathscr{G}$ is a perfect colocalizing subcategory. We denote $E = s_{\mathscr{G}}(C)$. Then $\mathscr{G} = \{M \in \mathbf{M}^C \mid M \,\square_C E = 0\}$ and E is a subcoalgebra with E a left coflat C-comodule.

Proposition 5.3. *Let C be a coalgebra, $\mathscr{G}$ the Goldie torsion theory, and $E = s_{\mathscr{G}}(C)$. The following properties hold*:

(1) *E is a cosemisimple subcoalgebra of C.*

(2) *$\mathbf{M}^C/\mathscr{G}$ is equivalent to $\mathbf{M}^E$.*

(3) *E is the sum of all simple subcoalgebras B of C with the property that B is a sum of minimal left co-ideals isomorphic with simple left C-comodules of the form $M^* = \operatorname{Hom}(M, k)$, where M is a simple projective right C-comodule.*

Proof. (1) and (2). Any object of $\mathbf{M}^C/\mathscr{G}$ is injective (i.e., $\mathbf{M}^C/\mathscr{G}$ is a spectral category, [7]) and contains a nonzero simple object. Thus $\mathbf{M}^C/\mathscr{G}$ is a semisimple category. Since $\mathscr{G}$ is a perfect colocalizing subcategory, $\mathbf{M}^C/\mathscr{G}$ is equivalent to $\mathbf{M}^E$. Hence $\mathbf{M}^E$ is also a semisimple category and E is a cosemisimple coalgebra.

(3) Let E' be the sum of all the coalgebras with the mentioned property in (3). If M is a projective right C-comodule, then M^* is an injective left C-comodule. Therefore E' is an injective left C-comodule and it is left coflat. Let $\mathscr{G}' = \{M \in \mathbf{M}^C \mid M \,\square_C E' = 0\}$, which is clearly a perfect colocaling subcategory. Since $s_{\mathscr{G}}(C) = E$ and $s_{\mathscr{G}'}(C) = E'$, we only

have to prove that $\mathcal{G} = \mathcal{G}'$. Let $M \in \mathbf{M}^C$. Then $M \in \mathcal{G}$ if and only if $\mathrm{Com}_C(S, M) = 0$ for any simple projective right comodule S. However, $\mathrm{Com}_C(S, M) = M \square_C S^*$. Thus $M \in \mathcal{G}$ if and only if $M \square_C S^* = 0$ for any projective right comodule S, which coincides with $M \square_C E' = 0$, i.e., $M \in \mathcal{G}'$.

The next result gives a sufficient condition in order to have that the coalgebra E is a direct summand (as coalgebra) of C. Let $\mathcal{A}$ be a Grothendieck category and let $\mathcal{C}$ be a localizing subcategory of $\mathcal{A}$. We denote by $t_{\mathcal{C}}(M)$ the sum of all subobjects $N \subseteq M$ such that $N \in \mathcal{C}$. Since C is closed under direct sums and quotients it is clear that $t_{\mathcal{C}}(M) \in \mathcal{C}$. Also $t_{\mathcal{C}}(M/t_{\mathcal{C}}(M)) = 0$. Therefore the left exact functor $t_{\mathcal{C}}$ is a radical [7].

LEMMA 5.4. *Let $\mathcal{A}$ be a locally finite Grothendieck category and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two localizing subcategories of $\mathcal{A}$ with the following properties*:

(1) $\mathcal{C}_1 \cap \mathcal{C}_2 = 0$.

(2) *If S is a simple object in $\mathcal{A}$, then $S \in \mathcal{C}_1$ or $S \in \mathcal{C}_2$.*

(3) *$\mathcal{C}_1$ and $\mathcal{C}_2$ are closed under injective envelopes.*

Then we have the decomposition $M = M_1 \oplus M_2$, where $M_1 = t_{\mathcal{C}_1}(M)$ and $M_2 = t_{\mathcal{C}_2}(M)$.

Proof. By (1) and (2) clearly $M_1 \cap M_2 = 0$. Assume that $M_1 \oplus M_2 \neq M$. Then there exists a nonzero simple object $S \in \mathcal{C}$ and $M' \subseteq M$ such that $M_1 \oplus M_2 \subseteq M'$ and $M'/M_1 \oplus M_2 \simeq S$. Since $M_i = t_{\mathcal{C}_i}(M)$ for $i = 1, 2$, we can assume $M' = M$.

Let N_1 be a maximal subobject with the properties $M_1 \subseteq N_1$ and $N_1 \cap M_2 = 0$ (N_1 always exits by Zorn's lemma). If $X \subseteq N_1$ is a nonzero simple object, then $X \cap M_2 = 0$. Hence $X \in \mathcal{C}_1$ and $X \subseteq M_1$. Thus N_1 is an essential extension of M_1 and by (3) it follows that $N_1 \in \mathcal{C}_1$. We obtain that $M_1 = N_1$. Analogously it results that M_2 is maximal with the property that $M_2 \cap M_1 = 0$.

Assume now that $S \in \mathcal{C}_1$. Since M_1 is maximal with the property $M_1 \cap M_2 = 0$, then the canonical monomorphism $0 \to M_2 \to M/M_1$ is essential. By (3) it follows that $M/M_1 \in \mathcal{C}_2$. The exact sequence $M/M_1 \to M/M_1 \oplus M_2 \simeq S \to 0$ yields $S \in \mathcal{C}_2$. Hence $S \in \mathcal{C}_1 \cap \mathcal{C}_2 = 0$. This is a contradiction.

If C is a coalgebra with the property that the Goldie torsion theory $\mathcal{G}$ of $\mathbf{M}^C$ is exactly the whole category $\mathbf{M}^C$, then we say that C is a *singular right coalgebra*. Following [1] or [8], $C^* = \mathrm{Hom}_k(C, k)$ has a natural ring structure. Indeed if $f, g \in C^*$, then $fg = (f \otimes g)\Delta$ (here $k \otimes k \simeq k$). The co-unit ϵ is the identity element of this ring. In [1] or [8] it is proved that the category $\mathbf{M}^C$ is isomorphic with a subcategory of left C^*-modules, denoted by Rat($C^* -$ Mod). The objects of Rat($C^* -$ Mod) are called left rational C^*-modules. In the case when $\dim C < \infty$, then $\mathbf{M}^C$ is isomorphic to $C^* -$ Mod. In this case C is a singular right coalgebra if and only if the ring C^* is left singular [7]. We have the very satisfactory decomposition:

PROPOSITION 5.5. *If the ring C^* is reduced (i.e., C^* has no nonzero nilpotent elements), then the coalgebra C is a direct sum of two subcoalgebras $C = D \oplus E$, where D is right singular coalgebra and E is cosemisimple.*

Proof. We first show that if S is a simple projective right C-comodule, then S is also an injective object in $\mathbf{M}^C$. By [2, Proposition 4] it follows that S is a left projective simple C^*-module. Therefore $C^* = I \oplus J$, where I and J are left ideals in C^* and $I \simeq S$ (as left C^*-modules). Then there exit two orthogonal idempotents $e, f \in C^*$ such that $1 = e + f$, $I = C^*e$, $J = C^*f$, and $ef = fe = 0$. If $b \in J$ then $b = \lambda f$ for some $\lambda \in C^*$. Since $(e\lambda f)^2 = e\lambda fe\lambda f = 0$, by hypothesis $e\lambda f = 0$; hence $IJ = 0$. Analogously we have $JI = 0$; hence I and J are two-sided ideals. Since $JS = 0$, then S is a left C^*/J-module. However, $C^*/J \simeq I \simeq S$ is a division ring; therefore S is a left (and right) projective module over the ring C^*/J. On the other hand the canonical ring morphism $\pi: C^* \to C^*/J$ is left and right flat. Hence S is also left injective as the C^*-module. Thus S is injective in $\mathbf{M}^C$.

We denote by $\mathscr{C}$ the localizing subcategory of $\mathbf{M}^C$ generated by simple projective right C-comodules. By Proposition 5.2 it follows that $\mathscr{G} \cap \mathscr{C} = 0$. The preceding result implies that $\mathscr{C}$ is closed under injective envelopes. Now we can apply Lemma 5.4, $C = D \oplus E$ where $D = t_{\mathscr{G}}(C_C)$ and $E = t_{\mathscr{C}}(C_C)$. By [5, Theorem 4.2], D and E are subcoalgebras. Clearly D is a right singular coalgebra and E is a cosemisimple coalgebra.

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