Generalized Bäcklund–Darboux Transformation: Spectral Properties and Nonlinear Equations

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INTRODUCTION

The Bäcklund–Darboux transformations (BDTs) are named after the pioneering discoveries by Bäcklund [6] and Darboux [14, 15] in differential geometry. The problems under consideration were related to the sine-Gordon equation $\frac{\partial^2 u}{\partial t \partial x} = \sin u$. In particular, a transformation producing a new solution of the sine-Gordon equation from a given solution was introduced. After being an active research area for some time after the turn of the (previous) century, Bäcklund transformations were mostly forgotten until they came back with a vengeance in the 1970s following the discovery of complete integrability of nonlinear PDEs such as the sine-Gordon equation, the Korteweg–de Vries equation, and the nonlinear Schrödinger equation, of their soliton solutions and of the inverse scattering method. See [2, 13, 30, 31, 45] for the achievements of the last few decades. The name Bäcklund transformation refers sometimes to any system of functional equations involving two functions $u$ and $\tilde{u}$ and their derivatives, such that if $u$ satisfies one given PDE then $\tilde{u}$ will satisfy another given PDE (often coinciding with the first PDE) and conversely (see, e.g., [2]). The functional equations typically contain a constant playing the role of a “spectral parameter.” On the other hand a similar transformation is essential in the spectral theory of the ODE: Crum–Krein–Agranovich–Marchenko formulas, insertion of additional bound states, trace formulas, commutation methods, and bispectrality (see [4, 17–19, 21, 31]). An eigenvalue $\lambda_0$ can be...
removed, for instance, from the spectrum of the Sturm–Liouville operator $L = -d^2 + u(x)$ [15, 17, 18]. The transformed operator $\tilde{L} = -d^2 + \tilde{u}(x)$, i.e., the potential $\tilde{u}$ in $\tilde{L}$, is constructed explicitly via the initial potential $u$ and the eigenfunction $\chi$ of the equation $L\chi = \lambda_0\chi$. The fundamental solution $\tilde{y}$ of the equation $\tilde{L}\tilde{y} = \lambda\tilde{y}$ is obtained from the initial fundamental solution. The generalized Bäcklund–Darboux transformation (GBDT) studied in this paper develops this approach. The methods of the characteristic function theory [10, 28, 39, 44] and of the system theory [7–9, 26, 40, 43] are used for this purpose.

We consider here the first order systems $w'(x, \lambda) = G(x, \lambda)w(x, \lambda)(w' = \frac{d}{dx}w)$. The solutions $\tilde{w}$ of the transformed systems $\tilde{w}'(x, \lambda) = \tilde{G}(x, \lambda)\tilde{w}(x, \lambda)$ are connected with the solutions $w$ by the relation $\tilde{w} = w_Aw$. The matrix function $w_A$ (the so-called gauge transformation) is written down for each $x$ in the form of the transfer matrix function $w_A(x, \lambda) = I + C(x)(\lambda I - A)^{-1}B(x)$ that goes back to Kallman [26]. In various examples $w_A$ proves to be a characteristic matrix function by Livšic [28]. The particular form of the transfer matrix function and at the same time a generalization of the characteristic matrix function that is used here was introduced by L. Sakhnovich in [40] (see also [41, 43] and references therein). Parameter matrices and the corresponding “generalized” eigenfunctions are used in constructing $w_A$ instead of the eigenvalues and eigenfunctions in the standard BDTs. The transfer matrix function type representation of $w_A$ allows one, in particular, to separate the dependence on the variable $x$ and spectral parameter $\lambda$. In this way the explicit solutions of the spectral problems and matrix nonlinear equations are expressed in a general and analytically and computationally optimal form. Such a representation proves to be important also in the study of the algebraic surfaces [12] and bispectrality [48]. The GBDT approach was developed in the papers [33–37] (see also the related papers [22–24]). This paper is dedicated to the detailed proofs and new applications of the GBDT. For simplicity we shall consider $G(x, \lambda)$ polynomial in the spectral parameter $\lambda$ and in $\lambda^{-1}$.

In Section 1 the GBDT is constructed. In Section 2 spectral properties of the GBDT are studied. Applications to nonlinear equations are given in Section 3.

1. GENERALIZED BÄCKLUND–DARBOUX TRANSFORMATION

Consider an $m \times m$ first order system

$$w'(x, \lambda) = G(x, \lambda)w(x, \lambda),
\begin{equation}
G(x, \lambda) = - \sum_{k=-r}^{r} \lambda^k q_k(x) \quad \left( w' = \frac{d}{dx}w, \quad x \in \Gamma \right),
\end{equation}$$

$$w'(x, \lambda) = G(x, \lambda)w(x, \lambda).$$
where the coefficients \( q_k(x) \) are \( m \times m \) matrix functions locally summable in \( \Gamma \), which will always denote either \([0, c)\) or \((-c, 0)\) or \((-c, 0]\) \((c \leq \infty)\).

The results stated in the paper for \( \Gamma \) are true for each of these domains. The function \( w \) in (1.1) is an absolutely continuous matrix function (it may be either fundamental solution or vector function, in particular). The GBDT of the system will be determined by the three square matrices \( A_1, A_2, \) and \( S(0) (\det S(0) \neq 0) \) of order \( n \) and by the two \( n \times m \) matrices \( \Pi_1(0) \) and \( \Pi_2(0) \), satisfying the operator identity

\[
A_1 S(0) - S(0) A_2 = \Pi_1(0) \Pi_2(0)^*.
\]

(1.2)

Suppose that such parameter matrices are fixed. Then we can introduce matrix functions \( S(x), \Pi_1(x), \) and \( \Pi_2(x) \) with the values \( \Pi_1(0), \Pi_2(0), \) and \( S(0) \) at \( x = 0 \) as the solutions of the linear differential equations

\[
\Pi_1'(x) = \sum_{p=-r}^{r} A_1^{p} \Pi_1(x) q_p(x), \quad \Pi_2'(x) = - \sum_{p=-r}^{r} (A_2^{p})^* \Pi_2(x) q_p(x)^* ,
\]

(1.3)

\[
S'(x) = \sum_{p=1}^{r} \sum_{j=1}^{p} A_1^{p-j} \Pi_1(x) q_p(x) \Pi_2(x)^* A_2^{j-1}

- \sum_{p=-r}^{p} \sum_{j=p+1}^{0} A_1^{p-j} \Pi_1(x) q_p(x) \Pi_2(x)^* A_2^{j-1}.
\]

Notice that Eqs. (1.3) are chosen in such a way that the identity

\[
A_1 S(x) - S(x) A_2 = \Pi_1(x) \Pi_2(x)^*
\]

(1.4)

follows from (1.2) and (1.3) for all \( x \) in the domain \( \Gamma \), where the coefficients \( q_k \) are defined. (The relation is obtained by the direct differentiation of the both sides of (1.4).) Assuming that \( \det S(x) \neq 0 \) we can define a matrix function

\[
w_A(x, \lambda) = I_m - \Pi_2(x)^* S(x)^{-1} (A_1 - \lambda I_m)^{-1} \Pi_1(x),
\]

(1.5)

where \( I_m \) is an \( m \times m \) identity matrix. The matrix \( w_A \) is the so-called transfer matrix function, which is a well known tool in system theory. Matrices of the form (1.5) with the additional property (1.4) were introduced by L. Sakhnovich in the context of his method of operator identities [40–43].

**Theorem 1.1.** Let the parameter matrices \( A_1 \) and \( A_2 \) be invertible:

\[
det A_1 \neq 0, \quad det A_2 \neq 0.
\]

(1.6)

Suppose that matrix functions \( w, \Pi_1, \Pi_2, \) and \( S \) satisfy equations (1.1)–(1.3). Then in the points of invertibility of \( S \) the matrix function

\[
\tilde{w}(x, \lambda) = w_A(x, \lambda) w(x, \lambda)
\]

(1.7)
satisfies the system
\[ \ddot{\varphi}(x, \lambda) = \tilde{G}(x, \lambda) \dot{\varphi}(x, \lambda), \quad \tilde{G}(x, \lambda) = - \sum_{k=-r}^{r} \lambda^k \tilde{q}_k(x) \]  
(1.8)

with coefficients
\[ \tilde{q}_k(x) = q_k(x) - \sum_{p=k+1}^{r} \left( q_p(x) Y_{p-k-1}(x) - X_{p-k-1}(x) q_p(x) \right) \]
\[ + \sum_{j=k+2}^{p} X_{p-j}(x) q_p(x) Y_{j-k-2}(x) \]  
(1.9)
for \( k \geq 0 \),
\[ \tilde{q}_k(x) = q_k(x) + \sum_{p=-r}^{k} \left( q_p(x) Y_{p-k-1}(x) - X_{p-k-1}(x) q_p(x) \right) \]
\[ - \sum_{j=p+1}^{k+1} X_{p-j}(x) q_p(x) Y_{j-k-2}(x) \]  
(1.9)
for \( k < 0 \),

where
\[ X_k(x) = \Pi_2(x)^* S(x)^{-1} A_1^k \Pi_1(x), \]
\[ Y_k(x) = \Pi_2(x)^* A_2^k S(x)^{-1} \Pi_1(x). \]
(1.10)

**Proof.** From (1.4) it follows that
\[ S(x)^{-1} A_1 - A_2 S(x)^{-1} = S(x)^{-1} \Pi_1(x) \Pi_2(x)^* S(x)^{-1}, \]
\[ A_2^{-1} S(x)^{-1} - S(x)^{-1} A_1^{-1} = A_2^{-1} S(x)^{-1} \Pi_1(x) \Pi_2(x)^* S(x)^{-1} A_1^{-1}. \]
(1.11)

By induction, taking into account (1.11), we obtain
\[ A_2^k S(x)^{-1} = S(x)^{-1} A_1^k - \sum_{p=0}^{k-1} A_2^{k-p-1} \]
\[ \times S(x)^{-1} \Pi_1(x) \Pi_2(x)^* S(x)^{-1} A_1^p \]  
(1.12)
for \( k \geq 0 \),
\[ A_2^k S(x)^{-1} = S(x)^{-1} A_1^k - \sum_{p=k}^{r} A_2^{k-p-1} \]
\[ \times S(x)^{-1} \Pi_1(x) \Pi_2(x)^* S(x)^{-1} A_1^p \]  
(1.12)
for \( k < 0 \).

Now we shall differentiate \( \Pi_2 S^{-1} \). By (1.3) and (1.10) we have
\[ (\Pi_2 S^{-1})'(x) = -\Pi_2(x)^* S(x)^{-1} S'(x) S(x)^{-1} - \sum_{p=-r}^{r} q_p(x) \Pi_2(x)^* A_2^p S(x)^{-1} \]
\[ = R_1(x) + R_2(x), \]  
(1.13)
where

\[
R_1(x) = - \sum_{p=1}^{r} \sum_{j=1}^{p} X_{p-j}(x)q_p(x)\Pi_2(x)^*A_2^{-1}S(x)^{-1}
\]

\[
- \sum_{p=0}^{r} q_p(x)\Pi_2(x)^*A_2^{-1}S(x)^{-1},
\]

\[
R_2(x) = \sum_{p=-r}^{-1} \sum_{j=p+1}^{0} X_{p-j}(x)q_p(x)\Pi_2(x)^*A_2^{-1}S(x)^{-1}
\]

\[
- \sum_{p=-r}^{-1} q_p(x)\Pi_2(x)^*A_2^{-1}S(x)^{-1}.
\]

In view of (1.10) and (1.12) we rewrite (1.14) in the form

\[
R_1(x) = - \sum_{p=1}^{r} \sum_{j=1}^{p} X_{p-j}(x)q_p(x)\Pi_2(x)^*S(x)^{-1}A_1^{-1}
\]

\[
+ \sum_{p=1}^{r} \sum_{j=0}^{p-2} X_{p-j}(x)q_p(x)Y_{j-1}(x)\Pi_2(x)^*S(x)^{-1}A_1^{-1}
\]

\[
- \sum_{p=0}^{r} q_p(x)\Pi_2(x)^*S(x)^{-1}A_1^{-1}
\]

\[
+ \sum_{p=0}^{r} \sum_{j=0}^{p-1} q_p(x)Y_{p-j-1}(x)\Pi_2(x)^*S(x)^{-1}A_1^{-1},
\]

(1.15)

\[
R_2(x) = \sum_{p=-r}^{-1} \sum_{j=p+1}^{0} X_{p-j}(x)q_p(x)\Pi_2(x)^*S(x)^{-1}A_1^{-1}
\]

\[
+ \sum_{p=-r}^{-1} \sum_{j=p+1}^{0} X_{p-j}(x)q_p(x)Y_{j-1}(x)\Pi_2(x)^*S(x)^{-1}A_1^{-1}
\]

\[
- \sum_{p=-r}^{-1} q_p(x)\Pi_2(x)^*S(x)^{-1}A_1^{-1}
\]

\[
- \sum_{p=-r}^{-1} \sum_{j=p+1}^{0} q_p(x)Y_{p-j-1}(x)\Pi_2(x)^*S(x)^{-1}A_1^{-1}.
\]

(1.16)
Calculate now the coefficients before the terms $\Pi_2^r S^{-1} A_k^r (-r \leq k \leq r)$ in (1.15) and (1.16):

$$R_1(x) = \sum_{k=0}^{r} q_k(x) - \sum_{p=k+1}^{r} q_p(x) Y_{p-k-1}(x) - X_{p-k-1}(x) q_p(x)$$

$$+ \sum_{j=k+2}^{p} X_{p-j}(x) q_p(x) Y_{j-k-2}(x)) \Pi_2(x)^* S(x)^{-1} A_k^r,$$

$$R_2(x) = - \sum_{k=-r}^{1} q_k(x) + \sum_{p=-r}^{k} q_p(x) Y_{p-k-1}(x) - X_{p-k-1}(x) q_p(x)$$

$$- \sum_{j=p+1}^{k+1} X_{p-j}(x) q_p(x) Y_{j-k-2}(x)) \Pi_2(x)^* S(x)^{-1} A_k^r.$$

In view of (1.9) and (1.13) the last relations yield the result

$$\left(\Pi_2^r S^{-1}\right)(x) = - \sum_{p=-r}^{r} \tilde{q}_p(x) \Pi_2(x)^* S(x)^{-1} A_k^r. \quad (1.17)$$

Taking into account (1.3) and (1.17) and using the equality $A_k^r = A_k^r - \lambda^k \rho_n + \lambda^k I_n$, differentiate the right-hand side of (1.5):

$$w'_A(x, \lambda) = \sum_{k=-r}^{r} \lambda^k \tilde{q}_k(x) \Pi_2(x)^* S(x)^{-1} (A_1 - \lambda I_n)^{-1} \Pi_1(x) + \tilde{q}_k(x) \Pi_2(x)^*$$

$$\times S(x)^{-1} (A_1^r - \lambda^r I_n) (A_1 - \lambda I_n)^{-1} \Pi_1(x) - \lambda^k \Pi_2(x)^*$$

$$\times S(x)^{-1} (A_1 - \lambda I_n)^{-1} \Pi_1(x) \tilde{q}_k(x) - \Pi_2(x)^* S(x)^{-1}$$

$$\times \left(A_1^r - \lambda^r I_n\right) (A_1 - \lambda I_n)^{-1} \Pi_1(x) \tilde{q}_k(x) \right]. \quad (1.18)$$

Recall now that

$$G(x, \lambda) = - \sum_{k=-r}^{r} \lambda^k \tilde{q}_k(x), \quad \widetilde{G}(x, \lambda) = - \sum_{k=-r}^{r} \lambda^k \tilde{q}_k(x). \quad (1.19)$$

By (1.10), (1.18), and (1.19) we get

$$w'_A(x, \lambda) = \widetilde{G}(x, \lambda) (w_A(x, \lambda) - I_m) - (w_A(x, \lambda) - I_m) \widetilde{G}(x, \lambda)$$

$$+ \sum_{k=0}^{r} \left(\tilde{q}_k(x) \sum_{j=0}^{k-1} \lambda^j X_{k-j-1}(x) - \left(\sum_{j=0}^{k-1} \lambda^j X_{k-j-1}(x)\right) \tilde{q}_k(x)\right)$$

$$- \sum_{k=-r}^{-1} \left(\tilde{q}_k(x) \sum_{j=k}^{-1} \lambda^j X_{k-j-1}(x) - \left(\sum_{j=k}^{-1} \lambda^j X_{k-j-1}(x)\right) \tilde{q}_k(x)\right)$$

$$= \widetilde{G}(x, \lambda) w_A(x, \lambda) - w_A(x, \lambda) G(x, \lambda) + \sum_{p=-r}^{r} \lambda^p c_p(x). \quad (1.20)$$
where

\[ c_p(x) = \tilde{q}_p(x) - q_p(x) + \sum_{k=p+1}^{r} (\tilde{q}_k(x)X_{k-p-1}(x) - X_{k-p-1}(x)q_k(x)) \]

for \( p \geq 0 \) \hspace{1cm} (1.21)

\[ c_p(x) = \tilde{q}_p(x) - q_p(x) - \sum_{k=-r}^{p} (\tilde{q}_k(x)X_{k-p-1}(x) - X_{k-p-1}(x)q_k(x)) \]

for \( p < 0 \) \hspace{1cm} (1.22)

We shall show that \( c_p(x) = 0 \) for \( 0 \leq p \leq r \). To simplify the formulas we shall omit the variable \( x \) during these calculations and shall write, for instance, just \( c_p \). According to (1.9) and (1.21) we have for \( p \geq 0 \)

\[ c_p = - \sum_{k=p+1}^{r} \left( q_k Y_{k-p-1} - X_{k-p-1} q_p + \sum_{j=p+2}^{k} X_{k-j} q_k Y_{j-p-2} \right) \]

\[ + \sum_{k=p+1}^{r} \left( \tilde{q}_k X_{k-p-1} - X_{k-p-1} q_k \right) \]

\[ = \sum_{k=p+1}^{r} \left\{ \left( q_k - \sum_{j=k+1}^{r} \left( q_j Y_{j-k-1} - X_{j-k-1} q_j + \sum_{i=k+2}^{j} X_{j-i} q_j Y_{i-k-2} \right) \right) \right\} \]

\[ \times X_{k-p-1} - q_k Y_{k-p-1} - \sum_{j=p+2}^{k} X_{k-j} q_k Y_{j-p-2} \} . \hspace{1cm} (1.23) \]

Notice now that by (1.10) and (1.11) the equality

\[ Y_{l-i} X_{j} = \Pi_2^* A_2^{t-i} S^{-1} \Pi_1 \Pi_2 S^{-1} A_1^t \Pi_1 \]

\[ = \Pi_2^* A_2^{t-i} S^{-1} A_1^{t+i} \Pi_1 - \Pi_2^* A_2^{t-i+1} S^{-1} A_1^t \Pi_1 \hspace{1cm} (t \geq l \geq 0) \hspace{1cm} (1.24) \]

is true. Taking into account (1.24) one easily gets

\[ \sum_{l=0}^{t} Y_{t-l} X_{l} = \Pi_2^* S^{-1} A_1^{t+1} \Pi_1 - \Pi_2^* A_2^{t+1} S^{-1} \Pi_1 \]

\[ = X_{t+1} - Y_{t+1} \hspace{1cm} (t \geq 0). \hspace{1cm} (1.25) \]
Changing the order of the summation and using (1.25) we obtain
\[
\sum_{k=p+1}^{r} \sum_{j=k+1}^{r} q_j Y_{j-k+1} X_{k-p-1} = \sum_{j=p+2}^{r} q_j (X_{j-p-1} - Y_{j-p-1}),
\]
\[
\sum_{k=p+1}^{r} \sum_{j=k+1}^{r} \sum_{i=k+2}^{j} X_{j-i} q_j Y_{i-k-2} X_{k-p-1}
= \sum_{j=p+2}^{r} \sum_{i=p+3}^{j} X_{j-i} q_j (X_{i-p-2} - Y_{i-p-2}).
\] (1.26)

By virtue of (1.26) and taking into account \(X_0 = Y_0\) we transform (1.23) into the equality
\[
c_p = \sum_{k=p+1}^{r} \left( \sum_{j=k+1}^{r} X_{j-k-1} q_j X_{k-p-1} - \sum_{j=p+2}^{k} X_{k-j} q_j Y_{j-p-2} \right) - \sum_{k=p+2}^{r} \sum_{j=p+2}^{k} X_{k-j} q_k (X_{j-p-2} - Y_{j-p-2}).
\] (1.27)

Changing the order of the summation we see also that
\[
\sum_{k=p+1}^{r} \sum_{j=k+1}^{r} X_{j-k-1} q_j X_{k-p-1} = \sum_{j=p+2}^{r} \sum_{k=p+1}^{j-1} X_{j-k-1} q_j X_{k-p-1}
= \sum_{k=p+2}^{r} \sum_{j=p+2}^{k} X_{k-j} q_k X_{j-p-2}. \] (1.28)

Formulas (1.27) and (1.28) yield the result
\[
c_p \equiv 0 \quad (p \geq 0). \] (1.29)

Consider now analogously the case \(p < 0\). From (1.22) and the second equality in (1.9) we obtain
\[
c_p = - \sum_{k=r}^{p} \left\{ \left( q_k + \sum_{j=r}^{k} \left( q_j Y_{j-k-1} - X_{j-k-1} q_j - \sum_{i=j+1}^{k+1} X_{j-i} q_j Y_{i-k-2} \right) \right) \right\} 
\times X_{k-p-1} - q_k Y_{k-p-1} + \sum_{j=k+1}^{p+1} X_{k-j} q_k Y_{j-p-2} \} \quad \] (p < 0). (1.30)

From the second equality in (1.11) we get
\[
Y_{t-l} X_l = \Pi_2^+ A_2^{l-1} S^{-1} A_1^{l+1} \Pi_1 - \Pi_2^+ A_2^{l+1} S^{-1} A_1^l \Pi_1 \quad (t < l < 0).
\]
Hence the equality
\[ \sum_{l=-1}^{-1} Y_{t-l} X_l = Y_{t+1} - X_{t+1} \quad (t < -1) \] (1.31)
is true. By (1.31) we have
\[
\sum_{k=-r}^p \sum_{j=-r}^k q_j Y_{j-k+1} X_{k-p-1} = \sum_{j=-r}^p q_j \sum_{k=j}^p Y_{j-k+1} X_{k-p-1} \\
= - \sum_{j=-r}^p q_j (X_{j-p-1} - Y_{j-p-1}),
\]
(1.32)
From (1.30) and (1.32) it follows that
\[
c_p = \sum_{k=-r}^p \sum_{j=-r}^k X_{j-k+1} q_j X_{k-p-1} - \sum_{j=-r}^p \sum_{i=j+1}^{p+1} X_{j-i} q_j X_{i-p-2} \\
= 0 \quad (p < 0).
\] (1.33)
According to (1.20), (1.29), and (1.33) we write down the final expression for the derivative of the transfer matrix function:
\[
w_A'(x, \lambda) = \tilde{G}(x, \lambda) w_A(x, \lambda) - w_A(x, \lambda) G(x, \lambda).
\] (1.34)
Formulas (1.1), (1.7), and (1.34) immediately yield (1.8).
\( \tilde{q}_k(x) \equiv 0 \) for all \(-r \leq k \leq -r_1 < 0\). In particular, starting with \( G(x, \lambda) \) polynomial in \( \lambda \) we can relax the condition (1.6).

**Theorem 1.2.** Let matrix functions \( w, \Pi_1, \Pi_2, \) and \( S \) satisfy the identity (1.2) and the equations

\[
\begin{align*}
    w'(x, \lambda) &= G(x, \lambda)w(x, \lambda), \\
    G(x, \lambda) &= -\sum_{k=0}^{r} \lambda^k q_k(x) \left( w' = \frac{d}{dx} w, \ x \in \Gamma \right), \quad (1.35) \\
    \Pi_1(x) &= \sum_{p=0}^{r} A_1^p \Pi_1(x)q_p(x), \\
    \Pi_2(x) &= -\sum_{p=0}^{r} (A_2^p)^* \Pi_2(x)q_p(x)^*, \quad (1.36) \\
    S'(x) &= \sum_{p=1}^{r} \sum_{j=1}^{p} A_1^p \Pi_1(x)q_p(x)^* A_2^{-1}. 
\end{align*}
\]

Then in the points of invertibility of \( S \) the matrix function \( \tilde{w}(x, \lambda) = w_A(x, \lambda)w(x, \lambda) \) satisfies the system

\[
\begin{align*}
    \tilde{w}'(x, \lambda) &= \tilde{G}(x, \lambda)\tilde{w}(x, \lambda), \\
    \tilde{G}(x, \lambda) &= -\sum_{k=0}^{r} \lambda^k \tilde{q}_k(x), \quad (1.37) 
\end{align*}
\]

where the coefficients \( \tilde{q}_k \) are given by the first formula in (1.9) (the case \( k \geq 0 \)).

With the restrictions (1.6) ignored, we can consider nilpotent matrices \( A_1 \) and \( A_2 \) and construct rational solutions of the nonlinear equations. (See [22–24, 38] for a detailed discussion on rational solutions and bispectral property.)

**Remark.** In many important examples in this paper as well as in [22, 33, 34] we have \( r = 1, q_{-1} \equiv 0, q_1 = (-1)^{ij} (j = j^* = j^{-1}; l = 0, 1), A := A_1 = A_2^*, \) and \( \Pi := \Pi_1 = i\Pi_2j. \) Therefore (1.3) yields \((-1)^{j+i}S' = \Pi_2 \Pi_1^* \geq 0. \) Starting with \( S(0) > 0 \) we have \( S(x) > 0 \) on \((-\infty, 0] \) for \( l = 0 \) and \( S(x) > 0 \) on \([0, \infty)\) for \( l = 1. \) Then \( S(x) \) is invertible and moreover \( w_A \) is the characteristic function corresponding to the matrix \( S(x)^{-1/2}A S(x)^{1/2}. \) We get \( S' \geq 0 \) for the GBDT of the canonical system \( w' = i\lambda H w, H \geq 0 \) also [37].

In some other examples the identity (1.4) takes the form \( AS(x) - S(x) A^* = i\Pi(x) \Pi(x)^* \) and hence \( A^* \ker S(x) \subseteq \ker S(x), \) \( \Pi(x)^* \ker S(x) = 0. \) These relations can be used to derive the invertibility of \( S(x) \) as well (see Section 4 of [33]). Finally notice that the singularities of the potentials (zeros of \( \det S(x) \)) are of mathematical and physical interest also (see [5, 42] and references therein).
One can easily invert the GBDT

**Theorem 1.3.** Let system (1.1) with the locally summable on \( \Gamma \) matrix coefficients \( q_k(x) \) be given. Let also the parameter matrices \( A_1, A_2, \Pi_1(0), \Pi_2(0), \) and \( S(0) \) be chosen so that identity (1.2) is true and the inequalities \( \det A_1 \neq 0, \det A_2 \neq 0, \) and \( \det S(x) \neq 0 \) (\( x \in \Gamma \)) hold. Then the GBDT of the transformed system (1.8), generated by matrices \( A_2, A_1, S(0)^{-1} \Pi_1(0), \) \( (S(0)^{-1})^* \Pi_2(0), \) and \( -S(0)^{-1} \), turns the transformed system (1.8) back into the initial one (1.1).

**Proof.** Let us show at first that

\[
(S^{-1} \Pi_1)'(x) = \sum_{p=-r}^{r} A^p_2 S(x)^{-1} \Pi_1(x) \tilde{q}_p(x). \tag{1.38}
\]

For this purpose consider the GBDT of system \( w' = \hat{G}w \) with

\[
\hat{G}(x, \lambda) = -\sum_{k=-r}^{r} \lambda^k \hat{q}_k(x), \quad \hat{q}_k(x) = -q_k(x)^*, \tag{1.39}
\]

generated by

\[
\hat{A}_1 = A_2^*, \quad \hat{A}_2 = A_1^*, \quad \hat{\Pi}_1(0) = \Pi_2(0), \quad \hat{\Pi}_2(0) = \Pi_1(0), \quad \hat{S}(0) = -S(0)^*. \tag{1.40}
\]

From (1.3), (1.39), and (1.40) it follows that

\[
\hat{\Pi}_1(x) = \Pi_2(x), \quad \hat{\Pi}_2(x) = \Pi_1(x), \quad \hat{S}(x) = -S(x)^*. \tag{1.41}
\]

According to (1.10) and (1.41) we have \( \hat{X}_k = -Y_k^*, \hat{Y}_k = -X_k^* \). Hence from (1.9) and (1.39) we obtain

\[
\hat{q}_k(x) = -\tilde{q}_k(x)^*. \tag{1.42}
\]

Notice now that by (1.17) the equation

\[
(\hat{\Pi}_2^* \hat{S}^{-1})'(x) = -\sum_{p=-r}^{r} \tilde{q}_p(x) \hat{\Pi}_2(x)^* \hat{S}(x)^{-1} \hat{A}^p_1 \tag{1.43}
\]

is true. In view of (1.40)–(1.42) taking adjoint for both sides of (1.43) we get (1.38). Let us denote by \( \Pi_1, \hat{\Pi}_2, \hat{S}, \) and \( \hat{w}_A \) matrix functions corresponding to the GBDT of the transformed system (1.8), generated by the parameter matrices \( \hat{A}_1 = A_2, \hat{A}_2 = A_1, \hat{\Pi}_1(0) = S(0)^{-1} \Pi_1(0), \hat{\Pi}_2(0) = (S(0)^{-1})^* \Pi_2(0), \) and \( \hat{S}(0) = -S(0)^{-1} \). The doubly transformed solutions
and coefficients we shall denote by $\tilde{w}$ and $\tilde{q}_k$. Taking into account (1.3), (1.17), and (1.38) we obtain

$$\tilde{\Pi}_1(x) = S(x)^{-1}\Pi_1(x), \quad \tilde{\Pi}_2(x) = (S(x)^{-1})^*\Pi_2(x)$$

(1.44)

and hence $\tilde{S}(x)$ satisfies the operator identity

$$A_2\tilde{S}(x) - \tilde{S}(x)A_1 = S(x)^{-1}\Pi_1(x)\Pi_2(x)^*S(x)^{-1}. \quad (1.45)$$

Compare (1.4) and (1.45). If $\sigma(A_1) \cap \sigma(A_2) = \emptyset$ ($\sigma$ is spectrum), then both identities (1.4) and (1.45) have unique solutions. On the other hand by (1.4) the matrix function $-S(x)^{-1}$ satisfies (1.45); i.e.,

$$\tilde{S}(x) = -S(x)^{-1}.$$  

(1.46)

If $\sigma(A_1) \cap \sigma(A_2) \neq \emptyset$, we can always choose sequences of matrices \{\Pi_1, I\}_l=0, \{\Pi_2, I\}_l=0, and \{\Pi, I\}_l=0 such that

$$\lim_{l \to \infty} \|A_k - A_{k,l}\| = 0,$$

$$\lim_{l \to \infty} \|\Pi_k(0) - \Pi_{k,l}(0)\| = 0 \quad (k = 1, 2), \quad (1.47)$$

$$A_1, S(0) - S(0)A_2 = \Pi_{1, l}(0)\Pi_{2, l}(0)^*,$$

$$\sigma(A_1, l) \cap \sigma(A_2, l) = \emptyset. \quad (1.48)$$

As $\det S(0) \neq 0$ we start to prove that in any neighbourhood of $A_k$ and $\Pi_k(0)$ there exist matrices $A_{k,l}$ and $\Pi_{k,l}(0)$, satisfying the operator identity and (1.48), by rewriting (1.2) in the form

$$A_1 - S(0)A_2S(0)^{-1} = \Pi_1(0)\Pi_2(0)^*S(0)^{-1}. \quad (1.49)$$

Then we consequently perturb $A_1$ and $S(0)A_2S(0)^{-1}$, reducing algebraic multiplicities of common eigenvalues to one, and obtain $A_{1,l}$ and $A_{2,l}$, satisfying the identity

$$A_{1,l} - S(0)A_{2,l}S(0)^{-1} = \Pi_{1,l}(0)\Pi_{2,l}(0)^*S(0)^{-1}. \quad (1.50)$$

Afterwards we remove common eigenvalues of $A_{1,l}$ and $A_{2,l}$ of multiplicity one by small perturbations of $\Pi_{1,l}(0)$, $\Pi_{2,l}(0)^*$ and corresponding additional perturbations of $S(0)A_{2,l}S(0)^{-1}$ and get

$$A_{1,l} - S(0)A_{2,l}S(0)^{-1} = \Pi_{1,l}(0)\Pi_{2,l}(0)^*S(0)^{-1},$$

$$\sigma(A_{1,l}) \cap \sigma(S(0)A_{2,l}S(0)^{-1}) = \emptyset. \quad (1.51)$$

We take also into account that $\sigma(S(0)A_{2,l}S(0)^{-1}) = \sigma(A)$. Apply now the GBDTs generated by these matrices to system (1.1) and the GBDTs generated by $A_{2,l}$, $A_{1,l}$, $S(0)^{-1}\Pi_1, I$, $(S(0)^{-1})^*\Pi_2, I$, and $-S(0)^{-1}$ to the corresponding transformed ones. According to (1.47), the matrix function $S(x)$
tends to $S(x)$ and $\tilde{S}'(x)$ tends to $\tilde{S}(x)$. By (1.48) we have $\tilde{S}'_i(x) = -S_i(x)^{-1}$. Hence (1.46) is true again. From (1.5), (1.44), and (1.46) it follows that

$$\tilde{w}_A(x, \lambda) = I_m - \tilde{\Pi}_2(x)^* \tilde{S}(x)^{-1} (\tilde{A}_1 - \lambda I_n)^{-1} \tilde{\Pi}_1(x)$$
$$= I_m + \Pi_2(x)^* (A_2 - \lambda I_n)^{-1} S(x)^{-1} \Pi_1(x). \quad (1.49)$$

In view of the identity (1.4) equality (1.49) yields \[40\] the equality

$$\tilde{w}_A(x, \lambda) = w_A(x, \lambda)^{-1}. \quad (1.50)$$

Formulas (1.7) and (1.50) show that $\tilde{w} = w$ and therefore $\tilde{q}_k = q_k$. 

Some important restrictions on the structure of the coefficients $q_k$ of system (1.35) hold under GBDT. We shall need

**Proposition 1.4.** Let the coefficients $q_k$ of system (1.35) satisfy the restrictions

$$q_k(x)^* = -J q_k(x) J^{-1} \quad (0 \leq k \leq r), \quad J^* = -J. \quad (1.51)$$

Let also the parameter matrices be chosen so that

$$A_1 = A_2^*, \quad \Pi_1(0) = \Pi_2(0) J, \quad S(0) = S(0)^*. \quad (1.52)$$

Then the transformed coefficients $\tilde{q}_k$ given by (1.9) satisfy the same restrictions

$$\tilde{q}_k(x)^* = -J \tilde{q}_k(x) J^{-1} \quad (0 \leq k \leq r). \quad (1.53)$$

**Proof.** From (1.36), (1.51), and (1.52) it follows that

$$\Pi_1(x) = \Pi_2(x) J, \quad S(x) = S(x)^*. \quad (1.54)$$

By (1.10), (1.52), and (1.54) we have

$$X_k(x)^* = J^* Y_k(x) J^{-1} = -J Y_k(x) J^{-1}, \quad Y_k(x)^* = -J X_k(x) J^{-1}. \quad (1.55)$$

Taking into account (1.9), (1.51), and (1.55) we see that (1.53) is true.

**Example.** Consider the Dirac type system

$$w'(x, \lambda) = -(\lambda q_1 + q_0(x))w(x, \lambda),$$

$$q_1 = ij, \quad j = \begin{bmatrix} I_h & 0 \\ 0 & -I_h \end{bmatrix}, \quad q_0(x) = \begin{bmatrix} 0 & u(x)^* \\ u(x) & 0 \end{bmatrix}, \quad (1.56)$$

$m = 2h.$ According to (1.56) this system has property (1.51) with $J = ij$.

From (1.9) and (1.56) it follows that

$$\tilde{q}_0(x) = \begin{bmatrix} 0 & \tilde{u}(x)^* \\ \tilde{u}(x) & 0 \end{bmatrix},$$

$$\tilde{u}(x) = u(x) + 2i (\Pi_2(x)^* S(x)^{-1} \Pi_1(x))_{21}, \quad (1.57)$$

where $(\Pi_2^* S^{-1} \Pi_1)_{21}$ is the $h \times h$ block of $\Pi_2^* S^{-1} \Pi_1$. 

2. THE SPECTRUM AND THE SCATTERING MATRIX

We call a vector function \( f \) an eigenfunction (bound state) of system (1.37) with eigenvalue \( \lambda \) if
\[
f'(x) = \tilde{G}(x, \lambda)f(x), \quad f \in L^2_m(\Gamma) .
\] (2.1)

In the next theorem we shall consider the cases \( \Gamma = (-\infty, 0] \) and \( \Gamma = [0, \infty) \). The related explicit and detailed results on the bound states of the Dirac type systems with the so-called pseudoexponential potentials one can find in [25].

**Theorem 2.1.** Let the system (1.35) be chosen to have matrix coefficients \( q_k(x) \) locally summable on \( \Gamma \) and choose the parameter matrices \( A_1, A_2, \Pi_1(0), \Pi_2(0), \) and \( S(0) \) so that identity (1.2) is true, the matrix function \( S(x) \) is invertible, \( q_r \equiv \text{const} \), and the transformed coefficients given by (1.9) tend to zero, i.e.,
\[
\lim_{x \to \infty} \| \tilde{q}_k(x) \| = 0 \quad (0 \leq k < r) \quad (2.2)
\]
in the case \( \Gamma = [0, \infty) \) and
\[
\lim_{x \to -\infty} \| \tilde{q}_k(x) \| = 0 \quad (0 \leq k < r) \quad (2.3)
\]
in case \( \Gamma = (-\infty, 0] \). Suppose also that \( g \) is an eigenvector of \( A_1 \) with an eigenvalue \( a \) and
\[
a'q_r + \bar{a}'q_r^* > 0 \quad (2.4)
\]
in the case \( \Gamma = [0, \infty) \) and
\[
a'q_r + \bar{a}'q_r^* < 0 \quad (2.5)
\]
in the case \( \Gamma = (-\infty, 0] \). Then \( f = \Pi_2 S^{-1} g \) is an eigenvector of system (1.37) considered either on \( \Gamma = [0, \infty) \) or on \( \Gamma = (-\infty, 0] \), respectively.

**Proof.** Equality (1.17) in the case of system (1.35) turns into
\[
(\Pi_2 S^{-1})'(x) = -\sum_{p=0}^r \tilde{q}_p(x)\Pi_2(x)^*S(x)^{-1}A_1^p .
\] (2.6)

According to (2.6) we have
\[
(\Pi_2 S^{-1})'(x)g = \tilde{G}(x, a)\Pi_2(x)^*S(x)^{-1}g^* ;
\] (2.7)
i.e., the first relation in the conditions (2.1) is true. Moreover for \( R(x) = g^*S(x)^{-1}\Pi_2(x)\Pi_2(x)^*S(x)^{-1}g \) from (2.7) it follows that
\[
R(x) = g^*S(x)^{-1}\Pi_2(x)(\tilde{G}(x, a) + \tilde{G}(x, a)^*)\Pi_2(x)^*S(x)^{-1}g .
\] (2.8)
In view of (2.8) there exist the values \( \varepsilon > 0 \) and \( x_0 \) such that
\[
R'(x) \leq -\varepsilon R(x) \quad \text{for all } x > x_0
\] (2.9)
if (2.2) and (2.4) hold, and
\[
R'(x) \geq \varepsilon R(x) \quad \text{for all } x < x_0
\] (2.10)
if (2.3) and (2.5) hold. Inequality (2.9) means that \( e^{\varepsilon x}R(x) \) monotonically decreases; i.e., \( R(x) \leq e^{-\varepsilon(x-x_0)}R(x_0) \) for \( x > x_0 \). Therefore
\[
\int_{x_0}^{\infty} R(x)dx < \infty \quad \text{and} \quad S^{-1}g \in L^2_\infty[0, \infty). \]
Quite analogously (2.10) yields \( R(x) \leq e^{\varepsilon(x-x_0)}R(x_0) \) for \( x < x_0 \) and \( S^{-1}g \in L^2_{\infty}(-\infty, 0) \).

Consider system (1.35) on the real axis \( \Gamma = (-\infty, \infty) \). Suppose
\[
q_k(x) \equiv iD, \quad D = \text{diag}\{d_1, d_2, \ldots, d_m\},
\]
\[
d_1 > d_2 > \cdots > d_m > 0,
\] (2.11)
\[
q_k(x) = -q_k(x)^* \quad (0 \leq k \leq r).
\]
Here \( \text{diag} \) means diagonal matrix. The scattering matrix of system (1.35) is defined by the equality
\[
\text{osc}(\lambda) = w_+(x, \lambda)^{-1}w_-(x, \lambda) \quad (\lambda = \lambda),
\] (2.12)
where \( w_\pm \) satisfy (1.35) and have the following asymptotics on the real axis:
\[
w_+(x, \lambda) = (I_m + o(1))e^{-i\lambda x}D \quad (x \to \infty),
\]
\[
w_-(x, \lambda) = (I_m + o(1))e^{i\lambda x}D \quad (x \to -\infty).
\] (2.13)
There is a simple connection between the scattering matrix of the initial and transformed systems. We shall denote by \( \pi_p \) the \( p \)th row of \( \Pi_1 \), by \( \mathbb{C} \) the complex plane, and by \( e_k \in \mathbb{C}^m \) the vector with only one \( k \)th nonzero entry, which equals 1.

**Theorem 2.2.** Let the system (1.35) be chosen to have matrix coefficients satisfying (2.11) on \( \Gamma = (-\infty, \infty) \). Suppose the parameter matrices are chosen so that
\[
A_1 = A_2^*, \quad \Pi_1(0) = i\Pi_2(0), \quad S(0) = S(0)^*
\] (2.14)
and \( \det S(x) \neq 0 \). Then \( \tilde{q}_k = -\tilde{q}_k^* \quad (0 \leq k \leq r) \). Suppose additionally that the scattering matrix \( \text{osc}(\lambda) \) exists,
\[
A_1 = \text{diag}\{a_1, \ldots, a_n\}, \quad a_k \neq a_j \quad (k \neq j),
\]
\[
a_k \neq \bar{a}_j, \quad \Im a_k^* < 0 \quad (1 \leq k \leq l), \quad \Re a_k^* > 0 \quad (k > l)
\] (2.15)
and

\[ q_k(x) \in L^1_{m \times m}(-\infty, \infty), \quad \pi_{\pm}(p) \neq 0 \quad (1 \leq p \leq m), \quad (2.16) \]

where \( a = (a - \bar{a})/2i \),

\[ \pi_{\pm}(p) = \lim_{x \to \pm \infty} e^{-ia_p^x \pi_p(x)e_{p\pm}}, \quad (2.17) \]

\( p_+ = 1 \) for \( p \leq l \), \( p_+ = m \) for \( p > l \), \( p_- = m \) for \( p \leq l \), \( p_- = 1 \) for \( p > l \). Then the scattering matrix \( \tilde{S}(\lambda) \) of the transformed system (1.37) has the form

\[ \tilde{S}(\lambda) = D_+(\lambda)S(\lambda)D_-(\lambda)^{-1}, \quad (2.18) \]

where

\[ D_{\pm}(\lambda) = \text{diag}\{\alpha_{\pm}(\lambda), 1, \ldots, 1, \alpha_{\mp}(\lambda)\}, \quad (2.19) \]

\[ \alpha_{\pm}(\lambda) = \prod_{p=1}^{l} \frac{\lambda - \bar{a}_p}{\lambda - a_p}, \quad \alpha_{\mp}(\lambda) = \prod_{p=l+1}^{n} \frac{\lambda - \bar{a}_p}{\lambda - a_p}. \]

**Proof.** The property \( \tilde{q}_k = -\tilde{q}_k^* \) is immediate from (2.11), (2.14), and Proposition 1.4. By (1.9) and (2.11) we have also \( q_r = q_r = iD \). Hence, taking into account (1.36) we see that the \( p \)th row \( \pi_p \) of \( \Pi_1 \) satisfies the equation

\[ \pi_p'(x) = \pi_p(x)\left(ia_p^x D + \sum_{k=0}^{r-1} a_p^k q_k(x)\right). \quad (2.20) \]

Notice that \( \Im a_p^k d_k \neq \Im a_j^k d_j \) (\( k \neq j \)) and \( q_k(x) \in L^1_{m \times m}(-\infty, \infty) \). Thus (2.20) belongs to the class of systems with thoroughly investigated asymptotics (see, for instance, [32]), and we easily obtain the existence of the limit on the right-hand side of (2.17) and the representation

\[ \pi_p(x) = e^{ia_p^x \pi_{\pm}(p)}e_{p\pm} + o(1) \quad (x \to \pm \infty). \quad (2.21) \]

As \( \pi_{\pm}(p) \neq 0 \), relations (1.54) for the case \( J = iI_m \) and formula (2.21) yield

\[ \Pi_1(x) = D_{\pm}(x)\left[e_{p\pm}^n\right]_{p=1}^{\infty} + o(1), \]

\[ \Pi_1(x)\Pi_2(x)^* = iD_{\pm}(x)\left[\begin{array}{cc} g_1 g_1^* & 0 \\
0 & g_2 g_2^* \end{array}\right] + o(1) \quad (x \to \pm \infty). \quad (2.22) \]
where $D_\pm(x) = \text{diag}\{\pi_{\pm}(1)e^{i\theta_d z^k}, \ldots, \pi_{\pm}(n)e^{i\theta_d z^k}\}$, $g_1 = \text{col}[1 \cdots 1] \in \mathbb{C}^l$, $g_2 = \text{col}[1 \cdots 1] \in \mathbb{C}^{n-1}$; col means column. From (1.4), (2.14), (2.15), and (2.22) it follows that

$$S(x) = iD_\pm(x)\left(\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} + o(1)\right)D_\pm(x)^* \quad (x \to \pm\infty),$$

$$S_1 = \left\{\frac{1}{a_k - \bar{a}_j}\right\}_{k,j=1}^l, \quad S_2 = \left\{\frac{1}{a_{k+l} - \bar{a}_{j+l}}\right\}_{k,j=1}^{n-l}. \quad (2.23)$$

Under the condition $a_k \neq a_j$ ($k \neq j$) the invertibility of the matrices $S_k$ is a well established fact. In view of (1.5), (2.22), and (2.23) now we get

$$w_A(x, \lambda) = \text{diag}\{\alpha_1(\lambda), 1, \ldots, 1, \alpha_2(\lambda)\} + o(1) \quad (x \to \infty),$$

$$w_A(x, \lambda) = \text{diag}\{\alpha_2(\lambda), 1, \ldots, 1, \alpha_1(\lambda)\} + o(1) \quad (x \to -\infty), \quad (2.24)$$

where

$$\alpha_k(\lambda) = 1 - g_k^*S_k^{-1}(A_{1k} - \lambda I)^{-1}g_k, \quad (2.25)$$

and $A_{1k}$ ($k = 1, 2$) are the blocks of

$$A_1 = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{12} \end{bmatrix}. \quad (2.26)$$

Notice that the functions $\alpha_k$ have poles at each eigenvalue of $A_{1k}$. Indeed, it is easily seen that the operator identity

$$A_{1k}S_k - S_kA_{1k}^* = g_kg_k^* \quad (k = 1, 2) \quad (2.27)$$

is true. Consider $\alpha_1(\lambda)$ and suppose that $\alpha_1$ has no singularity at $\lambda = \lambda_p$ ($p \leq l$). It means that $g_k^*S_k^{-1}e_p = 0$. On the other hand from (2.26) it follows that

$$(S_k^{-1}A_{11} - A_{11}^*S_k^{-1})e_p = S_k^{-1}S_k^{-1}g_k^*S_k^{-1}e_k.$$ 

Therefore we have $(S_k^{-1}A_{11} - A_{11}^*S_k^{-1})e_p = 0$, i.e., $(a_pI - A_{11}^*)S_k^{-1}e_p = 0$, which contradicts the inequality $a_p \neq \bar{a}_p$. The case of $\alpha_2(\lambda)$ is quite analogous. Hence functions $\alpha_k(\lambda)$ are rational functions with poles of the first order and precisely at the eigenvalues of $A_{1k}$. Analogously to (1.50) from (2.26) we get also that $\alpha_k(\lambda)\alpha_k(\bar{\lambda}) = 1$. So the zeros of $\alpha_k(\lambda)$ coincide with the poles of $\alpha_k(\lambda)$, i.e., with the eigenvalues of $A_{1k}$. Finally we obtain

$$\alpha_1(\lambda) = \prod_{p=1}^l \frac{\lambda - \bar{a}_p}{\lambda - a_p} = \alpha_+(\lambda), \quad \alpha_2(\lambda) = \prod_{p=l+1}^n \frac{\lambda - \bar{a}_p}{\lambda - a_p} = \alpha_-(\lambda). \quad (2.27)$$

According to (2.19), (2.24), and (2.27) we have

$$w_A(x, \lambda) = D_\pm(\lambda) + o(1) \quad (x \to \pm\infty). \quad (2.28)$$
From Theorem 1.2 and formulas (2.13) and (2.28) it follows that the matrix functions
\[ \tilde{w}_\pm(x, \lambda) = w_\pm(x, \lambda) w_\pm(x, \lambda) \mathbb{D}_\pm(\lambda)^{-1} \]  
(2.29)
satisfy the transformed system (1.37) and tend to \( e^{-i\lambda x^D} \) when \( x \) tends to \( \pm\infty \), respectively. Hence the scattering matrix \( \tilde{S}(\lambda) \) equals \( \tilde{w}_+(x, \lambda)^{-1} \tilde{w}_-(x, \lambda) \) and by (2.12) and (2.29) satisfies (2.18).

Consider now system (1.35) and put \( m = 2h, r = 1 \),
\[ q_1 = \begin{bmatrix} 0 & 0 \\ I_h & 0 \end{bmatrix}, \quad q_0(x) = -\begin{bmatrix} 0 & I_h \\ 0 & 0 \end{bmatrix} u(x). \]  
(2.30)
Solution \( w \) of system (1.35) with the coefficients given by (2.30) can be written down in the block form \( w = [ I ] \). Hence we rewrite (1.35) as \( y'(x, \lambda) = \tilde{y}(x, \lambda), y'(x, \lambda) = -\lambda y(x, \lambda) + u(x)y(x, \lambda); i.e., \)
\[ y''(x, \lambda) - u(x)y(x, \lambda) + \lambda y(x, \lambda) = 0. \]  
(2.31)
So system (1.35), (2.30) is equivalent to the Sturm–Liouville system (2.31).

**Proposition 2.3.** Let a matrix function \( y(x, \lambda) \) satisfy the Sturm–Liouville system (2.31) and put
\[ \tilde{y}(x, \lambda) = \begin{bmatrix} I_h & 0 \end{bmatrix} \tilde{w}(x, \lambda), \]  
(2.32)
where \( \tilde{w} \) is the GBDT of the solution \( w = [ \chi \gamma ] \) of system (1.35), (2.30). Then \( \tilde{y} \) satisfies the Sturm–Liouville system
\[ \tilde{y}''(x, \lambda) - \tilde{u}(x)\tilde{y}(x, \lambda) + \lambda \tilde{y}(x, \lambda) = 0, \]  
(2.33)
where
\[ \tilde{u}(x) = u(x) + 2(X_{011}(x) - X_{022}(x) + X_{012}(x)^2), \]  
(2.34)
and \( X_{0kj} \) are the \( h \times h \) blocks of the matrix \( X_0 \) given by (1.10).

**Proof.** According to (1.9) we have \( \tilde{q}_1 = q_1 \),
\[ \tilde{q}_0(x) = q_0(x) + X_0(x)q_1 - q_1 X_0(x) \]
\[ = q_0(x) + \begin{bmatrix} X_{012}(x) \\ X_{022}(x) - X_{011}(x) & -X_{012}(x) \end{bmatrix} \]  
(2.35)
By Theorem 1.2 and relations (1.37) and (2.35) the matrix function \( \tilde{y} \) of the form (2.32) satisfies the equation \( \tilde{y}'(x, \lambda) = [ I_h ] \tilde{w}(x, \lambda) - X_{012}(x)\tilde{y}(x, \lambda); i.e., \)
\[ \tilde{y}''(x, \lambda) = -\lambda \tilde{y}(x, \lambda) + (u(x) + X_{011}(x) - X_{022}(x))\tilde{y}(x, \lambda) \]
\[ + X_{012}(x)^2\tilde{y}(x, \lambda) - X_{012}(x)\tilde{w}(x, \lambda) \]  
(2.36)
This means that $\tilde{y}$ satisfies (2.33) with
\[ \ddot{u}(x) = u(x) + X_{011}(x) - X_{022}(x) + X_{012}(x)^2 - X'_{012}(x). \] (2.36)
Moreover from (1.10), (1.17), and (1.36) if follows that
\[ X_0'(x) = -\tilde{q}_0(x)X_0(x) - q_1Y_1(x) + X_0(x)q_0(x) + Y_1(x)q_1. \] (2.37)
Therefore we obtain
\[ X_{012}(x) = X_{022}(x) - X_{011}(x) - X_{012}(x)^2. \] (2.38)
Relations (2.36) and (2.38) yield (2.34).

Remark. Consider now the GBDT of system (2.31) with $A_1 = A_2 = \lambda_0I_n$. From (1.36) we obtain
\[ \Pi_1(x) = \lambda_0\Pi_1(x)q_1 + \Pi_1(x)q_0(x), \]
\[ \Pi_2(x)^* = -\lambda_0q_1\Pi_2(x)^* - q_0(x)\Pi_2(x)^*. \] (2.39)
In view of (2.30) and (2.39) we get the block representations
\[ \Pi_1(x) = \begin{bmatrix} -Z'(x) & Z(x) \end{bmatrix}, \quad \Pi_2(x)^* = \begin{bmatrix} \Psi(x) \\ \Psi'(x) \end{bmatrix}, \] (2.40)
with $Z$ and $\Psi$ satisfying the equations
\[ Z''(x) - Z(x)u(x) + \lambda_0Z(x) = 0, \]
\[ \Psi''(x) - u(x)\Psi(x) + \lambda_0\Psi(x) = 0. \] (2.41)
Taking into account (1.36) and (2.40) we obtain
\[ S'(x) = \Pi_1(x)q_1\Pi_2(x)^* = Z(x)\Psi(x). \] (2.42)
As $A_1S - SA_2 = \lambda_0S - \lambda_0S = 0$, we need
\[ Z(0)\Psi'(0) - Z'(0)\Psi(0) = 0, \] (2.43)
i.e., $\Pi_1(0)\Pi_2(0)^* = 0$, for the identity (1.2) to be true. By (2.34), (2.40), and (2.43) we have
\[ \ddot{u}(x) = u(x) + 2((\Psi(x)S(x)^{-1}Z(x))^2 - \Psi'(x)S(x)^{-1}Z(x) \]
\[ - \Psi(x)S(x)^{-1}Z'(x)). \] (2.44)
Formula (2.44) with $\Psi$, $Z$, and $S$ given by (2.41) and (2.42) defines under condition (2.43) the transformation from [4, Theorem 6.2.1], which was used there to remove singularity from the potential $u$ at some point $x = x_0$.
(See also [3] on BDT and regularization of isospectral operators.)

Some inverse spectral problems are treated in a way related to GBDT in [22–24, 37].
3. NONLINEAR EQUATIONS

Various integrable nonlinear equations may be considered (see [1, 46] and references in [45]) in the form

\[ G_t(x, t, \lambda) - F_x(x, t, \lambda) + [G(x, t, \lambda), F(x, t, \lambda)] = 0 \]  \hspace{1cm} (3.1)

\[ \left( G_t = \frac{\partial}{\partial t} G, \quad [G, F] = GF - FG \right), \]

which is equivalent to the famous Lax equality. We shall suppose in this section again that \( G \) and \( F \) are polynomial in \( \lambda \) and \( \lambda^{-1} \),

\[ G(x, t, \lambda) = - \sum_{k=-r}^{r} \lambda^k q_k(x, t), \quad F(x, t, \lambda) = - \sum_{k=-l}^{l} \lambda^k Q_k(x, t) \]  \hspace{1cm} (3.2)

and that the coefficients \( q_k \) and \( Q_k \) are continuous together with their first derivatives, \( x \in \Gamma_1, t \in \Gamma_2 \), the sets \( \Gamma_k \) are either \( (-c_k, c_k) \) or \( [0, c_k) \). Equation (3.1) is the condition of compatibility for the auxiliary systems

\[ w_t(x, t, \lambda) = G(x, t, \lambda) w(x, t, \lambda), \]

\[ w_t(x, t, \lambda) = F(x, t, \lambda) w(x, t, \lambda). \]  \hspace{1cm} (3.3)

Formulas (3.2) and (3.3) show that now we have to add variable \( t \) in the considerations of Section 1. The matrix functions \( \Pi_1(x, t), \Pi_2(x, t), \) and \( S(x, t) \) (and the corresponding GBDT) will be defined by the five parameter matrices and by the coefficients \( q_k(x, t) \) and \( Q_k(x, t) \) depending on the variables \( x \) and \( t \). Five parameter matrices are now \( A_1, A_2, \Pi_1(0, 0), \Pi_2(0, 0), \) and \( S(0, 0) \) (det \( S(0, 0) \neq 0 \)), satisfying the operator identity

\[ A_1 S(0, 0) + S(0, 0) A_2 = \Pi_1(0, 0) \Pi_2(0, 0)^*. \]  \hspace{1cm} (3.4)

The matrix functions \( \Pi_1(x, t), \Pi_2(x, t), \) and \( S(x, t) \) are given by their initial values at \( x = 0, t = 0, \) by the differential equations (1.3) with respect to \( x \), and by the differential equations

\[ \frac{\partial}{\partial t} \Pi_1(x, t) = \sum_{p=-l}^{l} A_1^p \Pi_1(x, t) Q_p(x, t), \]

\[ \frac{\partial}{\partial t} \Pi_2(x, t) = - \sum_{p=-l}^{l} (A_2^*)^p \Pi_2(x, t) Q_p(x, t)^*, \]

\[ S(x, t) = \sum_{p=1}^{l} \sum_{j=1}^{p} A_1^{p-j} \Pi_1(x, t) Q_p(x, t) \Pi_2(x, t)^* A_2^{j-1} \]

\[ - \sum_{p=-l}^{l} \sum_{j=p+1}^{l} A_1^{p-j} \Pi_1(x, t) Q_p(x, t) \Pi_2(x, t)^* A_2^{j-1} \]  \hspace{1cm} (3.5)

with respect to \( t \).
Proposition 3.1. Suppose compatibility condition (3.1), where $G$ and $F$ are given by relations (3.2), is valid. Let also the parameter matrices satisfy (3.4) and $\det S(0, 0) \neq 0$. Then the first, second, and third equations, respectively, in formulas (1.3) and (3.5) are compatible.

Proof. Equation (3.1) is equivalent to the system

$$\frac{\partial}{\partial x} Q_k(x, t) - \frac{\partial}{\partial t} q_k(x, t) + \sum_{p+s=k} [q_p(x, t), Q_s(x, t)] = 0,$$

where we put $q_k \equiv 0$ in the case $|k| > r$ and $Q_k \equiv 0$ in the case $|k| > l$. From (3.6) the compatibility of the equations defining $\Pi_1^1$ and $\Pi_2^1$, respectively, follows. The matrix function $S(x, t)$ is uniquely defined by the third equation in (1.3),

$$S_x(x, t) = \sum_{p=1}^{r} \sum_{j=1}^{p} A^{-j}_1 \Pi_1^1(x, t) q_p(x, t) \Pi_2^1(x, t)^* A_2^{-1}$$

$$- \sum_{p=-r}^{-1} \sum_{j=p+1}^{0} A^{-j}_1 \Pi_1^1(x, t) q_p(x, t) \Pi_2^1(x, t)^* A_2^{-1} \quad (3.7)$$

for $t = 0$, and by the third equation in (3.5),

$$S_t(x, t) = \sum_{p=1}^{l} \sum_{j=1}^{p} A^{-j}_1 \Pi_1^1(x, t) Q_p(x, t) \Pi_2^1(x, t)^* A_2^{-1}$$

$$- \sum_{p=-l}^{-1} \sum_{j=p+1}^{0} A^{-j}_1 \Pi_1^1(x, t) Q_p(x, t) \Pi_2^1(x, t)^* A_2^{-1} \quad (3.8)$$

Moreover in view of (3.4) we have for $S$ defined in this way the operator identity

$$A_1 S(x, t) - S(x, t) A_2 = \Pi_1(x, t) \Pi_2(x, t)^*, \quad (x, t) \in \Gamma_1 \times \Gamma_2. \quad (3.9)$$

On the other hand the solution $\hat{S}$ of the operator identity (3.9) may be obtained from the equations

$$\hat{S}(0, t) = S(0, t),$$

$$\hat{S}_x(x, t) = \sum_{p=1}^{r} \sum_{j=1}^{p} A^{-j}_1 \Pi_1(x, t) q_p(x, t) \Pi_2(x, t)^* A_2^{-1}$$

$$- \sum_{p=-r}^{-1} \sum_{j=p+1}^{0} A^{-j}_1 \Pi_1(x, t) q_p(x, t) \Pi_2(x, t)^* A_2^{-1}$$

$$\hat{S}_t(x, t) = \sum_{p=1}^{l} \sum_{j=1}^{p} A^{-j}_1 \Pi_1(x, t) Q_p(x, t) \Pi_2(x, t)^* A_2^{-1}$$

$$- \sum_{p=-l}^{-1} \sum_{j=p+1}^{0} A^{-j}_1 \Pi_1(x, t) Q_p(x, t) \Pi_2(x, t)^* A_2^{-1} \quad (3.10)$$
for each fixed \( t \). If

\[
\sigma(A_1) \cap \sigma(A_2) = \emptyset
\]  

(3.10)

identity (3.9) had a unique solution and \( S(x, t) = \tilde{S}(x, t) \); i.e., \( S(x, t) \), given by (3.7) for \( t = 0 \) and by (3.8), satisfies (3.7) for each \( t \). Hence Eqs. (3.7) and (3.8) in the case (3.10) are compatible. The compatibility in the general case is obtained by the approximation of the parameter matrices by the matrices \( A_{1,t}, A_{2,t}, \Pi_{1,t}(0,0), \) and \( \Pi_{2,t}(0,0) \), where \( \sigma(A_{1,t}) \cap \sigma(A_{2,t}) = \emptyset \), similarly to the proof of Theorem 1.3. Then we have

\[
\Pi_{k,t}(x, t) \rightarrow \Pi_k(x, t) \quad (k = 1, 2),
\]

(3.11)

\[
S_l(x, t) \rightarrow S(x, t) \quad (l \rightarrow \infty)
\]

(3.12)

uniformly on compact subsets of the interior of \( \Gamma_1 \times \Gamma_2 \), where \( S(x, t) \) is defined by (3.7) \( (t = 0) \) and by (3.8). As \( S_l \) satisfies (3.7) for each \( t \) we obtain for \( S_l \) the equality

\[
S_l(x, t) = S_l(0, t) + \int_0^t \sum_{p=1}^r \sum_{j=1}^s A_{1,t}^{p-j} \Pi_{1,t}(y, t) q_p(y, t) \Pi_{2,t}(y, t)^* A_{2,t}^{j-1} dy
\]

\[
- \sum_{p=-r}^{-1} \sum_{j=p+1}^s A_{1,t}^{p-j} \Pi_{1,t}(y, t) q_p(y, t) \Pi_{2,t}(y, t)^* A_{2,t}^{j-1} dy.
\]

(3.13)

From (3.11) and (3.12) it follows that

\[
S(x, t) = \lim_{l \rightarrow \infty} S_l(x, t)
\]

\[
= S(0, t) + \int_0^x \sum_{p=1}^r \sum_{j=1}^s A_{1}^{p-j} \Pi_{1}(y, t) q_p(y, t) \Pi_{2}(y, t)^* A_{2}^{j-1} dy
\]

\[
- \sum_{p=-r}^{-1} \sum_{j=p+1}^s A_{1}^{p-j} \Pi_{1}(y, t) q_p(y, t) \Pi_{2}(y, t)^* A_{2}^{j-1} dy.
\]

(3.14)

Therefore (3.7) is valid for \( S \) again. \( \blacksquare \)

The immediate corollary of Theorem 1.1 and Proposition 3.1 is

**Theorem 3.2.** Suppose matrix functions \( G \) and \( F \) of the form (3.2) satisfy Eq. (3.1). Let also the parameter matrices satisfy (3.4) and let \( \det S(0, 0) \neq 0 \). Then systems (3.3) are compatible and systems (1.3) and (3.5) are compatible. Moreover in the points of invertibility of \( S(x, t) \) the matrix function

\[
\tilde{w}(x, t, \lambda) = w_A(x, t, \lambda) w(x, t, \lambda),
\]

(3.13)

with \( w \) satisfying (3.3) and

\[
w_A(x, t, \lambda) = I_m - \Pi_2(x, t)^* S(x, t)^{-1} (A_1 - \lambda I_n)^{-1} \Pi_1(x, t),
\]

(3.14)
satisfies the differential equations in partial derivatives
\[
\begin{align*}
\tilde{w}_r(x, t, \lambda) &= \tilde{G}(x, t, \lambda)\tilde{w}(x, t, \lambda), \\
\tilde{w}_l(x, t, \lambda) &= \tilde{F}(x, t, \lambda)\tilde{w}(x, t, \lambda),
\end{align*}
\] (3.15)
where
\[
\begin{align*}
\tilde{G}(x, t, \lambda) &= - \sum_{k=-r}^{r} \lambda^k \tilde{q}_k(x, t), \\
\tilde{F}(x, t, \lambda) &= - \sum_{k=-l}^{l} \lambda^k \tilde{Q}_k(x, t),
\end{align*}
\] (3.16)
\[
\begin{align*}
\tilde{q}_k(x, t) &= q_k(x, t) \\
&\quad - \sum_{p=k+1}^{r} \left( q_p(x, t)Y_{p-k-1}(x, t) - X_{p-k-1}(x, t)q_p(x, t) \right) \\
&\quad + \sum_{j=k+2}^{p} X_{p-j}(x, t)q_p(x, t)Y_{j-k-2}(x, t) \quad \text{for } k \geq 0,
\end{align*}
\] (3.17)
\[
\begin{align*}
\tilde{Q}_k(x, t) &= Q_k(x, t) \\
&\quad - \sum_{p=k+1}^{l} \left( Q_p(x, t)Y_{p-k-1}(x, t) - X_{p-k-1}(x, t)Q_p(x, t) \right) \\
&\quad + \sum_{j=k+2}^{p} X_{p-j}(x, t)Q_p(x, t)Y_{j-k-2}(x, t) \quad \text{for } k \geq 0,
\end{align*}
\] (3.18)
\[
\begin{align*}
\tilde{Q}_k(x, t) &= \tilde{Q}_k(x, t) \\
&\quad + \sum_{p=-l}^{k} \left( Q_p(x, t)Y_{p-k-1}(x, t) - X_{p-k-1}(x, t)Q_p(x, t) \right) \\
&\quad - \sum_{j=p+1}^{k+1} X_{p-j}(x, t)Q_p(x, t)Y_{j-k-2}(x, t) \quad \text{for } k < 0,
\end{align*}
\] (3.19)
\[
\begin{align*}
X_k(x, t) &= \Pi_2(x, t)^*S(x, t)^{-1}A_k^\lambda \Pi_1(x, t), \\
Y_k(x, t) &= \Pi_2(x, t)^*A_k^\lambda S(x, t)^{-1}\Pi_1(x, t).
\end{align*}
\]
The matrix functions $\tilde{G}$ and $\tilde{F}$ defined in this way satisfy the equation
\[ \tilde{G}_i(x,t,\lambda) - \tilde{F}_i(x,t,\lambda) + [\tilde{G}(x,t,\lambda), \tilde{F}(x,t,\lambda)] = 0. \] (3.18)

**Proof.** The compatibility of systems (1.3) and (3.5) follows from Proposition 3.1. Relations (3.15)–(3.17) follow from Theorem 1.1 applied to both variables $x$ and $t$. Finally differentiate the first system in (3.15) with respect to $t$ and the second with respect to $x$:
\[ \begin{align*}
\tilde{w}_{xt}(x,t,\lambda) &= (\tilde{G}_i(x,t,\lambda) + \tilde{G}(x,t,\lambda)\tilde{F}(x,t,\lambda))\tilde{w}(x,t,\lambda), \\
\tilde{w}_{tx}(x,t,\lambda) &= (\tilde{F}_i(x,t,\lambda) + \tilde{F}(x,t,\lambda)\tilde{G}(x,t,\lambda))\tilde{w}(x,t,\lambda).
\end{align*} \] (3.19)

Now let $w$ be the fundamental solution of (3.3) to derive (3.18) from the equality $\tilde{w}_{xt} = \tilde{w}_{tx}$, and from the formula (3.19). \blacksquare

Theorem 3.2 allows us to construct new solutions of the integrable nonlinear equations from the initial ones.

**Example 1.** The nonlinear Schrödinger equation (NSE)
\[ 2u_x(x,t) = i(u_{xx}(x,t) + 2(-1)^p u(x,t)) \quad (p = 0, 1) \] (3.20)

with an $h_2 \times h_1$ matrix function $u$ is equivalent \cite{47} to the compatibility conditions (3.1) for systems (3.3) with polynomial in $\lambda$ matrix functions $G$ and $F$, where $r = 1$, $l = 2$,
\[ q_1 = -Q_2 = ij, \quad q_0(x,t) = -Q_1(x,t) = j^{p+1}V(x,t), \]
\[ V_0(x,t) = \frac{i}{2}((-1)^pjV(x,t)^2 - j^pV_0(x,t)), \]
\[ j = \begin{bmatrix} I_{h_1} & 0 \\ 0 & -I_{h_2} \end{bmatrix}, \quad V(x,t) = \begin{bmatrix} 0 & u(x,t)^* \\ u(x,t) & 0 \end{bmatrix}. \] (3.21)

The parameter matrices should satisfy the additional conditions
\[ A_1 = A_2^*, \quad \Pi_1(0,0) = i\Pi_2(0,0)j^p, \quad S(0,0) = S(0,0)^*. \] (3.22)

According to (3.21) the coefficients $q_k$ and $Q_k$ satisfy relations (1.51) with $J = ij^p$. From (1.51) and (3.22) by Proposition 1.4 we obtain
\[ \Pi_1(x,t) = i\Pi_2(x,t)j^p, \quad S(x,t) = S(x,t)^*. \] (3.23)

We shall omit here indices “1” in the matrices $A_1$ and $\Pi_1$. The operator identity (3.9) now takes the form
\[ i\mathbb{S}(x,t) - S(x,t)A^* = i\Pi(x,t)j^p\Pi(x,t)^*, \quad (x,t) \in \Gamma_1 \times \Gamma_2. \] (3.24)
By (3.17) we have \( \tilde{q}_1 = -\tilde{Q}_2 = ij \). Analogously to the example after Proposition 1.4 we see

\[
\tilde{q}_0(x, t) = -\tilde{Q}_1(x, t) = j^{p+1}\tilde{V}(x, t),
\]

\[
\tilde{V}(x, t) = V(x, t) + ij^{p+1}[X_0(x, t), j]
= \begin{bmatrix}
0 & \tilde{u}(x, t)^* \\
\tilde{u}(x, t) & 0
\end{bmatrix},
\]

(3.25)

where

\[
\tilde{u}(x, t) = u(x, t) + 2(\Pi(x, t)^*S(x, t)^{-1}\Pi(x, t))_{21}.
\]

(3.26)

Let us now show that

\[
\tilde{Q}_0(x, t) = i^2((-1)^p j\tilde{V}(x, t) - j^p\tilde{V}_x(x, t)).
\]

(3.27)

Sometimes we shall omit mentioning dependence on \( x \) and \( t \) in the calculations of this section. By (3.17) (and in view of the property \( jV = -Vj, \ j\tilde{V} = -\tilde{V}j \)) we have

\[
\tilde{Q}_0 = Q_0 + [X_0, Q_1] + X_1Q_2 - Q_2Y_1 - X_0Q_2Y_0
= i^2((-1)^p jV^2 - j^pV_x) - X_0j^{p+1}V + j^{p+1}VX_0
- i(X_1j - jY_1) + iX_0jX_0.
\]

(3.28)

As by (1.25) the equality \( Y_1 = X_1 - X_0^2 \) holds, we rewrite (3.28) in the form

\[
\tilde{Q}_0 = i^2((-1)^p jV^2 - j^pV_x) - X_0j^{p+1}V + j^{p+1}VX_0
- i[X_1, j] + i[X_0, j]X_0.
\]

(3.29)

Notice now from (1.3) and (1.17) it follows that

\[
\frac{\partial}{\partial x}X_0 = -q_1X_1 - \tilde{q}_0X_0 + X_1q_1 + X_0q_0.
\]

(3.30)

In view of (3.25) and (3.30) we get

\[
-\frac{i}{2}j^p\tilde{V}_x = -\frac{i}{2}j^pV_x + \frac{1}{2}
\times j(i[[X_1, j], j] + [X_0j^{p+1}V, j] - [j^{p+1}\tilde{V}X_0, j])
= -\frac{i}{2}j^pV_x - i[X_1, j] - \frac{1}{2}(X_0j + jX_0)j^pV
\]

\[
\frac{1}{2}j^p\tilde{V}(X_0j + jX_0).
\]

(3.31)
Relations (3.29) and (3.31) after the change of the order of the summation yield
\[
\tilde{Q}_0 = -\frac{i}{2} j^p \tilde{V}_x + i \frac{1}{2} (-1)^p j V^2 + \left( \frac{1}{2} (X_0 j + j X_0) j^p V - X_0 j^{p+1} V \right) + (j^{p+1} V X_0 + i [X_0, j] X_0) + \frac{1}{2} j^p \tilde{V}(X_0 j + j X_0);
\]
i.e.,
\[
\tilde{Q}_0 = -\frac{i}{2} j^p \tilde{V}_x + i \frac{1}{2} (-1)^p j V^2 + \frac{1}{2} (-1)^p j (\tilde{V} - V) V - j^p \tilde{V}_x X_0 + \frac{1}{2} j^p \tilde{V}(X_0 j + j X_0)
\]
\[
= -\frac{i}{2} j^p \tilde{V}_x + i \frac{1}{2} (-1)^p j V^2 + \frac{1}{2} (-1)^p j (\tilde{V} - V) V + \frac{i}{2} (-1)^p j \tilde{V}(\tilde{V} - V),
\]
and (3.27) is true. From (3.25) and (3.27) we see that the transformed coefficients have the same structure as the initial ones and are obtained by substitution of \( \tilde{u} \) instead of \( u \) in the right-hand sides of formulas (3.21).

Therefore (3.18) is equivalent to the NSE as well as (3.1). From Theorem 3.2 it follows now:

**Corollary 3.1.** Let an \( h_2 \times h_1 \) matrix function \( u \) satisfy NSE (3.20) and be continuous together with its derivatives \( u_x, u_j, \) and \( u_{xt} \). Let also the parameter matrices satisfy relations (3.4), (3.22), and \( \det S(0, 0) \neq 0 \). Then the transformed matrix function \( \tilde{u} \) given by (3.26) satisfies the NSE also.

Put now \( h_1 = h_2 = 1, n = 1, u \equiv 0, A \neq \tilde{A}, \) and \( \Pi(0, 0) = [c_1 \ c_2] \) to construct the scalar soliton solution for \( p = 0 \) and its analog with singularity for \( p = 1 \). According to (1.36) and (3.5) the entries of \( \Pi = [\Phi_1 \ \Phi_2] \) are given by \( \Phi_1(x, t) = c_1 e^{i(\xi x - A^1 t)}, \Phi_2(x, t) = c_2 e^{-i(\xi x - A^2 t)}. \) By (3.24) we get
\[
S(x, t) = i(A - \tilde{A})^{-1} \Pi(x, t) j^p \Pi(x, t)^*.
\]
Then (3.26) takes the form
\[
\tilde{u} = -2i(A - \tilde{A}) \Phi_2^* \Phi_1 (|\Phi_1|^2 + (-1)^p |\Phi_2|^2)^{-1}.
\]
Suppose for simplicity \( c_2 \neq 0 \) and put \( \theta := c_1/c_2, \ A := \xi + i \eta \ (\xi = \tilde{\xi}, \eta = \tilde{\eta}). \) Then
\[
\exp\{ -i(Ax - A^2 t) \} = \exp\{ -i(\xi x + (\eta^2 - \xi^2) t) \} \exp\{ \eta(x - 2 \xi t) \}.
\]
So the expression for $\tilde{u}$ can be rewritten as 

$$\tilde{u}(x, t) = 4\eta^2 e^{2i(x + (\eta^2 - \xi^2)t)} e^{2\eta(x - 2\xi t)|\theta|^2} + (-1)^p e^{4\eta(x - 2\xi t)} \sqrt{SHT}. $$

The maximum (hump) of the $|\tilde{u}|$ in the case $p = 0$ and the singularity of $\tilde{u}$ in the case $p = 1$ are achieved when $x - 2\xi t = (4\eta)^{-1} \ln |\theta|$. In the same way as Corollary 3.1 the following corollaries and propositions in Examples 2–5 are proved.

**Example 2.** The modified nonlinear Schrödinger equation (MNSE)

$$u_t(x, t) + iu_{xx}(x, t) + 2(-1)^p (u(x, t)u(x, t)u(x, t))^2 = 0$$

(3.32)

with an $h_2 \times h_1$ matrix function $u$ ($m = h_1 + h_2$) is equivalent [27] to the compatibility condition (3.1), where $r = 2$, $l = 4$,

$$q_k \equiv 0 \quad (k \geq 0), \quad Q_k \equiv 0 \quad (k \geq 0),$$

(3.33)

$$q_{-2}(x, t) = Q_{-4}(x, t)/4 = 2ij,$$

$$q_{-1}(x, t) = Q_{-3}(x, t)/2 = -2V(x, t)j^{p+1},$$

$$Q_{-2}(x, t) = (-1)^{p+1}4iV(x, t)^2j,$$

$$Q_{-1}(x, t) = 2(iV(x, t)j^{p} - 2j^{p+1}V(x, t)^3).$$

(3.34)

According to (3.17) and (3.33) we have

$$\tilde{q}_k \equiv 0 \quad (k \geq 0), \quad \tilde{Q}_k \equiv 0 \quad (k \geq 0),$$

(3.35)

Put $U(x, t) = I_m - X_{-1}(x, t)$. By (1.31) it is true that $(I_m + Y_{-1}) (I_m - X_{-1}) = I_m$; i.e.,

$$U^{-1} = (I_m - X_{-1})^{-1} = I_m + Y_{-1}. $$

(3.36)

We shall suppose that equalities (3.4) and (3.22) are fulfilled. Suppose also that matrix $A = A_1$ ($\det A \neq 0$) is similar to $-A$ and there exists a matrix $T$ such that

$$TAT^{-1} = -A, \quad T\Pi(0, 0)j = \Pi(0, 0),$$

$$TS(0, 0)^* = -S(0, 0).$$

(3.37)

Then it may be shown that $U$ is unitary and block diagonal:

$$U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}, \quad U_1^* U_1 = I_{h_1}, \quad U_2^* U_2 = I_{h_2}. $$

(3.38)

Moreover the following Corollary 3.2 is valid.
Corollary 3.2. Let an \( h_2 \times h_1 \) matrix function \( u \) satisfy MNSE (3.32) and be continuous together with its derivatives \( u_x, u_y, \) and \( u_{xy} \). Let also the parameter matrices satisfy relations (3.4), (3.22), (3.37), and (3.43) relation (3.23) with formulas (3.42). Similar to the considerations of Example 1, according to (3.23), and (3.43) we see
\[
\tilde{u}(x, t) = U_2(x, t)u(x, t)U_1(x, t)^* + 2i(X_{-2}(x, t))_{21}U_1(x, t)^*
= U_2(x, t)u(x, t)U_1(x, t)^* + 2iU_2(x, t)(Y_{-2}(x, t))_{21}.
\] (3.39)
satisfies the MNSE also.

Example 3. The system
\[
u(x, t)^*B(v_0(x, t) + v_t(x, t) + 2iv(x, t)Du(x, t))
+ (v_0(x, t)^* + v_t(x, t)^* - 2iu(x, t)^*Du(x, t)^*)Bv(x, t) = 0,
\]
with self-adjoint \( m \times m \) matrices \( D = D^* = \text{diag}(d_1, \ldots), B = B^* \), and \( m \times m \) matrix functions \( u \) and \( v \) such that
\[
u(x, t)^*D = Du(x, t),
\] (3.41)
is a completely integrable model of classical field theory with nontrivial particle interaction [11]. This system is equivalent [11] to the compatibility condition (3.1) for systems (3.3), where \( l = r = 1 \),
\[
q_1 = Q_1 = -iD, \quad q_0(x, t) = Q_0(x, t) = -iDu(x, t),
q_{-1}(x, t) = -Q_{-1}(x, t) = iv(x, t)^*Bv(x, t).
\] (3.42)
Let the parameter matrices satisfy (3.22) with \( p = 0 \), i.e.,
\[
A_1 = A_2^*, \quad \Pi_1(0, 0) = i\Pi_2(0, 0), \quad S(0, 0) = S(0, 0)^*,
\] (3.43)
and put
\[
\tilde{u}(x, t) = u(x, t) - D^{-1}[D, X_0(x, t)],
\]
\[
\tilde{v}(x, t) = v(x, t)(I_m + Y_{-1}(x, t)).
\] (3.44)
In view of (3.17) and (3.44) we obtain coefficients \( \tilde{q}_k \) and \( \tilde{Q}_k \) \((-1 \leq k \leq 1)\) after the substitution of \( \tilde{u} \) and \( \tilde{v} \) instead of \( u \) and \( v \) into the right-hand sides of formulas (3.42). Similar to the considerations of Example 1, according to (3.43) relation (3.23) with \( p = 0 \) may be drawn. Taking into account (1.10), (3.23), and (3.43) we see \( X_0^* = -X_0 \). In particular, \( X_0^* = -X_0 \) and by (3.44) \( \tilde{u} \) has property (3.41): \( \tilde{u}^*D = D\tilde{u} \).
COROLLARY 3.3. Let \( m \times m \) matrix functions \( u \) and \( v \) satisfy (3.40) and (3.41) and be continuous together with the derivatives \( u_x, u_t, v_x, \) and \( v_t \). Let also the parameter matrices satisfy relations (3.4) and (3.43) and \( \det S(0,0) \neq 0 \). Then the transformed matrix functions \( \tilde{u} \) and \( \tilde{v} \) given by (3.44) satisfy the same equations (3.40) and (3.41) also.

Moreover one can easily check

PROPOSITION 3.3. Suppose the conditions of Corollary 3.3 are fulfilled and \( u \) and \( v \) satisfy the equation

\[
v_t(x,t) + v_x(x,t) + 2i u(x,t)Du(x,t) = 0,
\]

from which the second of equations (3.40) follows. Then \( \tilde{u} \) and \( \tilde{v} \) satisfy (3.45) also.

Remark. In the case where the initial solution \( v \) is unitary, formula (3.45) can be rewritten as

\[
u(x,t) = \frac{i}{2} D^{-1} v(x,t)^* (v_x(x,t) + v_t(x,t))
\]

and the first of equations (3.40) takes the Budagov–Takhtadzhyan form

\[
(v^* v_t)_x - [v^* v_x, v^* v_t] = 4 [v^* Bv, D],
\]

Taking into account \( Y_k^* = -X_k \) and (3.36) we see that the matrix function \( I_m + Y_{-1} \) is unitary and therefore if \( v \) is unitary, then \( \tilde{v} \) is unitary and satisfies (3.47) also.

It is important that matrices \( Y_{-1} \) and \( \tilde{v} \) in [11] have the real-valued entries.

Suppose additionally to the conditions of Proposition 3.3 that

\[
v = \tilde{v}, \quad B = \overline{B},
\]

and there exists \( T \) such that

\[
TAT^{-1} = -\overline{A}, \quad T\Pi(0,0) = \overline{\Pi}(0,0), \quad S(0,0) = TS(0,0)T^*.
\]

(Here \( \overline{A} \) is the matrix with the entries complexly conjugated to the entries of \( A \).) From (3.46) and (3.48) we draw \( u = -\tilde{u} \) and in view of (1.3), (3.5), and (3.49) the formulas \( T\Pi = \overline{\Pi}, \overline{S} = TST^* \) are true. Hence we obtain

\[
\overline{Y}_k = (-1)^{k+1} Y_k, \quad \tilde{v} = \tilde{v}.
\]

EXAMPLE 4. The famous Korteweg–de Vries (KdV) equation

\[
u_t(x,t) - 3u(x,t)u_x(x,t) - 3u_x(x,t)u(x,t) + u_{xxx}(x,t) = 0
\]
is equivalent to the compatibility condition (3.1) with matrix functions $G$ and $F$ equal to polynomials in $\lambda$, where $r = 1$, the coefficients $q_0$ and $q_1$ are given by (2.30), and

$$Q_2 = 4q_1 = \begin{bmatrix} 0 & 0 \\ 4I_h & 0 \end{bmatrix}, \quad Q_1(x, t) = -\begin{bmatrix} 0 & 4I_h \\ 2u(x, t) & 0 \end{bmatrix},$$

$$Q_0(x, t) = \begin{bmatrix} u_x(x, t) & -2u(x, t) \\ u_{xx}(x, t) - 2u(x, t)^2 & -u_x(x, t) \end{bmatrix},$$

(3.52)

$u$ is an $h \times h$ matrix function. By (3.17) we have

$$\tilde{Q}_2 = Q_2, \quad \tilde{Q}_1 = Q_1 + 4 \begin{bmatrix} X_{012} & 0 \\ X_{022} - X_{011} & -X_{012} \end{bmatrix},$$

$$\tilde{Q}_0 = Q_0 + X_1Q_2 - Q_2X_1 + [X_0, Q_1] - X_0Q_2X_0.$$

(3.53)

Consider the GBDT $\tilde{w} = wAw$ of $w$. From the proof of Proposition 2.3 it follows that

$$[0 \ I_h]\tilde{w} = \tilde{y}_x - X_{012}\tilde{y} \quad (\tilde{y} = [I_h \ 0]\tilde{w}).$$

(3.54)

Relations (3.53) and (3.54) yield

$$\tilde{y}_t = 4\lambda \tilde{y}_x + 2[u + 2(X_{011} - X_{022} + X_{012})] \tilde{y}_x - u_x \tilde{y} + 4\lambda X_{012} \tilde{y}$$

$$- 4AX_{012} \tilde{y} + 2(uX_{012} - 2X_{112} + X_{012}u + 2X_{011}X_{012}$$

$$- 2X_{021} - 2X_{022}X_{012} + 2X_{012}X_{011} + 2X_{012}^3) \tilde{y}.$$  (3.55)

In view of (2.34), (2.37), and (2.38) after some calculations we can rewrite (3.55) as

$$\tilde{y}_t = 4\lambda \tilde{y}_x + 2u\tilde{y}_x - \tilde{u}_x \tilde{y}.$$  (3.56)

KdV equation (3.51) is equivalent to the compatibility condition of (2.33) and (3.56) (see [45] and references therein).

**Corollary 3.4.** Let an $h \times h$ matrix function $u$ satisfy KdV equation (3.51) and be continuous together with its derivatives $u_x, u_{xx}, u_{xt}, u_{xxx}$, and $u_{xxt}$. Let also the parameter matrices satisfy operator identity (3.4) and let $\det S(0, 0) \neq 0$. Then the transformed matrix function

$$\tilde{u}(x, t) = u(x, t) + 2(X_{011}(x, t) - X_{022}(x, t) + X_{012}(x, t))^2$$

(3.57)

satisfies the KdV equation also.

The particular case of (3.57), when $u = 0$ and explicit solutions may be obtained, was treated in [24].
Example 5. A hierarchy of nonlinear equations is generated [20] by the $2 \times 2$ matrix function $G$ of the form

$$G(x, t, \lambda) = -(q_1 + q_0(x, t)), \quad q_1 = ij,$$

$$j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad q_0(x, t) = \begin{bmatrix} 0 & u_1(x, t) \\ u_2(x, t) & 0 \end{bmatrix}$$

(3.58)

and by the $2 \times 2$ matrix functions $F_i$ of the form

$$F_i(x, t, \lambda) = -\sum_{k=0}^l \lambda^k Q_{ki}(x, t), \quad \text{Tr} Q_{ki}(x, t) \equiv 0,$$

(3.59)

where Tr means matrix trace.

Proposition 3.4. For the hierarchy of equations generated by (3.58) and (3.59) the GBDT preserves the structure of coefficients $q_k$ and $Q_{ki}$; i.e.,

$$\text{Tr} \bar{q}_k(x, t) \equiv 0$$

and

$$\text{Tr} \bar{Q}_{ki}(x, t) \equiv 0.$$

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