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A new result on impulsive differential equations involving non-absolutely convergent integrals

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ABSTRACT

In this paper we obtain, as an application of a Darbo-type theorem, global solutions for differential equations with impulse effects, under the assumption that the function on the right-hand side is integrable in the Henstock sense. We thus generalize several previously given results in literature, for ordinary or impulsive equations.

1. Introduction

Quite recently, in the study of differential and integral problems, many authors have focused their interest in obtaining existence results or properties of solutions under hypothesis of integrability in a weaker sense than the classical Bochner (respectively Lebesgue in the one-dimensional case) and Pettis integrals. Such an approach is appropriate when the equations are governed by highly oscillating functions. In this direction, we recall the results obtained on the real line in [4–6, 21] using the Henstock–Kurzweil integral and in the general case of Banach spaces in [22–24] under Henstock–Lebesgue integrability assumptions or in [20] in Henstock setting.

On the other hand, the study of the dynamics of processes subjected to instantaneously perturbations (such as those appearing in physics, biology and many other fields) involves impulsive differential problems. These problems were extensively studied in Bochner integrability case (see [2] and references therein) and, recently, using the Henstock–Lebesgue integral (in [6]).

The goal of the present paper is to obtain, via Henstock-type integrals, the existence of global solutions to the differential problem with impulse effects

$$\dot{x}(t) = f\left(t, x(t), \int_0^t h(t,s)x(s)\,ds\right), \quad \forall t \in [0,1] \setminus \{t_1, \ldots, t_m\}, \tag{1}$$

$$\Delta x(t_i) = I_i\big(x(t_i)\big), \quad \forall i \in \{1, \ldots, m\}, \tag{2}$$

$$x(0) = x^0. \tag{3}$$

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Here $0 < t_1 < \cdots < t_m \leqslant 1$ are the pre-assigned moments of impulse, $\Delta x(t) = x(t+) - x(t-)$ denotes the jump of the function x at t and the discontinuity at the point t_i is described by the function $I_i : X \to X$.

To achieve this, we apply a Darbo-type fixed point theorem established in [15], under some much weaker assumptions than those previously imposed for similar results (see [15] and the papers cited there).

2. Notations and preliminary facts

Let $[0,1]$ be the unit interval of the real line equipped with the usual topology and the Lebesgue measure μ. Through this paper X is a separable Banach space with norm $\|\cdot\|$ and corresponding distance d and for some fixed point $x^0 \in X$ and a fixed $R > 0$, the symbol $T_R(x^0)$ denotes the closed X-ball of radius R and centered at x^0. By $C([0,1],X)$ we denote the space of continuous functions endowed with the usual (Banach space) norm $\|f\|_C = \sup_{t\in[0,1]}\|f(t)\|$ and by $L^\infty([0,1],\mathbb{R})$ the space of essentially bounded real functions with the essential supremum norm $\|\cdot\|_{L^\infty}$.

Let us now introduce some basic facts on Henstock-type integrals in Banach spaces, which are extensions of the notion of real valued Henstock–Kurzweil integral (for which the reader is referred to [9]).

A *tagged partition of* $[0,1]$, or simply a *partition of* $[0,1]$ is a finite collection of pairs $\{(I_i,t_i)\colon\ i = 1,\dots,p\}$, where $I_1,\dots,I_p$ are non-overlapping subintervals of $[0,1]$, $t_i \in I_i$, $i = 1,\dots,p$, and $\bigcup_{i=1}^p I_i = [0,1]$. A *gauge* δ on $[0,1]$ is a positive function on $[0,1]$. For a given gauge δ we say that a partition $\{(I_i,t_i)\colon\ i = 1,\dots,p\}$ is δ*-fine* if $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, $i = 1,\dots,p$. For any function $\Phi : [0,1] \to X$ and for any subinterval $I = [a,b]$ of $[0,1]$, we set $\Phi(I) = \Phi(b) - \Phi(a)$.

Definition 1. 1) A function $f : [0,1] \to X$ is said to be *Henstock-integrable on* $[0,1]$, if there exists a vector $(\mathrm{H})\int_0^1 f(s)\,ds \in X$ such that, for every $\varepsilon > 0$, there is a gauge δ_ε on $[0,1]$ satisfying

$$\left\| \sum_{i=1}^p f(t_i)\mu(I_i) - (\mathrm{H})\int_0^1 f(s)\,ds \right\| < \varepsilon$$

for every δ_ε-fine partition $\{(I_i,t_i)\colon\ i = 1,\dots,p\}$ of $[0,1]$.

If f is Henstock-integrable, then it has the same feature on any sub-interval of $[0,1]$ (but in general not on any measurable subset of $[0,1]$). The function $\Phi(t) = (\mathrm{H})\int_0^t f(s)\,ds$ is called the Henstock-primitive of f on $[0,1]$.

2) A function $f : [0,1] \to X$ is said to be *Henstock–Lebesgue-integrable* (see [3]) (shortly *HL-integrable*) on $[0,1]$, if there exists a function $\Phi : [0,1] \to X$ such that, for every $\varepsilon > 0$, there is a gauge δ_ε on $[0,1]$ satisfying

$$\sum_{i=1}^p \left\| f(t_i)\mu(I_i) - \Phi(I_i) \right\| < \varepsilon$$

for every δ_ε-fine partition $\{(I_i,t_i)\colon\ i = 1,\dots,p\}$ of $[0,1]$.

Note that the HL-integral is also called in the literature *variationally Henstock-integral* (see [17]), or *strongly Henstock–Kurzweil-integral* (see [21]).

Also in this case, if f is HL-integrable, then it is HL-integrable on any sub-interval of $[0,1]$, but in general not on all measurable subsets of $[0,1]$. We set $\Phi(t) = (\mathrm{HL})\int_0^t f(s)\,ds$ and call it the *HL-primitive* of f on $[0,1]$.

Remark 2. One of the main differences between the notions of Henstock-integral and HL-integral is the fact that the primitive in HL-sense is continuous and differentiable a.e., while the Henstock primitive is continuous, but in general is not differentiable a.e. (see [3]).

As about the relationship between these integrals and the classical ones, it is well known that:

(j) any Bochner integrable function is HL-integrable and the converse is not valid;
(jj) the HL-integrability implies the Henstock integrability;
(jjj) any Pettis integrable function, taking values in a separable Banach space, is Henstock integrable (see [8]), but the implication in the other sense is not true even in the real case;
(jv) there exist Henstock–Lebesgue-integrable functions that are not Pettis integrable (see the real case) and vice-versa (as Example 42 in [7] shows).

In finite dimensional spaces, the two notions (of Henstock-integral and HL-integral) are equivalent. In particular, in the real case, the previous (equivalent) definitions give the Henstock–Kurzweil (shortly HK-) integral.

The space of all Henstock-integrable X-valued functions is denoted by $\mathcal{H}([0,1],X)$ and is endowed with the Alexiewicz norm:

$$\|f\|_A = \sup_{t\in[0,1]} \left\| (\mathrm{H})\int_0^t f(s)\,ds \right\|.$$

We are concerned with the differential problem with impulse effects (1)–(3). Consider (in order to simplify the calculus) $t_0 = 0$ and $t_{m+1} = 1$.

In the sequel by the symbol $C^{\diamond}_{x^0}([0,1], X)$ we denote the collection of all functions $x : [0,1] \to X$ satisfying the following properties:

(k) *x is continuous at every $t \in [0,1] \setminus \{t_1, \ldots, t_m\}$;*
(kk) *x is left continuous at every $t \in \{t_1, \ldots, t_m\}$;*
(kkk) *at every $t \in \{t_1, \ldots, t_m\}$ there exists the right limit $x(t+)$;*
(kv) *$x(0) = x^0$.*

$C^{\diamond}_{x^0}([0,1], X)$ becomes a Banach space when we endow it with the norm $\|\cdot\|_C$ (since it is a closed subspace of the space of all regulated X-valued functions on $[0,1]$ which, endowed with the specified norm, is complete, see [13]).

Moreover given a vector $x^0 \in X$ and a fixed $R > 0$, by the symbol $B_R(x^0)$ we denote the closed ball of $C^{\diamond}_{x^0}([0,1], X)$ of radius R and centered at the constant function x^0.

Definition 3. A function $x \in C^{\diamond}_{x^0}([0,1], X)$ is called a solution of the problem (1)–(3) if it satisfies condition (1) for almost every $t \in [0,1] \setminus \{t_1, \ldots, t_m\}$ and conditions (2) and (3).

For any subset E of a metric space Y we denote by $\alpha(E)$ *the Hausdorff measure of non-compactness* of E, i.e. the infimum of all $r > 0$ such that there exists a finite number of balls covering E, of radius smaller than r. For its properties the reader is referred to [12]. The measure of non-compactness α will play an essential role in establishing the main result.

Theorem 4. *(See [1].) Let $\mathcal{K} \subset C([0,1], X)$ be bounded and equi-continuous. Then $\alpha(\mathcal{K}) = \sup_{t\in[0,1]} \alpha(\mathcal{K}(t))$.*

We deduce the following

Corollary 5. *Let $\mathcal{K} \subset C^{\diamond}_{x^0}([0,1], X)$ be bounded and equi-continuous on every interval $]t_i, t_{i+1}]$ where $i \in \{0, \ldots, m\}$. Then $\alpha(\mathcal{K}) = \sup_{t\in[0,1]} \alpha(\mathcal{K}(t))$.*

Proof. Since it is not difficult to see that $\alpha(\mathcal{K}) \geqslant \sup_{t\in[0,1]} \alpha(\mathcal{K}(t))$, only the other inequality has to be proved. On every interval $]t_i, t_{i+1}]$, where $i \in \{0, \ldots, m\}$, $\mathcal{K}$ is equi-continuous, therefore on each closed interval $J \subset]t_i, t_{i+1}]$ one can apply Theorem 4 in order to obtain that $\alpha(\mathcal{K}/J) = \sup_{t\in J} \alpha(\mathcal{K}(t)) \leqslant \sup_{t\in[0,1]} \alpha(\mathcal{K}(t))$. It follows that, for each $i \in \{0, \ldots, m\}$, $\alpha(\mathcal{K}/]t_i, t_{i+1}]) \leqslant \sup_{t\in[0,1]} \alpha(\mathcal{K}(t))$ and so, the assertion follows. □

Recall that

Proposition 6. *(See [18, Proposition 1.4].) Let X be a separable Banach space and $(X_q)_q$ an increasing sequence of finite dimensional subspaces with $X = \overline{\bigcup_{q\in\mathbb{N}} X_q}$. Then for every bounded countable set $M = (a_m)_m \subset X$,*

$$\alpha(M) = \lim_{q\to\infty} \overline{\lim_{m\to\infty}} d(a_m, X_q).$$

The following result (proved in [22] under some different assumptions) generalizes a similar inequality available for Bochner integrable functions, which can be found in [12] or [19]. By the symbol (H) $\int_0^t \mathcal{M}(s)\,ds$ we mean the collection of all Henstock-integrals of elements of $\mathcal{M}$.

Theorem 7. *Let $\mathcal{M} \subset \mathcal{H}([0,1], X)$ be a $\|\cdot\|_A$-bounded and a.e. pointwisely bounded countable family. Assume that there is an increasing sequence $(X_q)_q$ of finite dimensional subspaces with $X = \overline{\bigcup_{q\in\mathbb{N}} X_q}$, a natural $q_0 \in \mathbb{N}$ and $g \in L^1([0,1], \mathbb{R})$ such that for every $q \geqslant q_0$,*

$$d\big(x(t), X_q\big) \leqslant g(t) \quad \text{a.e. } \forall x \in \mathcal{M}.$$

Then $\alpha(\mathcal{M}(\cdot)) \in L^1([0,1], \mathbb{R})$ and

$$\alpha\left((\mathrm{H})\int_0^t \mathcal{M}(s)\,ds\right) \leqslant \int_0^t \alpha\big(\mathcal{M}(s)\big)\,ds, \quad \forall t \in [0,1].$$

Proof. Let $\mathcal{M} = \{x_m,\ m \in \mathbb{N}\}$. Then for every $t \in [0,1]$,

$$(\mathrm{H})\int_0^t \mathcal{M}(s)\,ds = \left\{(\mathrm{H})\int_0^t x_m(s)\,ds,\ m \in \mathbb{N}\right\}.$$

By Proposition 6,

$$\alpha\left((\mathrm{H})\int_0^t \mathcal{M}(s)\,ds\right) = \lim_{q\to\infty}\overline{\lim_{m\to\infty}}\, d\left((\mathrm{H})\int_0^t x_m(s)\,ds, X_q\right)$$

and

$$\alpha\big(\mathcal{M}(s)\big) = \lim_{q\to\infty}\overline{\lim_{m\to\infty}}\, d\big(x_m(s), X_q\big) \quad \text{a.e. on } [0,1].$$

I. Let us first prove that, for every $x \in \mathcal{M}$ and for each linear subspace Z of X,

$$d\left((\mathrm{H})\int_0^t x(s)\,ds, Z\right) \leqslant \int_0^t d\big(x(s), Z\big)\,ds, \quad \forall t \in [0,1].$$

If $\int_0^t d(x(s), Z)\,ds = +\infty$, the inequality holds true. If $\int_0^t d(x(s), Z)\,ds < +\infty$, then the function $d(x(\cdot), Z)$ is Lebesgue integrable and therefore Henstock–Kurzweil integrable. Because the function x is Henstock integrable, for every $m \in \mathbb{N}$ there is a gauge δ_m on $[0,t]$ satisfying

$$\left\|(\mathrm{H})\int_0^t x(s)\,ds - \sum_{i=1}^p x(\xi_i)\mu(I_i)\right\| < \frac{1}{m}$$

and

$$\left|\int_0^t d\big(x(s), Z\big)\,ds - \sum_{i=1}^p d\big(x(\xi_i), Z\big)\mu(I_i)\right| < \frac{1}{m},$$

for any δ_m-fine partition $\{(I_i, \xi_i):\ i = 1, \ldots, p\}$ of $[0,t]$. From the linearity of Z one deduces that

$$d\left(\sum_{i=1}^p x(\xi_i)\mu(I_i), Z\right) \leqslant \sum_{i=1}^p d\big(x(\xi_i), Z\big)\mu(I_i).$$

Therefore,

$$d\left((\mathrm{H})\int_0^t x(s)\,ds, Z\right) \leqslant \int_0^t d\big(x(s), Z\big)\,ds.$$

II. The positive function $\alpha(\mathcal{M}(\cdot)) = \lim_{q\to\infty}\overline{\lim}_{m\to\infty} d(x_m(\cdot), X_q)$ is measurable and bounded by $g(\cdot)$, so it is Lebesgue integrable. By the first step of the proof,

$$\lim_{q\to\infty}\overline{\lim_{m\to\infty}}\, d\left((\mathrm{H})\int_0^t x_m(s)\,ds, X_q\right) \leqslant \lim_{q\to\infty}\overline{\lim_{m\to\infty}}\int_0^t d\big(x_m(s), X_q\big)\,ds.$$

On the other hand, by the reverse Fatou's lemma, for each $q \geqslant q_0$,

$$\overline{\lim_{m\to\infty}}\int_0^t d\big(x_m(s), X_q\big)\,ds \leqslant \int_0^t \overline{\lim_{m\to\infty}}\, d\big(x_m(s), X_q\big)\,ds$$

and, by the monotone convergence theorem,

$$\lim_{q\to\infty}\int_0^t \overline{\lim_{m\to\infty}}\, d\big(x_m(s), X_q\big)\,ds = \int_0^t \lim_{q\to\infty}\overline{\lim_{m\to\infty}}\, d\big(x_m(s), X_q\big)\,ds.$$

Consequently,

$$\lim_{q\to\infty}\overline{\lim_{m\to\infty}}\, d\left((\mathrm{H})\int_0^t x_m(s)\,ds, X_q\right) \leqslant \int_0^t \lim_{q\to\infty}\overline{\lim_{m\to\infty}}\, d\big(x_m(s), X_q\big)\,ds$$

or, otherwise stated,

$$\alpha\left((\mathrm{H})\int_0^t \mathcal{M}(s)\,ds\right) \leqslant \int_0^t \alpha\big(\mathcal{M}(s)\big)\,ds, \quad \forall t \in [0,1]. \qquad \square$$

Remark 8. Previous result generalizes those already known in Bochner setting (see [19, Proposition 1.4]), where it is assumed $\|x(t)\| \leqslant g(t)$, for every $x \in \mathcal{M}$ with $g \in L^1([0,1],\mathbb{R})$. Indeed, obviously, this hypothesis implies that $\mathcal{M}$ is pointwisely bounded and also $\|\cdot\|_A$-bounded, and that for every $q \in \mathbb{N}$,

$$d\big(x(t), X_q\big) \leqslant \big\|x(t)\big\| \leqslant g(t), \quad \forall x \in \mathcal{M}.$$

We will also need

Lemma 9. *Let $\gamma : X \to X$ satisfy, for some $a > 0$, the property that*

$$\big\|\gamma(x) - \gamma(y)\big\| \leqslant a\|x - y\|, \quad \forall x, y \in X.$$

Then

$$\alpha\big(\gamma(A)\big) \leqslant a\alpha(A), \quad \forall A \subset X \text{ bounded}.$$

Proof. Consider $A \subset X$ a bounded subset and $\varepsilon > 0$. Then A is contained in a finite union of balls of radius smaller than $\alpha(A) + \varepsilon$. It follows that there are $x_1, \ldots, x_p \in X$ such that, for each $x \in A$, one can find x_i with $\|x - x_i\| < \alpha(A) + \varepsilon$, whence $\|\gamma(x) - \gamma(x_i)\| < a(\alpha(A) + \varepsilon)$ and so, $\gamma(A)$ is contained in a finite union of balls of radius smaller than $a(\alpha(A) + \varepsilon)$. As ε is arbitrary, the inequality is proved. □

Our main existence result will be proved by applying the following generalization of the Darbo's fixed point theorem given in [15]:

Lemma 10. *Let F be a closed convex subset of a Banach space and the operator $A : F \to F$ be continuous with $A(F)$ bounded. For any bounded $B \subset F$ set*

$$\tilde{A}^1(B) = A(B) \quad \text{and} \quad \tilde{A}^n(B) = A\big(\overline{\text{co}}\big(\tilde{A}^{n-1}(B)\big)\big), \quad \forall n \geqslant 2.$$

If there exist a constant $0 \leqslant k < 1$ and a natural number n_0 such that $\alpha(\tilde{A}^{n_0}(B)) \leqslant k\alpha(B)$ for every bounded $B \subset F$, then A has a fixed point.

3. Main result

With the same notations in the presentation of the differential problem (1)–(3), we give the main result of the paper.

Theorem 11. *Let X be a real separable Banach space, $f : [0,1] \times X^2 \to X$, $h : [0,1]^2 \to \mathbb{R}$ and $I : X \to X$ satisfy the following conditions*:

(i) *for each $t \in [0,1]$, $h(t,\cdot) \in L^\infty([0,1],\mathbb{R})$ and $t \mapsto h(t,\cdot)$ is $\|\cdot\|_{L^\infty}$-bounded*;

(ii) *for each $R > 0$ and each $i \in \{1,\ldots,m\}$, there exists $a_{i,R} > 0$ such that, for any $x_1, x_2 \in T_R(x^0)$,*

$$\big\|I_i(x_1) - I_i(x_2)\big\| \leqslant a_{i,R}\|x_1 - x_2\|;$$

(iii) *for every pair of functions $x, y \in C^\diamond_{x^0}([0,1],X)$, $f(\cdot, x(\cdot), y(\cdot))$ is Henstock-integrable and*:

(iii)(1) *for each $R > 0$ and $\varepsilon > 0$, one can find $k_R > 0$ and $0 < \delta_{\varepsilon,R} < 1$ such that*

$$\left\|(\mathrm{H})\int_{t_1}^{t_2} f\big(s, x(s), y(s)\big)\,ds\right\| \leqslant \varepsilon, \quad \forall |t_1 - t_2| \leqslant \delta_{\varepsilon,R},\ \forall x, y \in B_R\big(x^0\big),$$

and

$$\limsup_{R\to\infty} \frac{2}{R\delta_{1,R}} < \liminf_{R\to\infty} \frac{k_R}{\sup_{t\in[0,1]}\big\|h(t,\cdot)\big\|_{L^\infty} + 1}$$

with

$$k_R + \sum_{i=1}^m a_{i,R} < 1;$$

(iii)(2) *the map $(x,y) \mapsto f(\cdot, x(\cdot), y(\cdot))$ from $C^\diamond_{x^0} \times C^\diamond_{x^0}$ to $\mathcal{H}([0,1],X)$ is $\|\cdot\|_A$-uniformly continuous*;

(iii)(3) *for every $s \in [0,1]$ and every countable bounded $A, B \subset X$, $f(s, A, B)$ is bounded*;

(iv) *there exist two positive integrable functions* $L_i : [0,1] \to \mathbb{R}$, $i = 1, 2$, *an increasing sequence* $(X_q)_q$ *of finite dimensional subspaces with* $X = \overline{\bigcup_{q\in\mathbb{N}} X_q}$ *and a natural* $q_0 \in \mathbb{N}$ *such that for every* $q \geqslant q_0$, $t \in [0,1]$, $x, y \in X$,

$$d\big(f(t,x,y), X_q\big) \leqslant L_1(t) d(x, X_q) + L_2(t) d(y, X_q).$$

Then the integral equation with impulse effects

$$x(t) = x^0 + (\mathrm{H}) \int_0^t f\left(s, x(s), \int_0^s h(s,\tau) x(\tau)\, d\tau\right) ds + \sum_{0<t_i<t} \big(x(t_i+) - x(t_i)\big) \tag{4}$$

possess solutions in $C^{\diamond}_{x^0}([0,1], X)$.

Proof. Let us begin by showing

Lemma 12. *The hypotheses* (iv) *and* (iii)(3) *of Theorem* 11 *imply that for any bounded countable* $D_1, D_2 \subset X$ *and any* $t \in [0,1]$,

$$\alpha\big(f(t, D_1, D_2)\big) \leqslant \sum_{i=1}^{2} L_i(t) \alpha(D_i).$$

Proof. Let $D_1 = \{x_n,\ n \in \mathbb{N}\}$ and $D_2 = \{y_p,\ p \in \mathbb{N}\}$. By assumption (iii)(3), one can apply Proposition 6. Therefore, taking into account also assumption (iv), we obtain

$$\begin{aligned}\alpha\big(f(t, D_1, D_2)\big) &= \lim_{q\to\infty} \overline{\lim_{n,p\to\infty}} d\big(f(t, x_n, y_p), X_q\big) \leqslant \lim_{q\to\infty} \overline{\lim_{n,p\to\infty}} \big(L_1(t) d(x_n, X_q) + L_2(t) d(y_p, X_q)\big) \\ &= L_1(t) \lim_{q\to\infty} \overline{\lim_{n\to\infty}} d(x_n, X_q) + L_2(t) \lim_{q\to\infty} \overline{\lim_{p\to\infty}} d(y_p, X_q) = \sum_{i=1}^{2} L_i(t) \alpha(D_i). \qquad \square\end{aligned}$$

Proceed now to prove the main theorem. We follow the ideas of proof of Theorem 3.1 in [15]. By the hypothesis (i), $t \mapsto h(t,\cdot)$ is $\|\cdot\|_{L^\infty}$-bounded, and so, we can set $b = \sup_{t\in[0,1]} \|h(t,\cdot)\|_{L^\infty}$.

From (iii)(1), one can find $R_0 > \|x^0\|(b+1)$ and $0 < r < \frac{k_{R_0}}{b+1}$ such that for any $R \geqslant \max\{R_0, bR_0 + \|x^0\|(b+1)\}$,

$$\frac{2}{\delta_{1,R}} < rR.$$

Consider $A : C^{\diamond}_{x^0}([0,1], X) \to C^{\diamond}_{x^0}([0,1], X)$ defined by

$$Ax(t) = x^0 + (\mathrm{H}) \int_0^t f\left(s, x(s), \int_0^s h(s,\tau) x(\tau)\, d\tau\right) ds + \sum_{0<t_i<t} \big(x(t_i+) - x(t_i)\big).$$

We claim that A is a continuous operator that maps the closed ball $B_{R_0}(x^0)$ of $C^{\diamond}_{x^0}([0,1], X)$ into itself.

Let us firstly prove that its values are in $C^{\diamond}_{x^0}([0,1], X)$.

In order to show that the property (k) is satisfied, consider $t \in [0,1] \setminus \{t_1, \dots, t_m\}$. One can find $i_0 \in \{0, 1, \dots, m\}$ such that $t \in]t_{i_0}, t_{i_0+1}[$ (let us remind that $t_0 = 0$ and $t_{m+1} = 1$). Take $t' \in]t_{i_0}, t_{i_0+1}[$. Then

$$\big\|Ax(t) - Ax(t')\big\| = \left\|(\mathrm{H}) \int_t^{t'} f\left(s, x(s), \int_0^s h(s,\tau) x(\tau)\, d\tau\right) ds\right\|$$

which, thanks to the continuity of the primitive in the Henstock-sense, becomes less than some fixed ε for t' sufficiently close to t.

To prove the property (kk), take $t = t_i$ where $i \in \{1, \dots, m\}$ and $t' \in]t_{i-1}, t_i[$. Then

$$\big\|Ax(t) - Ax(t')\big\| = \left\|(\mathrm{H}) \int_{t'}^{t} f\left(s, x(s), \int_0^s h(s,\tau) x(\tau)\, d\tau\right) ds\right\|$$

and so, it becomes less than some fixed ε for t' sufficiently close to t. Finally, to show property (kkk), fix $i \in \{1, \dots, m\}$ and take $t > t_i$,

$$Ax(t) - Ax(t_i) = (\mathrm{H})\int_{t_i}^{t} f\left(s, x(s), \int_0^s h(s,\tau)x(\tau)\,d\tau\right) ds + \big(x(t_i+) - x(t_i)\big),$$

and so, there exists $\lim_{t\to t_i^+}(Ax(t) - Ax(t_i)) = x(t_i+) - x(t_i)$.

Let us now prove that the operator A maps the ball $B_{R_0}(x^0)$ into itself. The hypothesis (iii)(1) implies that for $R = \max\{R_0, bR_0 + \|x^0\|(b+1)\}$ one can find $1 > \delta_{1,\max\{R_0,bR_0+\|x^0\|(b+1)\}} > 0$ such that

$$\left\|(\mathrm{H})\int_{t_1}^{t_2} f\big(s, x(s), y(s)\big)\,ds\right\| \leqslant 1, \quad \forall |t_1 - t_2| \leqslant \delta_{1,\max\{R_0,bR_0+\|x^0\|(b+1)\}}, \ \forall x, y \in B_{\max\{R_0,bR_0+\|x^0\|(b+1)\}}\big(x^0\big).$$

For every $t \in [0,1]$ and for all $x, y \in B_{\max\{R_0,bR_0+\|x^0\|(b+1)\}}(x^0)$, let $N \in \mathbb{N}$ be the integer part of $\frac{t}{\delta_{1,\max\{R_0,bR_0+\|x^0\|(b+1)\}}}$. Then

$$\begin{aligned}\left\|(\mathrm{H})\int_0^t f\big(s, x(s), y(s)\big)\,ds\right\| &\leqslant \left\|(\mathrm{H})\int_0^{\delta_{1,\max\{R_0,bR_0+\|x^0\|(b+1)\}}} f\big(s, x(s), y(s)\big)\,ds\right\| + \cdots \\ &\quad + \left\|(\mathrm{H})\int_{(N-1)\delta_{1,\max\{R_0,bR_0+\|x^0\|(b+1)\}}}^{N\delta_{1,\max\{R_0,bR_0+\|x^0\|(b+1)\}}} f\big(s, x(s), y(s)\big)\,ds\right\| \\ &\quad + \left\|(\mathrm{H})\int_{N\delta_{1,\max\{R_0,bR_0+\|x^0\|(b+1)\}}}^{t} f\big(s, x(s), y(s)\big)\,ds\right\| \\ &\leqslant N + 1 \leqslant \frac{1}{\delta_{1,\max\{R_0,bR_0+\|x^0\|(b+1)\}}} + 1 \leqslant \frac{2}{\delta_{1,\max\{R_0,bR_0+\|x^0\|(b+1)\}}}.\end{aligned}$$

Since it is natural to suppose that, for each $i \in \{1,\dots,m\}$, $I_i(x^0) = 0$, from (iii)(1) and (ii) we deduce that, for any $x \in C^{\diamond}_{x^0}([0,1],X)$ with $\|x - x^0\|_C \leqslant R_0$,

$$\big\|Ax - x^0\big\|_C \leqslant \sup_{t\in[0,1]}\left\|(\mathrm{H})\int_0^t f\left(s, x(s), \int_0^s h(s,\tau)x(\tau)\,d\tau\right) ds\right\| + \sum_{i=1}^m \big\|I_i\big(x(t_i)\big) - I_i\big(x^0\big)\big\|.$$

As, for every $s \in [0,1]$,

$$\left\|\int_0^s h(s,\tau)x(\tau)\,d\tau - x^0\right\| \leqslant \left\|\int_0^s h(s,\tau)\big(x(\tau) - x^0\big)\,d\tau\right\| + \left\|x^0\int_0^s h(s,\tau)\,d\tau\right\| + \big\|x^0\big\| \leqslant bR_0 + \big\|x^0\big\|(b+1),$$

taking into account that $R_0 > \|x^0\|(b+1)$, we infer

$$\big\|Ax - x^0\big\|_C \leqslant \frac{2}{\delta_{1,\max\{R_0,bR_0+\|x^0\|(b+1)\}}} + \sum_{i=1}^m a_{i,R_0}\big\|x - x^0\big\|_C \leqslant \left(k_{R_0} + \sum_{i=1}^m a_{i,R_0}\right) R_0 < R_0.$$

Concerning the continuity, from the hypothesis (iii)(2) it follows that for every $\varepsilon > 0$ there is $\eta_\varepsilon > 0$ such that

$$\sup_{t\in[0,1]}\left\|(\mathrm{H})\int_0^t \Big(f\big(s, x_1(s), y_1(s)\big) - f\big(s, x_2(s), y_2(s)\big)\Big)\,ds\right\| < \frac{\varepsilon}{2}$$

for any $x_i, y_i \in C^{\diamond}_{x^0}([0,1],X)$ satisfying $\max\{\|x_1 - x_2\|_C, \|y_1 - y_2\|_C\} < \eta_\varepsilon \max\{1, b\}$. Then, for every $x_1, x_2 \in C^{\diamond}_{x^0}([0,1],X)$ with $\|x_1 - x_2\|_C < \min(\eta_\varepsilon, \frac{\varepsilon}{2\sum_{i=1}^m a_{i,R_0}})$,

$$\begin{aligned}\|Ax_1 - Ax_2\|_C &= \sup_{t\in[0,1]}\big\|Ax_1(t) - Ax_2(t)\big\| \\ &\leqslant \sup_{t\in[0,1]}\left\|(\mathrm{H})\int_0^t f\left(s, x_1(s), \int_0^s h(s,\tau)x_1(\tau)\,d\tau\right) - f\left(s, x_2(s), \int_0^s h(s,\tau)x_2(\tau)\,d\tau\right) ds\right\|\end{aligned}$$

$$+\sum_{i=1}^{m}\big\|I_i\big(x_1(t_i)\big)-I_i\big(x_2(t_i)\big)\big\|$$
$$<\frac{\varepsilon}{2}+\sum_{i=1}^{m}a_{i,R_0}\frac{\varepsilon}{2\sum_{i=1}^{m}a_{i,R_0}}=\varepsilon.$$

This comes from the fact that if $\|x_1-x_2\|_C<\eta_\varepsilon$, then, for all $s\in[0,1]$, $\|x_1(s)-x_2(s)\|<\eta_\varepsilon$ and $\|\int_0^s h(s,\tau)x_1(\tau)\,d\tau-\int_0^s h(s,\tau)x_2(\tau)\,d\tau\|<b\eta_\varepsilon$.

Now we are showing that $F=\overline{\text{co}}\,A(B_{R_0}(x^0))$ is equi-continuous on each interval $]t_i,t_{i+1}]$. From Lemma 2.1 in [16], it is enough to show that $A(B_{R_0}(x^0))$ is equi-continuous on each interval $]t_i,t_{i+1}]$. Let us then consider $\bar t,\bar{\bar t}\in\,]t_i,t_{i+1}]$. For all $x\in B_{R_0}(x^0)$,

$$\big\|Ax(\bar t)-Ax(\bar{\bar t})\big\|=\left\|(\mathrm{H})\int_{\bar t}^{\bar{\bar t}}f\left(s,x(s),\int_0^s h(s,\tau)x(\tau)\,d\tau\right)ds\right\|\leqslant\sup_{\bar t\leqslant t'<t''\leqslant\bar{\bar t}}\left\|(\mathrm{H})\int_{t'}^{t''}f\left(s,x(s),\int_0^s h(s,\tau)x(\tau)\,d\tau\right)ds\right\|.$$

So, thanks to (iii)(1), $\|Ax(\bar t)-Ax(\bar{\bar t})\|$ can be made less than some fixed ε for $\bar t,\bar{\bar t}$ with an appropriately small distance between them. Then the equi-continuity follows.

Obviously, $A:F\to F$ is bounded and continuous.

Let us prove in what follows, by the method of mathematical induction, that for every $B\subset F$ and any $n\in\mathbb{N}$, $\tilde A^n(B)\subset A(B_{R_0}(x^0))$, so it is bounded and equi-continuous on each interval $]t_i,t_{i+1}]$. For $n=1$, this is valid, since $A(B)\subset A(F)\subset A(B_{R_0}(x^0))$. Suppose now that this is true for $n-1$ and prove it for n:

$$\tilde A^n(B)=A\big(\overline{\text{co}}\big(\tilde A^{n-1}(B)\big)\big)\subset A\big(\overline{\text{co}}\big(A\big(B_{R_0}\big(x^0\big)\big)\big)\big)\subset A\big(\overline{\text{co}}\big(B_{R_0}\big(x^0\big)\big)\big)=A\big(B_{R_0}\big(x^0\big)\big).$$

By Corollary 5,

$$\alpha\big(\tilde A^n(B)\big)=\sup_{t\in[0,1]}\alpha\big(\tilde A^n(B)(t)\big),\quad\forall n\in\mathbb{N}.$$

Similarly to the second part of the proof of Theorem 3.1 in [15], one can show that there exist a constant $0\leqslant k<1$ and a positive integer n_0 such that for any $B\subset F$, $\alpha(\tilde A^{n_0}(B))\leqslant k\alpha(B)$.

Fix $\varepsilon>0$. As $L_1(s)+bL_2(s)\in L^1([0,1],\mathbb{R})$, one can find a continuous function ϕ such that $\int_0^1|L_1(s)+bL_2(s)-\phi(s)|\,ds<\varepsilon$. Choose $M>0$ with $\|\phi\|_C\leqslant M$ and denote by $c=\sum_{i=1}^m a_{i,R_0}<1$. Now we show, by mathematical induction, that, for all integer $p\geqslant1$,

$$\alpha\big(\tilde A^p(B)(t)\big)\leqslant\left((\varepsilon+c)^p+C_p^1(\varepsilon+c)^{p-1}\frac{Mt}{1!}+\cdots+\frac{(Mt)^p}{p!}\right)\alpha(B),\quad\forall t.$$

In order to prove it for $p=1$, let $(v_n)_n$ be an arbitrary countable subset of $\tilde A^1(B)=A(B)$. There exists a sequence $(x_n)_n\subset B$ such that $v_n=Ax_n$. Hypothesis (iii)(1) implies the $\|\cdot\|_A$-boundedness of $(f(\cdot,x_n(\cdot),\int_0^{(\cdot)}h(\cdot,\tau)x_n(\tau)\,d\tau))_n$ and (iii)(3) yields its pointwise boundedness and so, we are able to apply Theorem 7 and Lemma 9 and to obtain that

$$\begin{aligned}\alpha\big(\big\{v_n(t),\ n\in\mathbb{N}\big\}\big)&=\alpha\big(\big\{Ax_n(t),\ n\in\mathbb{N}\big\}\big)\\&=\alpha\left(\left\{x^0+(\mathrm{H})\int_0^t f\left(s,x_n(s),\int_0^s h(s,\tau)x_n(\tau)\,d\tau\right)ds+\sum_{0<t_i<t}I_i\big(x_n(t_i)\big),\ n\in\mathbb{N}\right\}\right)\\&\leqslant\int_0^t\alpha\left(f\left(s,\big\{x_n(s),\ n\in\mathbb{N}\big\},\left\{\int_0^s h(s,\tau)x_n(\tau)\,d\tau,\ n\in\mathbb{N}\right\}\right)\right)ds+\sum_{0<t_i<t}a_{i,R_0}\alpha\big(\big\{x_n(t_i),\ n\in\mathbb{N}\big\}\big).\end{aligned}$$

Indeed, by hypothesis (iv),

$$d\left(f\left(s,x_n(s),\int_0^s h(s,\tau)x_n(\tau)\,d\tau\right),X_q\right)\leqslant L_1(s)d\big(x_n(s),X_q\big)+L_2(s)d\left(\int_0^s h(s,\tau)x_n(\tau)\,d\tau,X_q\right)$$

and, using the first part of the proof of Theorem 7 and the fact that $(x_n)_n\subset B_{R_0}(x^0)$,

$$\begin{aligned}d\left(f\left(s,x_n(s),\int_0^s h(s,\tau)x_n(\tau)\,d\tau\right),X_q\right)&\leqslant L_1(s)\big(\big\|x^0\big\|+R_0\big)+L_2(s)\int_0^s h(s,\tau)d\big(x_n(\tau),X_q\big)\,d\tau\\&\leqslant\big(L_1(s)+bL_2(s)\big)\big(\big\|x^0\big\|+R_0\big).\end{aligned}$$

It follows, by Lemma 12, that

$$\alpha\big(\{v_n(t),\ n\in\mathbb{N}\}\big) \leqslant \int_0^t L_1(s)\alpha\big(\{x_n(s),\ n\in\mathbb{N}\}\big) + L_2(s)\alpha\left(\left\{\int_0^s h(s,\tau)x_n(\tau)\,d\tau,\ n\in\mathbb{N}\right\}\right)ds + \sum_{0<t_i<t} a_{i,R_0}\alpha\big(\{x_n(t_i),\ n\in\mathbb{N}\}\big).$$

Applying again Theorem 7 we infer

$$\begin{aligned}\alpha\big(\{v_n(t),\ n\in\mathbb{N}\}\big) &\leqslant \int_0^t \big(L_1(s)+bL_2(s)\big)\alpha\big(\{x_n(s),\ n\in\mathbb{N}\}\big)\,ds + \sum_{0<t_i<t} a_{i,R_0}\alpha\big(\{x_n(t_i),\ n\in\mathbb{N}\}\big)\\ &\leqslant \left[\int_0^t \big(L_1(s)+bL_2(s)\big)\,ds + \sum_{0<t_i<t} a_{i,R_0}\right]\alpha(B).\end{aligned}$$

Since the Banach space is separable and the Hausdorff measure of non-compactness is preserved when the set under discussion is replaced by its adherence, this implies that

$$\alpha\big(\tilde{A}^1(B)(t)\big) \leqslant \left[\int_0^t \big(L_1(s)+bL_2(s)\big)\,ds + \sum_{0<t_i<t} a_{i,R_0}\right]\alpha(B) \leqslant (\varepsilon + c + Mt)\alpha(B).$$

Suppose now that the inequality is valid for p and prove it for $p+1$. For any countable subset $(v_n)_n$ of $\tilde{A}^{p+1}(B) = A(\overline{\mathrm{co}}(\tilde{A}^p(B)))$, there exist $(x_n)_n \subset \overline{\mathrm{co}}(\tilde{A}^p(B))$ such that $v_n = Ax_n$. Then, as before,

$$\alpha\big(\{v_n(t),\ n\in\mathbb{N}\}\big) \leqslant \left[\int_0^t \big(L_1(s)+bL_2(s)\big)\,ds + \sum_{0<t_i<t} a_{i,R_0}\right]\alpha\big(\tilde{A}^p(B)\big),$$

whence

$$\begin{aligned}\alpha\big(\tilde{A}^{p+1}(B)(t)\big) &\leqslant \left[\int_0^t \big(L_1(s)+bL_2(s)\big)\,ds + \sum_{0<t_i<t} a_{i,R_0}\right]\alpha\big(\tilde{A}^p(B)\big)\\ &\leqslant (\varepsilon+c)\alpha\big(\tilde{A}^p(B)\big) + M\int_0^t \alpha\big(\tilde{A}^p(B)\big)\,ds\\ &\leqslant (\varepsilon+c)\left((\varepsilon+c)^p + C_p^1(\varepsilon+c)^{p-1}\frac{Mt}{1!} + \cdots + \frac{(Mt)^p}{p!}\right)\alpha(B)\\ &\quad + M\int_0^t \left((\varepsilon+c)^p + C_p^1(\varepsilon+c)^{p-1}\frac{Ms}{1!} + \cdots + \frac{(Ms)^p}{p!}\right)ds\,\alpha(B)\\ &= \left((\varepsilon+c)^{p+1} + C_{p+1}^1(\varepsilon+c)^p\frac{Mt}{1!} + \cdots + \frac{(Mt)^{p+1}}{(p+1)!}\right)\alpha(B),\end{aligned}$$

and so, the assertion is proved.

The rest of the calculus goes as in [15]: for some integer n_0 the evaluation term

$$(\varepsilon+c)^{n_0} + C_{n_0}^1(\varepsilon+c)^{n_0-1}\frac{Mt}{1!} + \cdots + \frac{(Mt)^{n_0}}{n_0!}$$

can be made less than 1 since one can choose ε such that $\varepsilon + c < 1$. By Lemma 10, the operator A has a fixed point, which is a global solution for Eq. (4). □

Corollary 13. *If in Theorem* 11 *the Henstock-integrability is replaced by the Henstock–Lebesgue integrability, then the differential equation with impulse effects* (1)–(3) *possess solutions in* $C^{\diamond}_{x_0}([0,1],X)$.

Remark 14. Previous Theorem 11 and Corollary 13 improve the related results given (in the non-impulsive case) in [10,11, 14–16], where the involved functions are supposed to be uniformly continuous with respect to all arguments. Moreover, our results are related to Theorem 5.1 of [6] that establishes, under conditions (2) and (3), an existence result for the impulsive equation $\dot{x}(t) = f(t, x(t))$ by imposing a pointwisely Lipschitz hypothesis with respect to the second argument on f.

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