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Discrete Analytic Continuation of Solutions of Difference Equations

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I. INTRODUCTION

In this paper we will consider the linear homogeneous difference equation

$$F(x+n)+a_{n-1}F(x+n-1)+\cdots+a_1F(x+1)+a_0F(x)=0 \tag{1.1}$$

where $a_0, a_1, \cdots, a_{n-1}$ are constants. For integer values of x the solution to (1.1) is (see Boole [1])

$$F(x)=\sum_{k=1}^{p}\sum_{j=0}^{m_k-1} C_{jk}x^j r_k^x \tag{1.2}$$

where the C_{jk}'s are arbitrary constants and $r_1, r_2, \cdots, r_p$ are the roots of

$$r^n+a_{n-1}r^{n-1}+\cdots+a_1r+a_0=0 \tag{1.3}$$

with multiplicities $m_1, m_2, \cdots, m_p$ respectively. The problem which we shall consider is to extend the solution (1.2) of (1.1) into the complex plane as a discrete analytic function. We treat this problem by a procedure which is analogous to extending solutions of real ordinary differential equations as analytic functions of a complex variable.

To carry out this extension we employ the concept of a discrete analytic function which has been studied by Isaacs [2], Ferrand [3], and Duffin [4]. Discrete analytic function theory concerns functions defined on the points of the complex plane whose coordinates are integers. The points $x+iy$ of the complex plane with integer x and y form a lattice which breaks up the plane into unit squares. The function $F(z)$ is said to be discrete analytic on one of these unit squares if for $z = x + iy$

$$LF(z) \equiv F(z) + iF(z+1) + i^2F(z+1+i) + i^3F(z+i) = 0 \qquad (1.4)$$

where z, $z+1$, $z+1+i$, $z+i$ are the points of this unit square.

Merely to require that the extension of the solution (1.2) be discrete analytic is not enough to make a rigid problem. We also shall insist that the difference equation (1.1) have permanence of form. In other words for each lattice point z

$$F(z+n) + a_{n-1}F(z+n-1) + \cdots + a_1F(z+1) + a_0F(z) = 0. \qquad (1.5)$$

Thus the problem that we treat is the solution of the pair of simultaneous difference equation (1.4) and (1.5). Under the restriction that the roots of (1.3) are not $\pm\, i$ or 0 we find that the solution (1.2) of (1.1) has a unique extension into the discrete complex plane. This extension may be expressed by an explicit rational formula.

Sections II through VI of this paper deal with "discrete derivative equations." The concept of the discrete derivative equation was introduced in [5] and is useful in motivating and handling some of the details of discrete analytic continuation of solutions of difference equations of the type (1.1). A discrete derivative equation is a difference equation but the increments are allowed to be imaginary as well as real.

II. Discrete Derivative Equations

In what is to follow we need a few definitions concerning the discrete complex plane. A ***region*** in the discrete complex plane is the union of unit squares. A ***chain*** of lattice points $z_0, z_1, \cdots, z_m$ is a set of points in the discrete complex plane such that $|z_j - z_{j-1}| = 1$. A region R is said to be ***connected*** if any two lattice points of the region R can be connected by a chain with every point of the chain in R. A ***simple region*** R is a connected region which is the union of a finite number of unit squares.

In [3] and [4] it has been shown that if f is discrete analytic in a simple region R and a and z are points of R connected by a chain in R with $z_0 = a$ and $z_m = z$, then

$$F(z) = \int_a^z f\delta z \equiv \sum_{n=1}^{m} \frac{f(z_n) + f(z_{n-1})}{2}(z_n - z_{n-1}).$$

is discrete analytic in R. In [5] the notation $f(z) = \delta F(z)/\delta z$ was introducted and $\delta F/\delta z$ was defined as the discrete derivative of $F(z)$. The discrete derivative $\delta F/\delta z$ of $F(z)$ is uniquely given only up to an arbitrary function $k^- = (-1)^{x+y}\bar{k}$, where k is an arbitrary constant and $\bar{k}$ is the complex conjugate of k. The function k^- is termed a biconstant.

The following theorem is taken from [4]. This theorem reveals a certain duality between a function and its discrete derivative. Needless to say, no such duality exists in the continuous case.

THEOREM 2.1. *Let $F(z)$ be a given analytic function in a simple region R. Let a and b be points of R and let k be an arbitrary constant. Then*

$$\frac{\delta F(z)}{\delta z} = \left(4\int_b^z F^- \delta z + k\right)^-$$

is analytic in R and

$$F(z) = \int_a^z \frac{\delta F}{\delta z}\,\delta z + F(a).$$

The integration paths are assumed to be in R.

Here $f^-(z)$ denotes the dual of $f(z)$ and is defined by the relation $f^-(z) = (-1)^{x+y}\bar{f}(z)$ where $\bar{f}$ is the complex conjugate of f. If f is discrete analytic so is f^-.

In [5] it has been shown that there exists a unique discrete analytic function $F(z)$ such that

$$\frac{\delta^n F(z)}{\delta z^n} + c_{n-1}\frac{\delta^{n-1}F(z)}{\delta z^{n-1}} + \cdots + c_1\frac{\delta F(z)}{\delta z} + c_0 F(z) = f(z), \tag{2.1}$$

if $f(z)$ is discrete analytic, ± 2 and $\pm 2i$ are not roots of

$$r^n + c_{n-1}r^{n-1} + c_{n-2}r^{n-2} + \cdots + c_1 r + c_0 = 0,$$

and $F(z)$ is given appropriate boundary conditions.

In the following sections we will exhibit solutions of equations similar to (2.1).

III. First Order Homogeneous Discrete Derivative Equations

The first equation which we shall consider is

$$\frac{\delta F(z)}{\delta z} - aF(z) = 0. \tag{3.1}$$

If $(a^4 - 16) \neq 0$ we shall term Eq. (3.1) regular. The integral form of this equation is

$$F(z) = a \int_c^z F(t)\, \delta t + F(a). \tag{3.2}$$

Replacing c by z and z by $z + h$, where h equals ± 1 or $\pm i$ we get

$$F(z + h) = a \int_z^{z+h} F(t)\, \delta t + F(z).$$

Using the definition of line integral, we get the stepping formula

$$F(z + h) = \left(\frac{2 + ha}{2 - ha}\right) F(z), \tag{3.3}$$

giving $F(z + h)$ in terms of $F(z)$.

If Eq. (3.1) is regular it can be shown, using stepping formula (3.3), that the general solution to (3.1) is

$$F(z) = C \left(\frac{2 + a}{2 - a}\right)^x \left(\frac{2 + ia}{2 - ia}\right)^y \tag{3.4}$$

where C is an arbitrary constant and $z = x + iy$. Uniqueness of this solution given an initial condition is clear from Eq. (3.3).

By analogy with the continuous case the discrete exponential function $e(z, a)$ is defined to be

$$e(z, a) = \left(\frac{2 + a}{2 - a}\right)^x \left(\frac{2 + ia}{2 - ia}\right)^y. \tag{3.5}$$

This function has been investigated previously by Ferrand [3] and Duffin [4]. One important property which it possesses is discrete analyticity in the entire discrete complex plane.

When Eq. (3.1) is not regular we see that a equals $\pm 2i$ or ± 2. We will now consider this case by again using stepping formula (3.3).

For the initial condition $F(0) = C$, an arbitrary constant, we have that:

1. For $a = 2$, the solution $F(z)$, where $z = x + iy$, is

$$F(z) = \begin{cases} 0 & \text{for} \quad x < 0 \\ i^y C & \text{for} \quad x = 0 \\ \text{undefined} & \text{for} \quad x > 0 \end{cases}$$

2. For $a = -2$

$$F(z) = \begin{cases} 0 & \text{for} \quad x > 0 \\ i^{-y} C & \text{for} \quad x = 0 \\ \text{undefined} & \text{for} \quad x < 0 \end{cases}$$

3. For $a = i2$

$$F(z) = \begin{cases} 0 & \text{for} \quad y > 0 \\ i^x C & \text{for} \quad y = 0 \\ \text{undefined} & \text{for} \quad y < 0 \end{cases}$$

4. For $a = -i2$

$$F(z) = \begin{cases} 0 & \text{for} \quad y < 0 \\ i^{-x} C & \text{for} \quad y = 0 \\ \text{undefined} & \text{for} \quad y > 0 \end{cases}$$

These irregular cases have been presented here for completeness. Throughout the rest of this paper these cases will be avoided in order to simplify the ideas involved.

IV. Higher Order Homogeneous Equations

Consider now the nth order discrete derivative equation

$$\frac{\delta^n F}{\delta z^n} + c_{n-1} \frac{\delta^{n-1} F}{\delta z^{n-1}} + \cdots + c_1 \frac{\delta F}{\delta z} + c_0 F = 0. \tag{4.1}$$

Noting that $(\delta/\delta z)\, e(z, r) = re(z, r)$ ignoring any biconstants k^-, we assume $F(z) = e(z, r)$ and get that

$$(r^n + c_{n-1} r^{n-1} + \cdots + c_1 r + c_0)\, e(z, r) = 0.$$

Since $e(z, r) \neq 0$ we must have

$$r^n + c_n r^{n-1} + \cdots + c_1 r + c_0 = 0 \tag{4.2}$$

in order for $e(z, r)$ to be a solution of (4.1). We shall call (4.2) the *characteristic equation* of (4.1). Let $a_1, a_2, \cdots, a_n$ be the roots of (4.2). If none of these roots is ± 2 or $\pm 2i$, (4.1) is said to be *regular*.

If the roots $a_1, a_2, \cdots, a_n$ are distinct and (4.1) is regular the general solution to (4.1) is

$$F(z) = \sum_{i=1}^{n} B_i e(z, a_i) \tag{4.3}$$

where the B_i's are arbitrary constants.

If one of the roots is repeated then (4.3) is not the most general solution because only $n - 1$ arbitrary constants would occur. To understand this case consider the second order equation

$$\frac{\delta^2 F}{\delta z} - (2a + \epsilon) \frac{\delta F}{\delta z} + a(a + \epsilon) F = 0.$$

The roots are a and $a + \epsilon$ where ϵ is a real number greater than zero. Since $e(z, a)$ and $e(z, a + \epsilon)$ are solutions,

$$\frac{e(z, a + \epsilon) - e(z, a)}{\epsilon}$$

is a solution. Letting $\epsilon \to 0$, we see that

$$\frac{d}{dr} e(z, r) \bigg|_{r=a}$$

is a solution of

$$\frac{\delta^2 F}{\delta z^2} - 2a \frac{\delta F}{\delta z} + a^2 F = 0. \tag{4.4}$$

The general solution to (4.4) is

$$F(z) = B_1 e(z, a) + B_2 \frac{d}{dr} e(z, r) \bigg|_{r=a}$$

For convenience we will define

$$\frac{d}{da} e(z, a) = \frac{d}{dr} e(z, r) \bigg|_{r=a}.$$

A similar procedure can be used to show that if (4.2) the characteristic equation of (4.1) has a root α of multiplicity $m > 1$ then (4.1) has the m solutions

$$e(z, \alpha), \qquad \frac{d}{d\alpha} e(z, \alpha), \qquad \frac{d^2}{d\alpha^2} e(z, \alpha), \qquad \cdots, \qquad \frac{d^{m-1}}{d\alpha^{m-1}} e(z, \alpha).$$

Thus we get the following:

THEOREM 4.1. *The general solution to the regular discrete derivative equation*

$$\frac{\delta^n F}{\delta z^n} + c_{n-1} \frac{\delta^{n-1} F}{\delta z^{n-1}} + \cdots + c_1 \frac{\delta F}{\delta z} + c_0 F = 0 \tag{4.1}$$

is

$$F(z) = \sum_{k=1}^{p} \sum_{j=0}^{m_k - 1} B_{kj} \frac{d^j}{da_k^j} e(z, a_k) \tag{4.5}$$

where $a_1, a_2, \cdots, a_p$ with multiplicities $m_1, m_2, \cdots, m_p$ respectively are the roots of the characteristic polynomial of (4.1), and the B_{kj} are arbitrary constants.

V. First Order Nonhomogeneous Equation

We now consider the nonhomogeneous equation

$$\frac{\delta F(z)}{\delta z} - aF(z) = b(z). \tag{5.1}$$

Again we assume that (5.1) is regular, meaning $(a^4 - 16) \neq 0$. The discrete function $b(z)$ is required to be discrete analytic in a simple region R of the complex plane where we desire to find $F(z)$.

Putting (5.1) into integral form we have

$$F(z) = \int_a^z [aF(t) + b(t)]\, \delta t + F(a).$$

Using the definition of the line integral, we obtain the stepping formula

$$F(z + h) = \left(\frac{2 + ha}{2 - ha}\right) F(z) + h\, \frac{b(z + h) + b(z)}{2 - ha}. \tag{5.2}$$

For z in R using (5.2), we obtain

$$LF(z) = F(z)\left[1 + i\left(\frac{2 + a}{2 - a}\right) + i^2\left(\frac{2 + a}{2 - a}\right)\left(\frac{2 + ia}{2 - ia}\right) + i^3\left(\frac{2 + ia}{2 - ia}\right)\right] + \frac{i}{2 - ia} Lb(z) = 0.$$

Hence if (5.1) has a solution it is discrete analytic.

From stepping formula (5.2) starting at $z = 0$ with $F(0) = C$, a constant, we obtain the solution in the first quadrant as

$$\begin{aligned} F(z) = Ce(z, a) + e(z, a) \Bigg\{ &\sum_{k=1}^{x} \tfrac{1}{4} [e(-k, a) + e(-k + 1, a)]\, [b(k) + b(k - 1)] \\ &+ \sum_{k=1}^{y} \frac{i}{4} [e(-x - ik, a) + e(-x - i(k - 1), a)] \\ &\times [b(x + ik) + b(x + i(k - 1))] \Bigg\}. \end{aligned} \tag{5.3}$$

Duffin in [4] gave the following definition of the line integral for the product of two functions. Let $a = z_0, z_1, \cdots, z_m = b$ denote a chain of lattice points and let f and g be discrete functions. Then the "double dot" line integral is defined as

$$\int_a^b f(z) : g(z)\, \delta z = \sum_{n=1}^{m} \tfrac{1}{4} [f(z_n) + f(z_{n-1})]\, [g(z_n) + g(z_{n-1})]\, (z_n - z_{n-1}). \quad (5.4)$$

For f and g analytic this integral is independent of the path of integration, but is not necessarily analytic as a function of the upper limit of integration.

Using this line integral the relation (5.3) may be written as

$$F(z) = Ce(z, a) + \int_0^z e(z - t, a) : b(t)\, \delta t. \quad (5.5)$$

Because of the stepping procedure used in solving this problem the solution is easily seen to be unique. The solution as given by (5.5) is not restricted to the first quadrant. If the simple region R has for its boundary a closed chain, (5.5) is guaranteed to be single valued, if not it may be multiple valued. A region R with a single closed chain for its boundary will be termed *simply connected.* The preceding work may be summarized in the following theorem:

THEOREM 3.3. *The general solution to the regular equation*

$$\frac{\delta F(z)}{\delta z} - aF(z) = b(z) \quad (5.1)$$

where $b(z)$ is a discrete analytic function in a simple region R containing the origin is

$$F(z) = Ce(z, a) + \int_0^z e(z - t, a) : b(t)\, \delta t \quad (5.5)$$

where C is an arbitrary constant. The solution $F(z)$ is defined and discrete analytic in R, and if R is simply connected $F(z)$ is single valued in R.

VI. SYSTEMS OF LINEAR COMPLEX DIFFERENCE EQUATIONS

The nth order nonhomogeneous system

$$\frac{\delta^n F}{\delta z^n} + c_{n-1} \frac{\delta^{n-1}}{\delta z^{n-1}} + \cdots + c_1 \frac{\delta F}{\delta z} + c_0 F = b(z) \quad (6.1)$$

may be reduced to a first order system of equations using the substitution

$$\phi_k(z) = \frac{\delta^{k-1}F(z)}{\delta z^{k-1}}, \qquad k = 1, \cdots, n. \tag{6.2}$$

This gives the system

$$\frac{\delta}{\delta z}\begin{bmatrix} \phi_1(z) \\ \cdot \\ \cdot \\ \cdot \\ \phi_n(z) \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \vdots & & \\ -c_0 & -c_1 & & \cdots & -c_{n-1} \end{bmatrix} \begin{bmatrix} \phi_1(z) \\ \cdot \\ \cdot \\ \cdot \\ \phi_n(z) \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ b(z) \end{bmatrix}. \tag{6.3}$$

This gives some motivation for studying the more general system

$$\frac{\delta \mathbf{F}(z)}{\delta z} + A\mathbf{F}(z) = \mathbf{b}(z). \tag{6.4}$$

Here $\mathbf{F}(z)$ and $\mathbf{b}(z)$ are n-dimensional vector valued functions and A is an n-*by*-n matrix of complex constants.

In the case of an n-*by*-m discrete matrix function $\Phi(z)$ we define $\Phi(z)$ to be discrete analytic at z_0 if

$$L\Phi(z_0) = \Phi(z_0) + i\Phi(z_0 + 1) - \Phi(z_0 + 1 + i) - i\Phi(z_0 + i) = 0.$$

In what is to follow the line integrals and discrete derivatives of matrix functions are interpreted as operations on each entry separately.

A. Homogeneous Systems

Let A be an n-by-n matrix of complex constants such that ± 2 and $\pm i2$ are not eigenvalues of A. We desire to find the discrete function $\mathbf{F}(z)$ such that

$$\frac{\delta \mathbf{F}(z)}{\delta z} - A\mathbf{F}(z) = 0 \tag{6A.1}$$

with initial condition $\mathbf{F}(0) = \mathbf{C}$ a vector of complex constants. Because of the restriction on A we termed (6A.1) a *regular system.* Using the notation Coddington and Levinson [6] we associate with (6A.1) the complex matrix difference equation

$$\frac{\delta \Phi(z)}{\delta z} - A\Phi(z) = 0. \tag{6A.2}$$

We now look for a matrix $\Phi(z)$ satisfying (6A.2) such that $\det \Phi(z) \neq 0$. The columns of such a matrix are linearly independent solutions of (6A.1).

The solution of the matrix equation (6A.2) is the matrix function

$$e(z, A) = (2I + A)^x (2I - A)^{-x} (2I + iA)^y (2I - iA)^{-y}. \quad (6A.3)$$

This matrix function is analogous to the matrix exponential e^{zA} defined in Coddington and Levinson [6] by

$$e^{zA} = I + \sum_{m-1}^{\infty} \frac{z^m}{m!} A^m. \quad (6A.4)$$

Note that the continuous matrix exponential requires a power series representation, while the discrete matrix exponential has a rational representation.

Using (6A.3) we get that

$$\mathbf{F}(z) = e(z, A)\,\mathbf{F}(0) \quad (6A.5)$$

is the desired solution of (6A.1) since $e(0, A) = I$.

B. Nonhomogeneous Systems

Let $\mathbf{b}(z)$ be a discrete analytic n-dimensional vector function in a simple region R of the complex plane. We now desire to obtain the solution to the nonhomogeneous regular system

$$\frac{\delta \mathbf{F}(z)}{\delta z} - A\mathbf{F}(z) = \mathbf{b}(z). \quad (6B.1)$$

Motivated by the one dimensional case of (6.2.1) we will prove the following:

THEOREM 6.2. *The general solution to the non-homogeneous regular system*

$$\frac{\delta \mathbf{F}(z)}{\delta z} - A\mathbf{F}(z) = \mathbf{b}(z) \quad (6B.1)$$

where $\mathbf{b}(z)$ *is a discrete analytic n-vector function in a simple region R containing the origin is*

$$\mathbf{F}(z) = e(z, A)\,\mathbf{C} + \int_0^z e(z - t, A) : \mathbf{b}(t)\,\delta t, \quad (6B.2)$$

where $\mathbf{C}$ *is an arbitrary constant n-vector. The solution* $\mathbf{F}(z)$ *is defined and discrete analytic in R, and if R is simply connected* $\mathbf{F}(z)$ *is single valued in R.*

PROOF. The integral form for (6B.1) is

$$\mathbf{F}(z) = \int_{z_0}^{z} [A\mathbf{F}(t) + \mathbf{b}(t)]\,\delta t + \mathbf{F}(z_0), \quad (6B.3)$$

which from the definition of the line integral is equivalent to finding $\mathbf{F}(z)$ such that

$$\frac{1}{h}(\mathbf{F}(z+h)-\mathbf{F}(z))=\tfrac{1}{2}[A(\mathbf{F}(z+h)+\mathbf{F}(z))+\mathbf{b}(z+h)+\mathbf{b}(z)]$$

where h equal ± 1 or $\pm i$. This expression may be put into the more convenient form

$$(2I-hA)\,\mathbf{F}(z+h)-(2I+hA)\,\mathbf{F}(z)=h(\mathbf{b}(z+h)+\mathbf{b}(z)). \tag{6B.4}$$

By substituting (6B.2) in the left side of (6B.4), for the four values of h we find that the $\mathbf{F}(z)$ of (6B.2) satisfies (6B.4), hence is the solution of (6B.1). This proves Theorem 6.2.

VII. Discrete Analytic Continuation of Homogeneous Difference Equations

Consider the real nth order homogeneous difference equation

$$F(x+n)+b_{n-1}F(x+n-1)+\cdots+b_1F(x+1)+b_0F(x)=0 \tag{7.1}$$

where the b_i's are constants. If the nonzero roots of

$$r^n+b_{n-1}r^{n-1}+\cdots+b_1r+b_0=0 \tag{7.2}$$

are $r_1, r_2, \cdots, r_p$ with multiplicity $m_1, m_2, \cdots, m_p$ respectively, then it is well known (see Boole [1]) that the general solution of (7.1) is

$$F(x)=\sum_{k=1}^{p}\sum_{j=0}^{m_k-1}C_{kj}x^jr_k^x. \tag{7.3}$$

If the n values $F(0), F(1), \cdots, F(n-1)$ are given for $F(x)$, then the constants C_{kj} may be determined and $F(x)$ is given uniquely for all x.

The problem which we shall consider is to extend the solution (7.3) of (7.1) into the discrete complex plane as a discrete analytic function, with the additional requirement that the extension $F(z)$ of $F(x)$ satisfy

$$F(z+n)+b_{n-1}F(z+n-1)+\cdots+b_1F(z+1)+b_0F(z)=0, \tag{7.4}$$

where $z=x+iy$.

Before looking at this problem for the general case consider the example

$$F(x+2)-5F(x+1)+6F(x)=0.$$

For x a real integer this has the general solution

$$F(x)=C_12^x+C_23^x.$$

We desire to extend $F(x)$ to a discrete analytic function $F(z)$ with the additional requirement that

$$F(z+2) - 5F(z+1) + 6F(z) = 0.$$

From this difference equation we see that

$$F(z) = C_1(y)\, 2^x + C_2(y)\, 3^x$$

where $C_1(y)$ and $C_2(y)$ are no longer constant but depend on y. Using the requirement that $LF(z) = 0$ for all z we get that

$$C_1(y) = \left(\frac{1+i2}{2+i}\right)^y C_1 \qquad \text{and} \qquad C_2(y) = \left(\frac{1+i3}{3+i}\right)^y C_2 .$$

Thus one such extension is

$$F(z) = C_1 2r \left(\frac{1+i2}{2+i}\right)^y + C_2 3^x \left(\frac{1+i3}{3+i}\right)^y.$$

We shall now consider the general case of this problem.

A. *Existence and Uniqueness of Continuation*

The nth order difference equation

$$F(x+n) + b_{n-1}F(x+n-1) + \cdots + b_1F(x+1) + b_0F(x) = 0 \tag{7A.1}$$

can be treated as a system of first order difference equations. This may be accomplished by letting

$$w_k(x) = F(x+k-1), \qquad k = 1, 2, \cdots, n. \tag{7A.2}$$

Then (7A.1) and (7A.2) can be written in the matrix form

$$\begin{bmatrix} w_1(x+1) \\ w_2(x+1) \\ \vdots \\ \\ w_n(x+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \\ 0 & & \cdots & & 1 \\ -b_0 & -b_1 & & \cdots & -b_{n-1} \end{bmatrix} \begin{bmatrix} w_1(x) \\ w_2(x) \\ \vdots \\ \\ w_n(x) \end{bmatrix}. \tag{7A.3}$$

If $\mathbf{w}(x)$ denotes the vector with components $w_j(x)$, the relation (7A.3) can be abreviated as

$$\mathbf{w}(x+1) = C\mathbf{w}(x) \tag{7A.4}$$

where C is a square matrix. The following theorem does not assume that C is restricted to the form shown in (7A.3).

THEOREM 7.1. *In the system of difference equation (7A.4) let C be a constant matrix which is arbitrary except that C does not have the eigenvalues* 0 *or* $\pm i$. *Then any solution* $\mathbf{w}(x)$ *on the real axis may be extended to be a discrete analytic function* $\mathbf{w}(z)$ *in the complex plane by the formula*

$$\mathbf{w}(z) = C^x(I + iC)^y (C + iI)^{-y} \mathbf{w}(0) \tag{7A.5}$$

where I is the identity matrix and $z = x + iy$. This extension is unique if it is required that

$$\mathbf{w}(z + 1) = C\mathbf{w}(z). \tag{7A.6}$$

PROOF. By the hypothesis that 0 or $\pm i$ are not eigenvalues of C it is seen that (7A.5) defines $f(z)$ for all lattice points of the complex plane. It is also seen that

$$L\mathbf{w}(z) = [(I + iC) - C(I + iC)(C + iI)^{-1} - i(I + iC)(C + iI)^{-1}]\, \mathbf{w}(z)\,.$$

The right side vanishes so $\mathbf{w}(z)$ is discrete analytic. Obviously (7A.5) satisfies (7A.6). Let $\mathbf{w}'(z)$ be any other solution such that $\mathbf{w}'(0) = \mathbf{w}(0)$. Let $\mathbf{G}(z) = \mathbf{w}'(z) - \mathbf{w}(z)$. Then it follows from (7A.4) that $\mathbf{G}(x) = 0$ for all real x. Since $L\mathbf{G}(z) = 0$, we have

$$L\mathbf{G}(x) = \mathbf{G}(x) + i\mathbf{G}(x + 1) - \mathbf{G}(x + 1 + i) - i\mathbf{G}(x + i) = 0.$$

But

$$\mathbf{G}(x) = 0 \qquad \text{and} \qquad \mathbf{G}(x + 1) = 0,$$

so

$$\mathbf{G}(x + 1 + i) + i\mathbf{G}(x + i) = 0. \tag{7A.7}$$

Then (7A.7) and (7A.6) imply

$$(C + iI)\,\mathbf{G}(x + i) = 0.$$

Since $-i$ is not an eigenvalue of C it follows that $\mathbf{G}(x + i) = 0$ for all x. This process may be repeated to show that $\mathbf{G}(x + iy) = 0$ for y a positive integer. A similar process which makes use of the fact that $+i$ is not an eigenvalue of C show that $\mathbf{G}(x + iy) = 0$ for y a negative integer. This shows that $\mathbf{G}(z)$ vanishes identically and the proof is complete.

In the notation of matrix functions

$$e(z, A) = (2I + A)^x (2I - A)^{-x} (2I + iA)^y (2I - iA)^{-y} \tag{6A.3}$$

we may write (7A.5) as

$$\mathbf{w}(z) = e(z, A)\, \mathbf{w}(0) \tag{7A.8}$$

where A is obtained from the relation

$$C = (2I + A)(2I - A)^{-1},$$

or, more directly,

$$A = 2(C - I)(C + I)^{-1}. \tag{7A.9}$$

From the definition of $e(z, A)$ and (7A.5) we see that if 0 were an eigenvalue of C then -2 would be an eigenvalue of A and if C had $\pm i$ as eigenvalues then A would have $\pm 2i$ as eigenvalues. No finite eigenvalue of C corresponds to the eigenvalue 2 for A.

COROLLARY 7.2. *In the* nth *order difference equation*

$$F(x + n) + b_{n-1}F(x + n - 1) + \cdots + b_1F(x + 1) + b_0F(x) = 0 \tag{7A.1}$$

let the b_j *be arbitrary constants except that none of the roots of*

$$r^n + b_{n-1}r^{n-1} + \cdots + b_1r + b_0 = 0 \tag{7A.10}$$

are $\pm i$. *Then any solution* $F(x)$ *on the real axis can be extended to be a discrete analytic function* $F(z)$ *in the complex plane. This extension is unique if it is required that*

$$F(z + n) + b_{n-1}F(z + n - 1) + \cdots + b_1F(z + 1) + b_0F(z) = 0. \tag{7A.11}$$

PROOF. There is no loss of generality in assuming that $b_0 \neq 0$. Then the matrix C is constructed as in (7A.3). It is seen that if r is an eigenvalue of C then r satisfies (7A.10). By hypothesis $r \neq \pm i$. Also $b_0 \neq 0$, so $r \neq 0$. Thus the matrix C of Eq. (7A.3) satisfies the conditions of Theorem 7.1. Under the correspondence (7.A.2) it is seen that $\mathbf{w}(z)$ satisfying (7A.6) is equivalent to $F(z)$ satisfying (7A.11). Moreover $\mathbf{w}(z)$ being discrete analytic is equivalent to $F(z)$ being discrete analytic. Thus the Corollary follows from Theorem 7.1

The following theorem brings out the connection between the extension of the nth order difference equation and the solution of the nth order complex discrete derivative equations.

THEOREM 7.3. *Let* $F(z)$ *be the unique extension of* $F(x)$ *a solution of (7A.1) subject to the restriction*

$$F(z + n) + b_{n-1}F(z + n - 1) + \cdots + b_1F(z + 1) + b_0F(z) = 0. \tag{7A.12}$$

If $r_1, r_2, \cdots, r_k$ *the nonzero roots of (7A.10) with multiplicities* $m_1, m_2, \cdots, m_p$ *are such that* $r_j \neq \pm i$ *or* -1, *then the extension* $F(z)$ *may be represented in the form*

$$F(z) = \sum_{k=1}^{p} \sum_{j=0}^{m_k - 1} B_{kj} \frac{d^j}{d\alpha_k^j} e(z, \alpha_k) \tag{7A.13}$$

where

$$\alpha_k = 2\left(\frac{r_k - 1}{r_k + 1}\right) \qquad \textit{for} \qquad k = 1, 2, \cdots, p.$$

Before proving this we state the following

LEMMA 7.4. *For α finite and $\alpha \neq \pm 2i$ or ± 2*

$$\frac{d^n}{d\alpha^n} e(z, \alpha) = \frac{d^n}{dt^n} e(z, t)\Big|_{t=\alpha} = Q_n(x, y)\, e(z, \alpha) \tag{7A.14}$$

where $Q_n(x, y)$ is a polynomial in x of degree exactly n for fixed y and visa-versa. The proof of this Lemma is straightforward. The restriction $\alpha \neq \pm 2i$ *or* ± 2 is obviously necessary from the definition of $e(z, \alpha)$.

PROOF OF THEOREM 7.3. Referring back to Eq. (7.3), we know $F(x)$ has the form

$$F(x) = \sum_{k=1}^{p} \sum_{j=0}^{m_k-1} C_{kj} x^j r_k^x .$$

Since $r_k \neq \pm i$ or -1 we have $\alpha_k = 2(r_k - 1)/(r_k + 1)$. Thus

$$F(x) = \sum_{k=1}^{p} \sum_{j=0}^{m_k-1} C_{kj} x^j \left(\frac{2 + \alpha_k}{2 - \alpha_k}\right)^x . \tag{7A.15}$$

By (7A.14) we know that

$$\frac{d^n}{d\alpha_k^n} e(x, \alpha_k) = (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)\, e(x, \alpha_k).$$

Thus we conclude that B_{kj}'s can be found so that

$$\sum_{j=0}^{m_k-1} C_{kj} x^j \left(\frac{2 + \alpha_k}{2 - \alpha_k}\right)^x = \sum_{j=0}^{m_k-1} B_{kj} \frac{d^j}{d\alpha_k^j} e(x, \alpha_k). \tag{7A.16}$$

Having chosen the B_{kj}'s so that (7A.16) holds for $k = 1, 2, \cdots, p$, let

$$F(z) = \sum_{k=1}^{p} \sum_{j=0}^{m_k-1} B_{kj} \frac{d^j}{d\alpha_k^j} e(z, \alpha_k).$$

$F(z)$ is obviously a discrete analytic function. For fixed y by (7A.14) we may write

$$F(z) = \sum_{k=1}^{p} \sum_{j=0}^{m_k-1} D_{kj}(y)\, x^j \left(\frac{2 + \alpha^k}{2 - \alpha_k}\right)^x$$

or

$$F(z) = \sum_{k=1}^{p} \sum_{j=0}^{m_k - 1} D_{kj}(y)\, x^j r_k^x .$$

Thus $F(z)$ satisfies (7A.12) and by Theorem 7.1 is the desired unique extension. This completes the proof of Theorem 7.3.

Note that the requirement $r_i \neq -1$ in Theorem 7.3 does not appear in Corollary 7.2. This means that in some way the matrix form of the problem circumvents the difficulty of the indeterminancy for $r_i = -1$. If the root -1 appears for (7A.10) with multiplicity one no trouble results if we define

$$e(z,\infty) = (-1)^{x+y}.$$

Trouble occurs when -1 is a root of multiplicity greater than one.

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