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Fractional Part Sums and Divisor Functions*

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Elementary methods are used to study sums of the form $\sum_{d\leqslant x} d^p\{x/d\}^t$ for integers p and t, $t>0$, where $\{x\}$ denotes the fractional part of x. These sums are then used to study sums of the form $\sum_{d\leqslant x} d^p P_t(x/d)$ for integers p and t, $t>0$, where $P_t(x)=B_t(\{x\})$ and $B_t(x)$ are Bernoulli polynomials. Finally, these sums and some general results on sums of error terms are used to study sums of the form $\sum_{n\leqslant x} n^t\sigma_a(n)$ and $\sum_{n\leqslant x} E_t(n)$ for integers t and a, $a\geqslant 0$, where $\sigma_a(n)$ is the sum of the ath powers of the divisors of n and $E_t(x)$ is the error term in the sum $\sum_{n\leqslant x} n^t\sigma_a(n)$.

1. INTRODUCTION

For real $x\geqslant 1$, let $\{x\}$ and k denote, respectively, the fractional part of x and the quantity $\sqrt{x}-\{\sqrt{x}\}$. Let B_t and $B_t(x)$ denote respectively the Bernouilli numbers and polynomials, and define $P_t(x)$ by

$$P_t(x)=B_t(\{x\}).$$

The object of the first four sections of this work is to obtain expansions of the sums $\sum_{d\leqslant x} d^p\{x/d\}^t$ and $\sum_{d\leqslant x} d^p P_t(x/d)$ for integers p and t, $t>0$, in terms of powers of x and the basic functions $\sum_{d\leqslant k} d^p P_t(x/d)$. These rather curious functions have received a fair amount of study recently. Trivially we have

$$\sum_{d\leqslant k} d^p P_t\left(\frac{x}{d}\right)=O(x^{p/2+1/2}). \tag{i}$$

But because of the oscillatory nature of $P_t(x/d)$ one expects to be able to do better than (i). Indeed, Chowla and Walum [3, 4], have conjectured that, for non-negative integers p and t,

$$\sum_{d\leqslant k} d^p P_t\left(\frac{x}{d}\right)=O(x^{p/2+1/4+\varepsilon}) \qquad \text{for } \varepsilon>0. \tag{ii}$$

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They have proven the somewhat stronger special case

$$\sum_{d \leqslant k} dP_2\left(\frac{x}{d}\right) = O(x^{3/4}). \tag{iii}$$

Segal, in [8], proves that if $E_0(x)$ is defined by

$$\sum_{n \leqslant x} \sigma(n) = \frac{\zeta(2)}{2} x^2 + E_0(x),$$

then

$$\sum_{n \leqslant x} E_0(n) = \tfrac{1}{4}(\zeta(2) - 1)\, x^2 + O(x^{5/4}), \tag{iv}$$

while Walfiscz, in [9, p. 99], shows that

$$E_0(x) = O(x(\log x)^{2/3}) \tag{v}$$

and that

$$\sum_{n \leqslant x} \frac{\sigma(n)}{n} = \zeta(2)\, x - \frac{1}{2} \log x + O(\log^{2/3} x). \tag{vi}$$

In Theorem 6, we will show

$$\sum_{n \leqslant x} E_0(n) = \frac{1}{4}(\zeta(2) - 1)\, x^2 - \frac{1}{2} x \sum_{d \leqslant k} P_2\left(\frac{x}{d}\right) - \frac{1}{2} \sum_{d \leqslant k} d^2 P_2\left(\frac{x}{d}\right)$$
$$- \left(\frac{1}{2} - P_1(x)\right) x \sum_{d \leqslant k} \frac{1}{d} P_1\left(\frac{x}{d}\right) + O(x), \tag{vii}$$

and that

$$E_0(x) = -x \sum_{d \leqslant k} \frac{1}{d} P_1\left(\frac{x}{d}\right) - \frac{1}{2} x - \sum_{d \leqslant k} dP_1\left(\frac{x}{d}\right) + O(x^{1/2}), \tag{viii}$$

while in Theorem 8 we will show

$$\sum_{n \leqslant x} \frac{\sigma(n)}{n} = \zeta(2)\, x - \frac{1}{2} \log x - \sum_{d \leqslant k} \frac{1}{d} P_1\left(\frac{x}{d}\right) - \frac{1}{2} \log(2\pi + \gamma)$$
$$- \frac{1}{x} \sum_{d \leqslant k} dP_1\left(\frac{x}{d}\right) + O(x^{-1/2}). \tag{ix}$$

From (iv) and (vii) we have

$$x \sum_{d \leqslant k} P_2\left(\frac{x}{d}\right) + \sum_{d \leqslant k} d^2 P_2\left(\frac{x}{d}\right) = O(x^{5/4}), \tag{x}$$

while from (v) and (viii) or from (vi) and (ix)

$$\sum_{d \leqslant k} \frac{1}{d} P_1\left(\frac{x}{d}\right) = O(\log^{2/3} x). \tag{xi}$$

Buchstab, in [1], has shown that

$$x \sum_{n \leqslant x} \frac{\sigma(n)}{n} - \sum_{n \leqslant x} \sigma(n) = \frac{\zeta(2)}{2} x^2 - \frac{1}{2} x \log x - \frac{1}{2}(\gamma - 1 + \log 2\pi) x + O(x^{2/7} \log x). \tag{xii}$$

In Lemma 20, we will show

$$x \sum_{n \leqslant x} \frac{\sigma(n)}{n} - \sum_{n \leqslant x} \sigma(n) = \frac{\zeta(2)}{2} x^2 - \frac{1}{2} x \log x - \frac{1}{2}(\gamma - 1 + \log 2\pi) x - \frac{1}{2}\frac{1}{x} \sum_{d \leqslant k} d^2 P_2\left(\frac{x}{d}\right) - \frac{1}{2} \sum_{d \leqslant k} P_2\left(\frac{x}{d}\right) + O(1). \tag{xiii}$$

From (x) and (xiii) we see that Buchstab's result can be improved by the replacement of 2/7 by 1/4. Results (iii) and (x) would suggest the slightly stronger (for $t = 2$) conjecture

$$\sum_{d \leqslant k} d^p P_2\left(\frac{x}{d}\right) = O(x^{p/2 + 1/4}). \tag{xiv}$$

By our Theorem 7, solving the "Dirichlet divisor problem" is equivalent to verifying (ii) for $p = 0$ and $t = 1$, where it is known that the exponent ε cannot be dispensed with (see, e.g., Chandrasekharan [2, Theorem 3, p. 205]).

Thus, it seems that our basic functions are of some independent interest; in any case, the expansions in terms of basic functions are essential to our subsequent study of sums of error terms.

Typical of our results would be the following:

$$\begin{aligned}\sum_{d \leqslant x} d^p P_t\left(\frac{x}{d}\right) &= E_{p,t} x^{p+1} + (-1)^t t \sum_{t-1 \leqslant m \leqslant p} \frac{(-1)^{m+1}}{m+1} x^{p-m} \binom{p-t+1}{m-t+1} \\ &\quad \times \sum_{d \leqslant k} \frac{1}{d^{p-m-t+1}} P_{m+1}\left(\frac{x}{d}\right) + \sum_{d \leqslant k} d^p P_t\left(\frac{x}{d}\right) \\ &\quad + \sum_{0 \leqslant m \leqslant p/2} \frac{(-1)^{m+1}}{m+1} x^{p-m} P_{m+1}(x) \sum_{u=0}^{t} \binom{t}{u} \binom{p-t+u}{m} B_u \\ &\quad + O(x^{p/2}), \qquad 1 \leqslant t \leqslant p,\end{aligned}$$

where

$$E_{p,t} = \sum_{u=0}^{t} \binom{t}{u} \frac{B_u}{p+1+u-t} - \binom{p+1}{t}^{-1} \zeta(p+2-t).$$

In the next section these results are used to obtain some general theorems concerning the sums $\sum_{n \leqslant x} n^t f(n)$ and $\sum_{n \leqslant x} E_t(n)$ and the integral $\int_1^x E_t(u)\,du$, where t is an integer, f is defined for arbitrary numerical functions c by $f(n) = \sum_{d/n} c(d)$, and $E_t(x)$ is the "error term" in the expansion $\sum_{n \leqslant x} n^t f(n)$.

In the following sections, these general theorems are in turn applied to the case where c is defined by $c(d) = d^a$.

For example, when we have $a = 0$, f becomes the standard divisor function, $f(n) = d(n)$, and we show

$$\sum_{n \leqslant x} n^t d(n) = \frac{1}{t+1} x^{t+1} \log x + \frac{1}{t+1}\left(2\gamma - \frac{1}{t+1}\right) x^{t+1}$$
$$- 2x^t \sum_{d \leqslant k} P_1\left(\frac{x}{d}\right) + O(x^t),$$

for $t \geqslant 0$, and on defining $E_t(x)$ by

$$\sum_{n \leqslant x} n^t\, d(n) = \frac{1}{t+1} x^{t+1} \log x + \frac{1}{t+1}\left(2\gamma - \frac{1}{t+1}\right) x^{t+1} + E_t(x),$$

we have

$$\sum_{n \leqslant x} E_t(n) = \frac{1}{2(t+1)} x^{t+1} \log x + \frac{1}{2(t+1)}\left(2\gamma - \frac{1}{t+1} + \frac{1}{2}\left[\frac{1}{t+1}\right]\right) x^{t+1}$$
$$- x^t \sum_{d \leqslant k} dP_2\left(\frac{x}{d}\right) + O(x^{t+1/2})$$

and

$$\int_1^x E_t(u)\,du = \frac{1}{4}\left[\frac{1}{t+1}\right] x^{t+1} - x^t \sum_{d \leqslant k} dP_2\left(\frac{x}{d}\right) + O(x^{t+1/2}),$$

with similar results for negative t, where $[u] = u - \{u\}$ is the integer part of u.

These could, of course, be thought of as providing information on the average order of some of the basic functions; in the above example, for $t = 0$ we see that $\sum_{d \leqslant k} P_1(x/d)$ has average order $-\frac{1}{4}\log x$.

Throughout the paper, no tools more powerful than summation by parts and the Euler-MacLaurin summation formula are required. In general, the proofs are only indicated or sketched. The reader interested in details may request a copy of the extended paper, MacLeod [5], for fairly complete explanations.

2. Bernoulli Polynomials and the Euler Summation Formula

Let the Bernoulli polynomials $B_r(x)$ and the Bernoulli numbers B_r be defined by

$$B_0(x) = 1, \tag{1}$$

$$B'_{q+1}(x) = (q+1)\, B_q(x), \qquad q = 0, 1, 2, \ldots, \tag{2}$$

$$B_r(0) = B_r, \qquad r = 0, 1, 2, \ldots, \tag{3}$$

$$B_r(1) = B_r, \qquad r = 2, 3, 4, \ldots, \tag{4}$$

$$B_1(1) = 1 + B_1. \tag{5}$$

Then by induction on q we have

$$B_q(x) = \sum_{k=0}^{q} \binom{q}{k} B_k x^{q-k}. \tag{6}$$

Putting (4) and (5) into (6) yields the recursion

$$\sum_{k=0}^{q-1} \binom{q}{k} B_k = 0, \qquad q = 2, 3, 4, \ldots$$
$$= 1, \qquad q = 1. \tag{7}$$

From (7) we obtain

$$B_0 = 1, \qquad B_1 = -\tfrac{1}{2}, \qquad B_2 = \tfrac{1}{6}$$
$$B_{2k+1} = 0, \qquad k = 1, 2, 3, \ldots. \tag{8}$$

It also follows from (6) that

$$B_q(x+1) = \sum_{k=0}^{q} \binom{q}{k} B_k(x). \tag{9}$$

Let the functions P_r be defined by

$$P_r(x) = B_r(\{x\}), \tag{10}$$

where as usual $\{x\} = x - [x]$ denotes the fractional part of x. The following inequalities are well known (see, e.g., Rademacher [6, p. 18]).

$$|P_{2r}(x)| \leqslant |B_{2r}| \leqslant \frac{(2r)!}{12(2\pi)^{2r-2}}, \tag{11a}$$

$$|P_r(x)| \leqslant \frac{r!}{12(2\pi)^{r-2}}. \tag{11b}$$

We shall make use of the following forms of the Euler–MacLaurin summation formula, proven by induction and integration by parts, together with (8), (10) and (11).

THEOREM (Euler–MacLaurin). *Let a and b be real numbers, and k a positive integer, with $a < b$, and suppose that f and its first k derivatives are continuous on $[a, b]$. Then*

$$\sum_{1+[a] \leqslant n \leqslant b} f(n)$$
$$= \int_a^b f(t)\,dt + \sum_{r=1}^{k} \frac{(-1)^r}{r!} (P_r(b) f^{(r-1)}(b) - P_r(a) f^{(r-1)}(a))$$
$$+ \frac{(-1)^{k+1}}{k!} \int_a^b P_k(t) f^{(k)}(t)\,dt.$$

COROLLARY 1. *Let x be real, $x > 1$, and k a positive integer, and suppose that f and its first k derivatives are continuous on $[1, x]$. Then*

$$\sum_{n \leqslant x} f(n) = \int_1^x f(t)\,dt + \sum_{r=1}^{k} \frac{(-1)^r}{r!} P_r(x) f^{(r-1)}(x) - \sum_{r=1}^{k} \frac{B_r}{r!} f^{(r-1)}(1)$$
$$+ \frac{(-1)^{k+1}}{k!} \int_1^x P_k(t) f^{(k)}(t)\,dt.$$

If in addition $\int_1^\infty |f^{(k)}(t)|\,dt$ converges, then

$$\sum_{n \leqslant x} f(n) = \int_1^x f(t)\,dt + A + \sum_{r=1}^{k} (-1)^r \frac{P_r(x)}{r!} f^{(r-1)}(x)$$
$$+ \frac{(-1)^k}{k!} \int_x^\infty P_k(t) f^{(k)}(t)\,dt,$$

where

$$A = -\sum_{r=1}^{k} \frac{B_r}{r!} f^{(r-1)}(1) + \frac{(-1)^{k+1}}{k!} \int_1^\infty P_k(t) f^{(k)}(t)\,dt$$

and

$$\sum_{n>x} f(n)=\int_x^{\infty} f(t)\,dt+\sum_{r=1}^{k}(-1)^{r+1}\frac{P_r(x)}{r!}f^{(r-1)}(x)$$
$$+\frac{(-1)^{k+1}}{k!}\int_x^{\infty} P_k(t)f^{(k)}(t)\,dt.$$

COROLLARY 2. *Let N and k be positive integers, and suppose that f and its first $2k$ derivatives are continuous on $[1,x]$. Then*

$$\sum_{n\leqslant N} f(n)=\int_1^{N} f(t)\,dt+\frac{f(N)}{2}+\sum_{r=1}^{k}\frac{B_{2r}}{(2r)!}f^{(2r-1)}(N)$$
$$+\frac{f(1)}{2}-\sum_{r=1}^{k}\frac{B_{2r}}{(2r)!}f^{(2r-1)}(1)$$
$$-\frac{1}{(2k)!}\int_1^{N} P_{2k}(t)f^{(2k)}(t)\,dt.$$

If in addition $\int_1^{\infty}|f^{(2k)}(t)|\,dt$ converges, then

$$\sum_{n\leqslant N} f(n)=\int_1^{N} f(t)\,dt+\frac{f(N)}{2}+\sum_{r=1}^{k}\frac{B_{2r}}{(2r)!}f^{(2r-1)}(N)$$
$$+A+\frac{1}{(2k)!}\int_N^{\infty} P_{2k}(t)f^{(2k)}(t)\,dt,$$

where

$$A=\frac{1}{2}f(1)-\sum_{j=1}^{k}\frac{B_{2j}}{(2j)!}f^{(2j-1)}(1)-\frac{1}{(2k)!}\int_1^{\infty} P_{2k}(t)f^{(2k)}(t)\,dt.$$

3. SOME COMBINATORIAL LEMMAS

LEMMA 1. *Interchange of summation.*

(i) *Let $b\leqslant a\leqslant n$. Then*

$$\sum_{r=a}^{n}\sum_{s=b}^{r} f(r,s)=\sum_{s=a}^{n}\sum_{r=s}^{n} f(r,s)+\sum_{s=b}^{a-1}\sum_{r=a}^{n} f(r,s).$$

(ii) *Let $a\leqslant n\leqslant m$. Then*

$$\sum_{r=a}^{n}\sum_{s=r}^{m} f(r,s)=\sum_{s=a}^{n}\sum_{r=a}^{s} f(r,s)+\sum_{s=n+1}^{m}\sum_{r=a}^{n} f(r,s).$$

In particular,

$$\sum_{r=a}^{n} \sum_{s=a}^{r} f(r, s) = \sum_{s=a}^{n} \sum_{r=s}^{n} f(r, s).$$

(iii) *Let $b \geqslant 0$. Then*

$$\sum_{r=a}^{n} \sum_{s=a}^{r+b} f(r, s) = \sum_{s=a+b}^{n+b} \sum_{r=s-b}^{n} f(r, s) + \sum_{s=a}^{a+b-1} \sum_{r=a}^{n} f(r, s).$$

LEMMA 2. *Let m, n, and p be non-negative integers. Then*

(i)
$$\sum_{(k)} (-1)^k \binom{n-k}{m-k} \binom{p}{k} = \binom{n-p}{m}, \qquad n \geqslant p$$
$$= (-1)^m \binom{p-n+m-1}{m}, \qquad n < p,$$

(ii)
$$\sum_{(k)} (-1)^{k+m} \binom{n+k}{k} \binom{p+m}{p+k} = \binom{n-p}{m}, \qquad n \geqslant p$$
$$= (-1)^m \binom{p-n+m-1}{m}, \qquad n < p,$$

(iii)
$$\sum_{(k)} (-1)^{k+m} \binom{m}{k} \binom{n+p+k}{p+k} = \binom{n+p}{m+p},$$

where all three sums are over the whole range.

Proofs. Lemma 1 is standard, and Lemma 2 appears on pp. 8–11 of Riordan [7].

LEMMA 3. Let p, s, t *and* v be integers, with $0 \leqslant s \leqslant t-p$ and $1 \leqslant v \leqslant t-s+1$. *Then*

(i)
$$\sum_{r=s}^{t-p} (-1)^{r+s} \binom{t}{r} \binom{r}{s} \binom{t-p-r+v-1}{t-p-r}$$
$$= (-1)^{t-p-s} \binom{t}{p} \binom{t-v-s}{t-p-s}, \qquad 1 \leqslant v \leqslant p$$
$$= 0, \qquad p+1 \leqslant v \leqslant t-s$$
$$= \binom{t}{s}, \qquad v = t-s+1,$$

$$\text{(ii)}\quad \sum_{r=s}^{t} (-1)^r \binom{t}{r}\binom{r}{s} = 0, \qquad t > s$$
$$= (-1)^t, \qquad t = s.$$

Proof. This follows easily from Lemma 2 on replacing $\binom{t}{r}\binom{r}{s}$ by $\binom{t}{s}\binom{t-s}{t-r}$ and changing the dummy variables.

LEMMA 4. *Let the sequences $\{a_m\}$ and $\{z_m\}$ be defined as follows:*

$$a_p(0) = z_p(0) = 0, \qquad a_0(n) = \frac{1}{n}, \qquad a_{p+1}(n) = \sum_{k=1}^{n} a_p(k),$$

$$z_0(n) = 1, \qquad z_{p+1}(n) = \sum_{k=1}^{n} z_p(k),$$

for $p = 0, 1, 2,\ldots,$ $n = 1, 2,\ldots.$ Then

$$\text{(i)}\quad a_p(n) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n+p-1}{k+p-1} \frac{1}{k},$$

$$\text{(ii)}\quad z_p(n) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n+p}{k+p} = \frac{n}{n+p} \binom{n+p}{p},$$

$$\text{(iii)}\quad pa_{p+1}(n) = (n+p)\, a_p(n) - z_p(n),$$

$$\text{(iv)}\quad na_{p+1}(n) = (n+p)\, a_{p+1}(n-1) + z_p(n),$$

$$\text{(v)}\quad a_{p+1}(n) = \binom{n+p}{p} \left(a_1(n) - \sum_{t=1}^{p} \frac{z_t(n)}{t \binom{n+t}{t}} \right)$$
$$= \binom{n+p}{p} (a_1(n+p) - a_1(p)),$$

$$\text{(vi)}\quad z_n(p) + z_p(n) = \binom{n+p}{p},$$

$$\text{(vii)}\quad a_{p+1}(n) = \sum_{k=1}^{n} a_0(k)\, z_p(n+1-k),$$

$$\text{(viii)}\quad a_{p+1}(n) = \sum_{k=1}^{n} \binom{n+p-k}{p} a_0(k),$$

$$\text{(ix)}\quad a_{p+1}(n) - a_{n+1}(p) = \binom{n+p}{p} (a_1(n) - a_1(p)).$$

Proof. These follow easily by induction, Lemmas 1 and 2, and earlier parts, where appropriate.

LEMMA 5. *Let w, p, and n be non-negative integers. Define $c_w^p(n)$ and $d_w^p(n)$ by*

$$c_w^p(n) = \sum_{v=0}^{n} (-1)^v \binom{n}{v} \binom{p-v}{w} \frac{1}{p-v}, \qquad p > n,$$

$$d_w^p(n) = \sum_{v=0}^{n} (-1)^v \binom{n}{v} \binom{p+v}{w} \frac{1}{p+v}.$$

Then we have

$$c_w^p(n) = \frac{(-1)^n}{\binom{p}{n}(p-n)}, \qquad w=0$$

$$= \frac{1}{w} \binom{p-n-1}{p-w}, \qquad w \geqslant 1.$$

$$d_w^p(n) = \frac{1}{p} \frac{1}{\binom{n+p}{n}}, \qquad w=0$$

$$= (-1)^n \frac{1}{w} \binom{p-1}{w-1+n}, \qquad w \geqslant 1.$$

In particular,

$$c_w^p(n) = 0, \qquad 1 \leqslant w \leqslant n.$$

Proof. For $w \geqslant 1$, the results follow easily from Lemma 2(i), while for $w=0$ and $n \geqslant 1$, it suffices to replace $\binom{n}{v}$ by $\binom{n-1}{v} + \binom{n-1}{v-1}$.

COROLLARY 5.1. *Let p, s, and t be non-negative integers.*

(i) *If $t-p \leqslant s$, then*

$$\sum_{r=s}^{t} (-1)^{r+s} \binom{t}{r} \binom{r}{s} \frac{1}{p+1+r-t} = \frac{1}{p+1} \frac{\binom{t}{s}}{\binom{p}{t-s}}.$$

In particular,

$$\sum_{r=0}^{t} (-1)^r \binom{t}{r} \frac{1}{p+1+r-t} = \frac{1}{(p+1-t)\binom{p+1}{t}}.$$

(ii) *If $s \leqslant t-p$, then*

$$\sum_{r=t-p}^{t} (-1)^{r+s} \binom{t}{r} \binom{r}{s} \frac{1}{p+1+r-t} = (-1)^{p+s+t} \binom{t}{s} a_{t-s-p}(p+1).$$

In particular,

$$\sum_{r=t-p}^{t} (-1)^{r} \binom{t}{r} \frac{1}{p+1+r-t} = (-1)^{p+t} a_{t-p}(p+1).$$

(iii) *If $s \leqslant t-p$, then*

$$\sum_{r=s}^{t-p-2} (-1)^{r+s} \binom{t}{r} \binom{r}{s} \frac{1}{t-p-r-1} = (-1)^{p+s+t} \binom{t}{s} a_{p+2}(t-s-p-1).$$

In particular,

$$\sum_{r=0}^{t-p-2} (-1)^{r} \binom{t}{r} \frac{1}{t-p-r-1} = (-1)^{p+t} a_{p+2}(t-p-1)$$

and

$$\sum_{r=1}^{t-p-2} (-1)^{r} \binom{t}{r} \frac{1}{t-p-r-1} = (-1)^{p+t} a_{p+2}(t-p-1) - \frac{1}{t-p-1}.$$

(iv) *If $p \geqslant 2$, then*

$$\sum_{r=s}^{t} (-1)^{r+s} \binom{t}{r} \binom{r}{s} \frac{1}{p+t-r-1} = (-1)^{s+t} \frac{1}{p-1} \frac{\binom{t}{s}}{\binom{t-s-p-1}{p-1}}$$

$$= (-1)^{s+t} \frac{1}{p-1} \frac{\binom{t+p-1}{s}}{\binom{t+p-1}{t}}.$$

4. Some Number-Theoretic Lemmas

Lemma 6. *Let c be any numerical function and t a positive integer, and define $C_r(x)$ by*

$$C_r(x) = \sum_{n \leqslant x} c(n)\, n^r.$$

Then for any integer k such that $1 \leqslant k \leqslant [x]$,

$$\sum_{n \leqslant x} c(n) \left\{\frac{x}{n}\right\}^t$$

$$= \sum_{n \leqslant x/k} c(n) \left\{\frac{x}{n}\right\}^t \left(C_{-t}(x) - C_{-t}\left(\frac{x}{k}\right)\right)$$

$$+ \sum_{r=1}^{t} \binom{t}{r} (-1)^r x^{t-r} \left(\sum_{s=1}^{r} \binom{r}{s} (-1)^{s+1} \sum_{d \leqslant k} d^{r-s} C_{-(t-r)}\left(\frac{x}{d}\right)\right.$$

$$\left. - k^r C_{-(t-r)}\left(\frac{x}{p}\right)\right).$$

Proof. In the interval $x/(d+1) < n \leqslant x/d$, we have $[x/n] = d$. Hence

$$\sum_{n \leqslant x} c(n) \left\{\frac{x}{n}\right\}^t$$

$$= \sum_{n \leqslant x/k} c(n) \left\{\frac{x}{n}\right\}^t + \sum_{d=1}^{k-1} \sum_{x/(d+1) < n \leqslant x/d} c(n) \left(\frac{x}{n} - d\right)^t$$

$$= \sum_{n \leqslant x/k} c(n) \left\{\frac{x}{n}\right\}^t + \sum_{d=1}^{k-1} \sum_{x/(d+1) < n \leqslant x/d} c(n) \sum_{r=0}^{t} (-1)^r d^r \binom{t}{r} \left(\frac{x}{n}\right)^{t-r}$$

$$= \sum_{n \leqslant x/k} c(n) \left\{\frac{x}{n}\right\}^t + \sum_{r=0}^{t} \binom{t}{r} (-1)^r x^{t-r}$$

$$\times \sum_{d=1}^{k-1} d^r \left(C_{-(t-r)}\left(\frac{x}{d}\right) - C_{-(t-r)}\left(\frac{x}{d+1}\right)\right).$$

From summation by parts we have

$$\sum_{d=1}^{k-1} d^r \left(C_{-(t-r)}\left(\frac{x}{d}\right) - C_{-(t-r)}\left(\frac{x}{d+1}\right)\right)$$

$$= \sum_{d=1}^{k} \sum_{s=1}^{r} (-1)^{s+1} d^{r-s} C_{-(t-r)}\left(\frac{x}{d}\right) - k^r C_{-(t-r)}\left(\frac{x}{k}\right), \qquad r \geqslant 1$$

$$= C_{-t}(x) - C_{-t}\left(\frac{x}{k}\right), \qquad r = 0.$$

Hence

$$\sum_{n\leqslant x} c(n)\left\{\frac{x}{n}\right\}^t = \sum_{n\leqslant x/k} c(n)\left\{\frac{x}{n}\right\}^t + x^t\left(C_{-t}(x) - C_{-t}\left(\frac{x}{k}\right)\right)$$
$$+\sum_{r=1}^{t}\binom{t}{r}(-1)^r x^{t-r}\left(\sum_{s=1}^{r}\binom{r}{s}(-1)^{s+1}\right.$$
$$\left.\times\sum_{d\leqslant k} d^{r-s}C_{-(t-r)}\left(\frac{x}{d}\right) - k^r C_{-(t-r)}\left(\frac{x}{k}\right)\right).$$

COROLLARY 6.1. *Let c be any numerical function with $c(1)=1$, let t be a positive integer, and define $C_r(x)$ by*

$$C_r(x) = \sum_{n\leqslant x} c(n)\, n^r.$$

Then

$$\sum_{n\leqslant x} c(n)\left\{\frac{x}{n}\right\}^t = x^t C_{-t}(x) + \sum_{r=1}^{t}\binom{t}{r}(-1)^r x^{t-r}$$
$$\times\sum_{s=1}^{r}\binom{r}{s}(-1)^{s+1}\sum_{d\leqslant x} d^{r-s}C_{-(t-r)}\left(\frac{x}{d}\right).$$

Proof. Put $k=[x]$ in Lemma 6, and use Lemma 2.

COROLLARY 6.2. *Let p and t be integers, with t positive, and define $L_r(x)$ by*

$$L_r(x) = \sum_{n\leqslant x} n^r.$$

Then

$$\sum_{n\leqslant x} n^p\left\{\frac{x}{n}\right\}^t = x^t L_{p-t}(x) + \sum_{r=1}^{t}\binom{t}{r}(-1)^r x^{t-r}$$
$$\times\sum_{s=1}^{r}\binom{r}{s}(-1)^{s+1}\sum_{d\leqslant x} d^{r-s}L_{p+r-t}\left(\frac{x}{d}\right).$$

LEMMA 7. *Let u, k and l be integers, with $k\geqslant 1$, $l\geqslant 1$. Define $L_u(x)$ by*

$$L_u(x) = \sum_{d\leqslant x} d^u.$$

Then, for arbitrary positive integer l,

$$\text{(i)}\quad L_u(x) = \frac{1}{u+1} \sum_{r=0}^{u+1} (-1)^r \binom{u+1}{r} P_r(x)\, x^{u+1-r} - \frac{1}{u+1} B_{u+1} - \left[\frac{1}{u+1}\right], \qquad u \geqslant 0$$

$$= \frac{1}{u+1} x^{u+1} - \sum_{r=0}^{u} \frac{(-1)^r}{r+1} \binom{u}{r} P_{r+1}(x)\, x^{u-r} - \frac{1}{u+1} B_{u+1} - \left[\frac{1}{u+1}\right], \qquad u \geqslant 0.$$

$$\text{(ii)}\quad L_u(k) = \frac{1}{u+1} \sum_{r=0}^{u} (-1)^r \binom{u+1}{r} B_r k^{u+1-r}, \qquad u \geqslant 0$$

$$= \frac{1}{u+1} \sum_{r=0}^{u} \binom{u+1}{r} B_r k^{u+1-r} + k^u, \qquad u \geqslant 1.$$

$$\text{(iii)}\quad L_{-1}(x) = \log x + \gamma - \sum_{r=1}^{l} \frac{1}{r} P_r(x)\, x^{-r} + O(x^{-(l+1)}).$$

$$\text{(iv)}\quad L_{-u}(x) = \zeta(u) - \frac{1}{u-1} x^{-(u-1)} - \sum_{r=1}^{l} \frac{1}{r} \binom{u+r-2}{r-1} \times P_r(x)\, x^{-(u+r-1)} + O(x^{-(u+l)}), \qquad u \geqslant 2.$$

$$\text{(v)}\quad L_{-1}(k) = \log k + \gamma - \sum_{r=1}^{l} \frac{1}{r} B_r k^{-r} + O(k^{-(l+1)}).$$

$$\text{(vi)}\quad L_{-u}(k) = \zeta(u) - \frac{1}{u-1} k^{-(u-1)} - \frac{1}{u-1} \sum_{r=1}^{l} \binom{u+r-2}{r} B_r k^{-(u+r-1)} + O(k^{-(u+l)}), \qquad u \geqslant 2.$$

Proof. These follow readily from the Euler–MacLaurin formulas, together with the properties of Bernoulli numbers in the Introduction.

LEMMA 8. *Let t and n be non-negative integers. Then*

$$\sum_{s=1}^{t} \binom{t+n}{s} (-1)^{s+1} L_{t-s}(k) = \sum_{w=1}^{t} l_t^n(w)\, k^w,$$

where

$$l_t^n(w) = (-1)^{t+w} \frac{(t+n)!}{w!\,n!\,(t-w+1)!} \sum_{v=0}^{t-w} \frac{\binom{t-w+1}{v}}{\binom{v+w-1+n}{n}} B_v. \tag{12}$$

In particular,

$$\sum_{s=1}^{t} \binom{t}{s} (-1)^{s+1} L_{t-s}(k) = k^t,$$

$$\sum_{s=1}^{t} \binom{t+1}{s} (-1)^{s+1} L_{t-s}(k) = \sum_{w=1}^{t} l_t(w)\, k^w, \tag{13}$$

where

$$l_t(w) = (-1)^{t+w} \binom{t+1}{w} \sum_{v=0}^{t-w} \binom{t+1-w}{v} \frac{B_v}{v+w}. \tag{14}$$

Proof. This follows from Lemma 7(ii), Lemma 1, and (7).

LEMMA 9. *Let p and t be non-negative integers with $t \geqslant p+1$. Then*

$$\sum_{r=1}^{t-p-1} \sum_{s=1}^{r} (-1)^{s+r} \frac{1}{t-r-p} \binom{t}{r} \binom{r}{s} L_{t-s-p}(k) = \sum_{w=1}^{t-p} e_t^p(w)\, k^w,$$

where

$$e_t^p(w) = (-1)^w \sum_{r=0}^{t-p-w} \frac{1}{r+w} \binom{t}{r+p+w-1} \binom{r+w}{r} B_r a_{p+1}(r+w-1)$$

$$= (-1)^w \sum_{s=w}^{t-p} \frac{1}{s} \binom{t}{s+p-1} \binom{s}{w} B_{s-w} a_{p+1}(s-1).$$

Proof. This follows moderately easily from Lemmas 7 and 4(i), with summation interchanges.

LEMMA 10. *Let p and t be positive integers, with $p > t$. Then*

$$\sum_{r=1}^{t} \sum_{s=1}^{r} (-1)^{s+r} \frac{1}{p+r-t} \binom{t}{r} \binom{r}{s} L_{-(p+s-t)}(k)$$

$$= \frac{t!}{p!} \sum_{m=0}^{t-1} \zeta(p-m) \frac{(p-m-1)!}{(t-m)!}$$

$$- \frac{1}{(p-t)\binom{p}{t}} \frac{1}{k^{p-t}} + \frac{1}{p(t+1)} \frac{1}{k^p} + O\left(\frac{1}{k^{p+1}}\right).$$

Proof. From Lemmas 1 and 5, we obtain

$$\sum_{r=1}^{t}\sum_{s=1}^{r}(-1)^{s+r}\frac{1}{p+r-t}\binom{t}{r}\binom{r}{s}L_{-(p+s-t)}(k)$$
$$=\frac{t!}{p!}\sum_{m=0}^{t-1}L_{-(p-m)}(k)\frac{(p-m-1)!}{(t-m)!}.$$

The result follows from Lemma 7(vi) after using (7) and making use of the induced telescoping effect.

LEMMA 11. *Let t be a positive integer. Then*

$$\sum_{r=1}^{t}\sum_{s=1}^{r}(-1)^{r+s}\frac{1}{r}\binom{t}{r}\binom{r}{s}L_{-s}(k)$$
$$=\log k+\left(\gamma+\sum_{s=2}^{t}\frac{\zeta(s)}{s}\right)+\frac{1}{t(t+1)}\frac{1}{k^t}+O\left(\frac{1}{k^{t+1}}\right).$$

Proof. From Lemmas 1 and 5, we have

$$\sum_{r=1}^{t}\sum_{s=1}^{r}(-1)^{r+s}\frac{1}{r}\binom{t}{r}\binom{r}{s}L_{-s}(k)=L_{-1}(k)+\sum_{s=2}^{t}\frac{1}{s}L_{-s}(k).$$

The result follows from Lemmas 7 and 1.

LEMMA 12. *Let p and t be positive integers, with $p < t$. Then*

$$\sum_{r=t-p}^{t}\sum_{s=1}^{r}(-1)^{s+r}\frac{1}{p+1+r-t}\binom{t}{r}\binom{r}{s}L_{t-p-1-s}(k)$$
$$=\sum_{m=1}^{t-p-1}(-1)^m k^m\sum_{s=m}^{t-p-1}\binom{t}{p+s}\binom{s}{m}\frac{1}{s}B_{s-m}a_s(p+1)$$
$$+\frac{1}{p+1}\binom{t}{p}\left\{\log k+D_{p,t}+\sum_{m=1}^{p+1}\frac{1}{mk^m}\sum_{s=1}^{t-p-1}\frac{\binom{t-p}{s}}{\binom{m+s}{s}}B_{m+s}\right\}$$
$$+\frac{1}{(p+1)(t+1)}\frac{1}{k^{p+1}}+O\left(\frac{1}{k^{p+2}}\right),$$

where

$$D_{p,t}=\gamma+\sum_{m=1}^{p}\frac{\zeta(m+1)}{\binom{m+t-p}{m}}.$$

Proof. This is similar to the proof in Lemma 11, if slightly more messy.

LEMMA 13. *Let u be a non-negative integer, and define $M_u(x)$ by*

$$M_u(x) = \sum_{d \leqslant x} d^u \log d.$$

Then we have, for arbitrary positive integer l,

$$\text{(a)} \quad M_u(x) = \frac{1}{u+1} \sum_{r=0}^{u} (-1)^r \binom{u+1}{r} \times P_r(x)\, x^{u+1-r}(\log x - a_1(u) - a_1(u+1-r)) + \frac{(-1)^{u+1}}{u+1} P_{u+1}(x) \log x + A_u + (-1)^u \sum_{r=u+2}^{l} \frac{u!(r-u-2)!}{r!} P_r(x) \frac{1}{x^{l-u-1}} + O\left(\frac{1}{x^{l-u-1}}\right);$$

$$\text{(b)} \quad M_{-1}(x) = \frac{1}{2} \log^2 x + A_{-1} + \sum_{r=1}^{l} \frac{P_r(x)}{r} x^{-r}(a_1(r-1) - \log x) + O(x^{-l} \log x);$$

$$\text{(c)} \quad M_{-u}(x) = -\zeta'(u) - \frac{1}{u-1}\left(\log x + \frac{1}{u-1}\right) x^{-(u-1)} - \sum_{r=1}^{l} \binom{u+r-2}{u-1} \frac{P_r(x)}{r} x^{-(u+r-1)} \times (\log x + a_1(u-1) - a_1(u+r-1)) + O(x^{-(u+l-1)} \log x), \qquad u \geqslant 2.$$

Proof. These follow easily from the Euler–MacLaurin formulas.

Note. We have by Stirling's formula that

$$A_0 = \tfrac{1}{2} \log 2\pi.$$

The other constants A_l do not appear to be well known in terms of known elementary constants. However, they may readily be calculated to a required degree of accuracy by the Euler–MacLaurin formula. For example, we compute A_2 to at least eight digits as follows:

$$\sum_{d\leqslant x} d^2 \log d$$
$$= \int_1^x t^2 \log t\, dt + A_2 + \sum_{r=1}^{8} (-1)^r \frac{P_r(x)}{r!} f^{(r-1)}(x)$$
$$+ \frac{1}{8!}\int_x^{\infty} P_8(t) f^{(8)}(t)\, dt$$
$$= \frac{x^3 \log x}{3} - \frac{x^3}{9} + \frac{1}{9} + A_2 + \frac{x^2 \log x}{2} + \frac{x}{12} + \frac{x \log x}{6} - \frac{1}{720}\frac{2}{x}$$
$$+ \frac{1}{30{,}240}\frac{4}{x^3} - \frac{1}{1{,}209{,}600}\frac{48}{x^5} + \frac{1}{8!}\int_x^{\infty} P_8(t)\left(-\frac{240}{t^6}\right) dt.$$

From (11) we have

$$\left|\frac{1}{8!}\int_x^{\infty} P_8(t)\left(-\frac{240}{t^6}\right) dt\right| \leqslant \frac{4}{(2\pi)^6}\frac{1}{x^5} \leqslant \frac{1}{14{,}000}\frac{1}{x^5}.$$

Hence we have

$$\frac{1}{9} + A_2 = \sum_{d\leqslant x} d^2 \log d - \frac{x^3 \log x}{x} + \frac{x^3}{9} - \frac{x^2}{2} - \frac{x}{12} - \frac{x \log x}{6}$$
$$+ \frac{1}{360}\frac{1}{x} - \frac{1}{7560}\frac{1}{x^3},$$

with an error bounded above by

$$\frac{1}{25{,}200}\frac{1}{x^5} + \frac{1}{14{,}000}\frac{1}{x^5} \leqslant \frac{1}{9{,}000}\frac{1}{x^5}.$$

For $x = 10$, we thus have at least eight digits, and to this accuracy,

$$A_2 = 0.03044845.$$

LEMMA 14. *Let p and t be positive integers with $0 \leqslant p < t$. Then*

$$\sum_{s=1}^{t-p} \binom{t-p}{s} (-1)^{s+1} M_{t-p-s}(k)$$
$$= k^{t-p} \log k - \frac{1}{t-p} k^{t-p} + \sum_{v=1}^{t-p-1} m_{t,p}(v)\, k^v$$
$$+ \frac{(-1)^{t-p-1}}{t-p+1} \log k + c_{t,p} + \sum_{w=1}^{p} n_{t,p}(w)\frac{1}{k^w} + O\left(\frac{1}{k^{p+1}}\right),$$

where

$$m_{t,p}(v)=\frac{(-1)^{t-p-v}}{t-p+1}\binom{t-p+1}{v}\sum_{j=0}^{t-p-v}\binom{t-p-v+1}{j}B_j a_1(v+j-1),$$

$$c_{t,p}=(-1)^{t-p-1}\sum_{u=0}^{t-p-1}(-1)^u\binom{t-p}{u}A_u,$$

$$n_{t,p}(w)=\frac{(-1)^{t-p-1}}{(t-p+1)\,w}\sum_{v=1}^{t-p}\frac{\binom{t-p+1}{v}}{\binom{w+v}{v}}B_{w+v}.$$

Proof. After using Lemmas 13 and 1, and (7), and repeated changes of variable, the result follows.

LEMMA 15. *Let p and t be non-negative integers. Then*

$$\sum_{n\leqslant x/k} n^p\left\{\frac{x}{n}\right\}^t$$

$$=\frac{1}{t+1}\sum_{r=1}^{t}\binom{t+1}{r}\sum_{n\leqslant x/k}n^pP_r\left(\frac{x}{n}\right)$$

$$+\frac{1}{(t+1)(p+1)}\sum_{r=1}^{p+1}(-1)^r\binom{p+1}{r}P_r\left(\frac{x}{k}\right)\left(\frac{x}{k}\right)^{p+1-r}$$

$$+\frac{1}{(t+1)(p+t)}\left(\frac{x}{k}\right)^{p+1}-\frac{1}{(t+1)(p+1)}B_{p+1}-\frac{1}{t+1}\left[\frac{1}{p+1}\right].$$

Proof. From (6), (9), and (7),

$$\frac{1}{t+1}\{B_{q+1}(x+1)-B_{t+1}(x)\}=x^t,$$

so that

$$x^t=\frac{1}{t+1}\sum_{k=0}^{t}\binom{t+1}{k}B_k(x).$$

Replacing x by $\{x/n\}$, multiplying by n^p, and summing for $n\leqslant x/k$, and interchanging order of summation, together with Lemma 7(i), gives the result.

5. Sums of Powers

In this section, we obtain expansions of the Euler–MacLaurin type for the sums $\sum_{d\leqslant x} d^p\{x/d\}^t$ for integers p and t, $t>0$. Perhaps not surprisingly, the answer varies somewhat depending upon the relation between p and t, and depending upon whether p is negative or non-negative.

Theorem 1. *Let p and t be non-negative integers. Let $k=[\sqrt{x}]$. Then*

$$\begin{aligned}
\sum_{d\leqslant x} d^p \left\{\frac{x}{d}\right\}^t &= C_{p,t}x^{p+1} + \sum_{m=0}^{p} \frac{(-1)^m}{m+1} x^{p-m} \\
&\quad \times \sum_{w=0}^{t-1} (-1)^w \binom{t}{w}\binom{p-w}{m-w} \sum_{d\leqslant k} \frac{1}{d^{p-m-w}} P_{m+1}\left(\frac{x}{d}\right) \\
&\quad + \frac{1}{t+1}\sum_{m=1}^{t}\binom{t+1}{m}\sum_{d\leqslant k} d^p P_m\left(\frac{x}{d}\right) \\
&\quad + \sum_{0\leqslant m\leqslant p/2} \frac{(-1)^{m+1}}{m+1}\binom{p-t}{m} x^{p-m}P_{m+1}(x) \\
&\quad + O(x^{p/2}), \qquad 0\leqslant t\leqslant p+1 \\
&= C_{p,t}x^{p+1} + \sum_{m=0}^{p} \frac{(-1)^m}{m+1} x^{p-m} \\
&\quad \times \sum_{w=0}^{m} (-1)^w \binom{t}{w}\binom{p-w}{m-w} \sum_{d\leqslant k} \frac{1}{d^{p-m-w}} P_{m+1}\left(\frac{x}{d}\right) \\
&\quad + \frac{1}{t+1}\sum_{m=1}^{t}\binom{t+1}{m}\sum_{d\leqslant k} d^p P_m\left(\frac{x}{d}\right) \\
&\quad + \frac{1}{t+1}\sum_{m=p+2}^{t}\binom{t+1}{m}\frac{1}{x^{m-(p+1)}}\sum_{d\leqslant k} d^{2m-(p+2)} P_m\left(\frac{x}{d}\right) \\
&\quad - \sum_{0\leqslant m\leqslant p/2} \frac{1}{m+1}\binom{t-p-1+m}{m} x^{p-m}P_{m+1}(x) \\
&\quad + O(x^{p/2}), \qquad t\geqslant p+2,
\end{aligned}$$

$$\begin{aligned}
\sum_{d\leqslant x} \frac{1}{d}\left\{\frac{x}{d}\right\}^t &= \frac{1}{t+1}\log x + C_{-1,t} + \frac{1}{t+1}\sum_{m=1}^{t}\binom{t+1}{m}\sum_{d\leqslant k}\frac{1}{d}P_m\left(\frac{x}{d}\right) \\
&\quad + \frac{1}{t+1}\sum_{m=1}^{t}\binom{t+1}{m}\frac{1}{x^m}\sum_{d\leqslant k} d^{2m-1}P_m\left(\frac{x}{d}\right) + O(x^{-1/2}),
\end{aligned}$$

$$\sum_{d\leqslant x}\frac{1}{d^p}\left\{\frac{x}{d}\right\}^t = C_{-p,t}+\frac{1}{t+1}\sum_{m=1}^{t}\binom{t+1}{m}\sum_{d\leqslant k}\frac{1}{d^p}P_m\left(\frac{x}{d}\right)$$
$$+\frac{1}{t+1}\sum_{m=1}^{t}\binom{t+1}{m}\frac{1}{x^{p+m-1}}\sum_{d\leqslant k}d^{2m+p-2}P_m\left(\frac{x}{d}\right)$$
$$+O(x^{-p/2}),\qquad p\geqslant 2,$$

where

$$C_{p,t}=\frac{1}{p+1-t}-\frac{t!}{(p+1)!}\sum_{m=0}^{t-1}\frac{(p-m)!}{(t-m)!}\zeta(p+1-m),\qquad 0\leqslant t\leqslant p$$
$$=a_1(p+1)-\gamma-\sum_{m=0}^{p-1}\frac{1}{p+1-m}\zeta(p+1-m),\qquad t=p+1$$
$$=\frac{1}{p+1-t}-\frac{\gamma}{p+1}\binom{t}{p}$$
$$-\frac{t!}{(p+1)!}\sum_{m=0}^{p-1}\frac{(p-m)!}{(t-m)!}\zeta(p+1-m)$$
$$+\left(\frac{1}{p+1}\right)\sum_{m=0}^{t-p-2}(-1)^m\binom{t-p-1}{m}A_m,\qquad t\geqslant p+2,$$

$$C_{-p,t}=-\frac{1}{t}+\frac{\gamma}{t+1}+\sum_{m=0}^{t-1}(-1)^m\binom{t}{m}A_m,\qquad p=1$$
$$=\frac{1}{t+1}\zeta(p),\qquad p\geqslant 2,$$

$$a_1(n)=\sum_{m=1}^{n}\frac{1}{m},$$

and A_m is the constant term in the Euler–MacLaurin expansion of

$$\sum_{d\leqslant k}d^m\log d.$$

Proof. For $t=0$, the result appears in Lemma 7. In [5] we give all details of the proofs for $1\leqslant t\leqslant p$ and $t\geqslant p+2$ (some 15 pages). The case $t=p+1$ is similar to that for $1\leqslant t\leqslant p$, while the cases involving negative powers of x are similar to, but easier than, the case $t\geqslant p+2$.

We see that, on putting $c(n)=n^p$ in Lemma 6, we obtain

$$\sum_{n\leqslant x} n^p \left\{\frac{x}{n}\right\}^t = \sum_{n\leqslant x/k} n^p \left\{\frac{x}{n}\right\}^t + x^t \left(L_{p-t}(x) - L_{p-t}\left(\frac{x}{k}\right)\right)$$
$$+ \sum_{r=1}^{t} (-1)^r \binom{t}{r} x^{t-r} \left(\sum_{s=1}^{r} (-1)^{s+1} \binom{r}{s} \sum_{d\leqslant k} d^{r-s} L_{p+r-t}\left(\frac{x}{d}\right)\right.$$
$$\left. - k^r L_{p+r-t}\left(\frac{x}{k}\right)\right). \tag{15}$$

It is then a matter of using the machinery built up in the lemmas, together with careful symbol manipulation, to obtain the results claimed.

LEMMA 16. *Define* $E_{p,t}$ *by* $E_{p,t} = \sum_{u=0}^{t} \binom{t}{u} B_u C_{p,t-u}$. *Then we have*

$$E_{p,t} = \frac{1}{p+1}, \qquad t=0$$
$$= \sum_{u=0}^{t} \binom{t}{u} \frac{B_u}{p+1+u-t} - \binom{p+1}{t}^{-1} \zeta(p+2-t), \qquad 1 \leqslant t \leqslant p$$
$$= \sum_{u=1}^{t} \binom{t}{u} \frac{B_u}{u} + a_1(t) - \gamma, \qquad t = p+1$$
$$= \sum_{\substack{u=0\\ u\neq t-p-1}}^{t} \binom{t}{u} \frac{B_u}{p+1+u-t}$$
$$+ \binom{t}{t-p-1} B_{t-p-1} a_1(p+1)$$
$$+ (-1)^{t-p}\, t \binom{t-1}{p+1} A_{t-p-2}, \qquad t \geqslant p+2.$$

Proof. The proof is based on repeated use of changes of variable, together with (7) and Lemma 1.

THEOREM 2. *Let p and t be non-negative integers. Let* $k = [\sqrt{x}]$. *Then*

$$\sum_{d\leqslant x} d^p P_t\left(\frac{x}{d}\right)$$
$$= E_{p,t} x^{p+1} + (-1)^t\, t \sum_{t-1\leqslant m\leqslant p} \frac{(-1)^{m+1}}{m+1} x^{p-m} \binom{p-t+1}{m-t+1}$$
$$\times \sum_{d\leqslant k} \frac{1}{d^{p-m-t+1}} P_{m+1}\left(\frac{x}{d}\right) + \sum_{d\leqslant k} d^p P_t\left(\frac{x}{d}\right)$$

$$+ \sum_{0 \leqslant m \leqslant p/2} \frac{(-1)^{m+1}}{m+1} x^{p-m} P_{m+1}(x) \sum_{u=0}^{t} \binom{t}{u} \binom{p-t+u}{m} B_u$$

$$+ O(x^{p/2}), \qquad 0 \leqslant t \leqslant p+1$$

$$= E_{p,t} x^{p+1} + \frac{1}{x^{t-p-1}} \sum_{d \leqslant k} d^{2t-p-2} P_t\left(\frac{x}{d}\right) + \sum_{d \leqslant k} d^p P_t\left(\frac{x}{d}\right)$$

$$+ \sum_{0 \leqslant m \leqslant p/2} \frac{(-1)^{m+1}}{m+1} x^{p-m} P_{m+1}(x) \sum_{u=0}^{t} \binom{t}{u} \binom{p-t+u}{m} B_u$$

$$+ O(x^{p/2}), \qquad t \geqslant p+2,$$

$$\sum_{d \leqslant x} \frac{1}{d^p} P_t\left(\frac{x}{d}\right)$$

$$= E_{-p,t} + \sum_{d \leqslant k} \frac{1}{d^p} P_t\left(\frac{x}{d}\right) + \frac{1}{x^{p+t-1}} \sum_{d \leqslant k} d^{2t+p-2} P_t\left(\frac{x}{d}\right) + O(x^{-p/2}),$$

where

$$E_{p,t} = \frac{1}{p+1}, \qquad t = 0$$

$$= \sum_{u=0}^{t} \binom{t}{u} \frac{B_u}{p+1+u-t} - \binom{p+1}{t}^{-1} \zeta(p+2-t), \qquad 1 \leqslant t \leqslant p$$

$$= \sum_{u=1}^{t} \binom{t}{u} \frac{B_u}{u} + a_1(t) - \gamma, \qquad t = p+1$$

$$= \sum_{\substack{u=0 \\ u \neq t-p-1}}^{t} \binom{t}{u} \frac{B_u}{p+1+u-t}$$

$$+ \binom{t}{t-p-1} B_{t-p-1} a_1(p+1)$$

$$+ (-1)^{t-p} t \binom{t-1}{p+1} A_{t-p-2}, \qquad t \geqslant p+2,$$

$$E_{-p,t} = -\sum_{u=0}^{t-1} \binom{t}{u} \frac{B_u}{t-u} + (-1)^{t-1} t A_{t-1}, \qquad p = 1$$

$$= 0, \qquad p \geqslant 2$$

and $a_1(n)$ and A_m are as in the previous theorem.

Proof. The result follows from Theorem 1 and Lemma 16, together with repeated use of (7) and Lemma 1.

6. Some General Theorems

In this section, we obtain some results which enable us to obtain readily expansions for sums involving certain arithmetic functions and for sums of the error terms in these expansions. In the next section, we shall apply these results to functions involving the number of, or sums of powers of, divisors.

We state for reference formulas involving summation by parts.

Lemma 17. *Let g and h be numerical functions, and let $H(x) = \sum_{n \leqslant x} h(n)$. Then*

(i) $\sum_{n \leqslant x} g(n)\, h(n) = \sum_{n \leqslant x} H(n)\, g(n) - (g(n+1)) + g([x]+1)\, H(x)$,

(ii) $\sum_{n \leqslant x} nh(n) = ([x]+1) \sum_{n \leqslant x} h(n) - \sum_{n \leqslant x} \sum_{k \leqslant n} h(k)$,

(iii) $\sum_{n \leqslant x} \sum_{k \leqslant n} k^t f(k) = ([x]+1) \sum_{n \leqslant x} n^t f(n) - \sum_{n \leqslant x} n^{t+1} f(n)$.

Proof. Formula (i) is the standard formula for summation by parts, formula (ii) is the special case $g(n) = n$ in (i), and formula (iii) is obtained from (ii) by putting $h(k) = k^t f(k)$.

Theorem 3. *Let c be any numerical function, let $f(n) = \sum_{d \mid n} c(d)$, and let t be a non-negative integer. Then*

$$\sum_{n \leqslant x} n^t f(n) = \frac{1}{t+1} x^{t+1} \sum_{d \leqslant x} \frac{c(d)}{d}$$

$$+ \frac{1}{t+1} \sum_{r=1}^{t+1} (-1)^r \binom{t+1}{r} x^{t+1-r} \sum_{d \leqslant x} d^{r-1} c(d)\, P_r\left(\frac{x}{d}\right)$$

$$- \left(\frac{1}{t+1} B_{t+1} + \left[\frac{1}{t+1}\right]\right) \sum_{d \leqslant x} d^t c(d).$$

Proof.

$$\sum_{n \leqslant x} n^t f(n) = \sum_{n \leqslant x} n^t \sum_{d \mid n} c(d)$$

$$= \sum_{md \leqslant x} m^t d^t c(d)$$

$$= \sum_{d \leqslant x} d^t c(d) \sum_{m \leqslant x/d} m^t$$

$$= \sum_{d \leqslant x} d^t c(d)\, L_t\left(\frac{x}{d}\right).$$

The result now follows from Lemma 7. (Results for negative t obtained similarly are not very useful in practice.)

THEOREM 4. *Let t be an arbitrary real number, and f any numerical function. Let $\sum_{n \leqslant x} n^t f(n) = g_t(x) + E_t(x)$. Then*

$$\text{(a)} \quad \sum_{n \leqslant x} E_t(n) = x g_t(x) - g_{t+1}(x) - \sum_{n \leqslant x} g_t(n) + (\tfrac{1}{2} - P_1(x))(g_t(x) + E_t(x)) - (E_{t+1}(x) - xE_t(x)),$$

$$\text{(b)} \quad \sum_{n \leqslant x} E_t(n) = x \sum_{n \leqslant x} n^t f(n) - \sum_{n \leqslant x} n^{t+1} f(n) - \sum_{n \leqslant x} g_t(n) + (\tfrac{1}{2} - P_1(x)) \cdot (g_t(x) + E_t(x)),$$

and (Segal [8])

$$\text{(c)} \quad \int_1^x E_t(u)\, du = x \sum_{n \leqslant x} n^t f(n) - \sum_{n \leqslant x} n^{t+1} f(n) - \int_1^x g_t(u)\, du,$$

if g_t is continuous,

$$\text{(d)} \quad \int_1^x E_t(u)\, du = \sum_{n \leqslant x} E_t(n) - \frac{1}{2} g_t(x) + \left(P_1(x) - \frac{1}{2}\right) E_t(x) - \sum_{r=1}^{l-1} \frac{(-1)^r}{(r+1)!} P_{r+1}(x)\, g_t^{(r)}(x) + O(1) + O(|g_t^{(l)}(x)|),$$

if $g_t(x)$ is l times continuously differentiable and of constant sign for sufficiently large x.

Proof. From Lemma 17(iii) we have

$$\sum_{n \leqslant x} \sum_{k \leqslant n} k^t f(k) = ([x] + 1) \sum_{n \leqslant x} n^t f(n) - \sum_{n \leqslant x} n^{t+1} f(n).$$

Hence

$$\sum_{n \leqslant x} E_t(n) + \sum_{n \leqslant x} g_t(n) = ([x] + 1)\, g_t(x) + ([x] + 1)\, E_t(x) - g_{t+1}(x) - E_{t+1}(x).$$

The first two results now follow.

Writing $F_t(x)=\sum_{n\leqslant x} n^t f(n)$, we have from Stieltjes integration by parts

$$\begin{aligned}\sum_{n\leqslant x} n^{t+1} f(n) &= \int_0^x u\, dF_t(u)\\ &= xF_t(x)-\int_0^x F_t(u)\, du\\ &= x\sum_{n\leqslant x} n^t f(n)-\int_1^x g_t(u)\, du-\int_1^x E_t(u)\, du.\end{aligned}$$

This proves part (c). Now, by the Euler–MacLaurin formula,

$$\begin{aligned}\sum_{n\leqslant x} g_t(n) &= \int_1^x g_t(u)\, du - P_1(x)\, g_t(x) + \sum_{r=2}^{l} (-1)^r \frac{P_r(x)}{r!} g_t^{(r-1)}(x)\\ &\quad + O(1) + O(|g_t^{(l)}(x)|).\end{aligned}$$

The fourth result now follows from the second.

THEOREM 5. *Let c be any numerical function, let* $f(n)=\sum_{d\mid n} c(d)$, *let* t *be an integer, and let* g_t *and* E_t *be defined by*

$$\sum_{n\leqslant x} n^t f(n) = g_t(x) + E_t(x).$$

Then

$$\begin{aligned}\sum_{n\leqslant x} E_t(n) &= \frac{1}{(t+1)(t+2)} x^{t+2} \sum_{d\leqslant x} \frac{c(d)}{d} - \sum_{n\leqslant x} g_t(n)\\ &\quad + \frac{1}{t+1} \sum_{r=2}^{t+1} (-1)^r \binom{t+1}{r} x^{t+2-r} \sum_{d\leqslant x} d^{r-1} c(d)\, P_r\left(\frac{x}{d}\right)\\ &\quad - \frac{1}{t+2} \sum_{r=2}^{t+2} (-1)^r \binom{t+2}{r} x^{t+2-r} \sum_{d\leqslant x} d^{r-1} c(d)\, P_r\left(\frac{x}{d}\right)\\ &\quad + \frac{B_{t+2}}{t+2} \sum_{d\leqslant x} d^{t+1} c(d) - x\left(\frac{B_{t+1}}{t+2} + \left[\frac{1}{t+1}\right]\right) \sum_{d\leqslant x} d^t c(d)\\ &\quad + \left(\frac{1}{2} - P_1(x)\right) \sum_{n\leqslant x} n^t f(n), \qquad t\geqslant 0.\end{aligned}$$

Proof. For $t \geqslant 0$, we have from Theorem 3

$$x \sum_{n \leqslant x} n^t f(n) - \sum_{n \leqslant x} n^{t+1} f(n)$$
$$= \frac{1}{t+1} x^{t+2} \sum_{d \leqslant x} \frac{c(d)}{d}$$
$$+ \frac{1}{t+1} \sum_{r=1}^{t+1} (-1)^r \binom{t+1}{r} x^{t+2-r} \sum_{d \leqslant x} d^{r-1} c(d) P_r\left(\frac{x}{d}\right)$$
$$- x\left(\frac{1}{t+1} B_{t+1} + \left[\frac{1}{t+1}\right]\right) \sum_{d \leqslant x} d^t c(d) - \frac{1}{t+2} x^{t+2} \sum_{d \leqslant x} \frac{c(d)}{d}$$
$$- \frac{1}{t+2} \sum_{r=1}^{t+2} (-1)^r \binom{t+2}{r} x^{t+2-r} \sum_{d \leqslant x} d^{r-1} c(d) P_r\left(\frac{x}{d}\right)$$
$$+ \left(\frac{1}{t+2} B_{t+2} + \left[\frac{1}{t+2}\right]\right) \sum_{d \leqslant x} d^{t+1} c(d).$$

The result now follows from Theorem 4 if we combine the first and fourth terms, note that the term for $r = 1$ cancels in the second and fifth terms (this is crucial), and note that $[1/(t+2)]$ is zero for $t \geqslant 0$.

7. Applications to Divisor Functions

LEMMA 18. *Let a and t be integers, $a \geqslant 0$, $t \geqslant 1$. Then*

$$\sum_{r=1}^{t+1} (-1)^r \binom{t+1}{r} E_{r+a-1,r}$$
$$= \frac{B_{t+1}}{t+1} - \frac{1}{t+1} + \gamma + \left[\frac{1}{t+1}\right], \qquad a = 0$$
$$= \frac{B_{t+1}}{t+1+a} - \frac{1}{a} + \frac{t+1}{t+1+a} \zeta(a+1) + \left[\frac{1}{t+1}\right] \frac{t+1}{t+1+a}, \qquad a \geqslant 1.$$

Proof. This depends upon Lemma 16, together with Lemmas 1 and 3.

THEOREM 6. *Let t and a be integers, $t \geqslant 0$, $a \geqslant 1$. Then*

$$\sum_{n \leqslant x} n^t \sigma_a(n) = \frac{\zeta(a+1)}{t+a+1} x^{t+a+1}$$
$$+ \sum_{0 \leqslant m \leqslant a/2} \frac{(-1)^{m+1}}{m+1} \binom{a+t}{m} x^{t+a-m} \sum_{d \leqslant k} \frac{1}{d^{a-m}} P_{m+1}\left(\frac{x}{d}\right)$$
$$- \frac{1}{2(t+1)} x^{t+1} - x^t \sum_{d < k} d^a P_1\left(\frac{x}{d}\right) + O(x^{t+a/2}).$$

Writing

$$\sum_{n\leq x} n^t\sigma_a(n)=\frac{\zeta(a+1)}{t+a+1}x^{t+a+1}+E_t(x),$$

we have

$$\begin{aligned}\sum_{n\leq x} E_t(n)=&\frac{\zeta(a+1)}{2(t+a+1)}x^{t+a+1}-\frac{1}{2(t+1)(t+2)}x^{t+2}\\&+\frac{\zeta(a+1)}{t+a+1}\sum_{1\leq r\leq a/2}\frac{(-1)^r}{r+1}\binom{t+a+1}{r}P_{r+1}(x)\,x^{t+a+1-r}\\&+\sum_{1\leq m\leq (a+1)/2}\frac{(-1)^m}{m+1}\binom{a+t}{m-1}x^{t+a+1-m}\\&\times\sum_{d\leq k}\frac{1}{d^{a-m}}P_{m+1}\left(\frac{x}{d}\right)\\&+\left(\frac{1}{2}-P_1(x)\right)\sum_{0\leq m\leq a/2}\frac{(-1)^{m+1}}{m+1}\binom{a+t}{m}x^{t+a-m}\\&\times\sum_{d\leq k}\frac{1}{d^{a-m}}P_{m+1}\left(\frac{x}{d}\right)-\frac{1}{2}x^t\sum_{d\leq k}d^{a+1}P_2\left(\frac{x}{d}\right)\\&+O(x^{t+a/2+1/2}),\end{aligned}$$

$$\begin{aligned}\int_1^x E_t(u)\,du=&\sum_{1\leq m\leq (a+1)/2}\frac{(-1)^m}{m+1}\binom{a+t}{m-1}x^{t+a+1-m}\sum_{d\leq k}\frac{1}{d^{a-m}}P_{m+1}\left(\frac{x}{d}\right)\\&-\frac{1}{2(t+1)(t+2)}x^{t+2}-\frac{1}{2}x^t\sum_{d\leq k}d^{a+1}P_2\left(\frac{x}{d}\right)\\&+O(x^{t+a/2+1/2}).\end{aligned}$$

In particular, for $a=1$, *where* $\sigma_a(n)=\sigma(n)$, *we have*

$$\begin{aligned}\sum_{n\leq x} n^t\sigma(n)=&\frac{\zeta(2)}{t+2}x^{t+2}-x^{t+1}\sum_{d\leq k}\frac{1}{d}P_1\left(\frac{x}{d}\right)\\&-\frac{1}{2(t+1)}x^{t+1}-x^t\sum_{d\leq k}dP_1\left(\frac{x}{d}\right)+O(x^{t+1/2}),\end{aligned}$$

$$\begin{aligned}\sum_{n\leq x} E_t(n)=&\frac{1}{2(t+2)}\left(\zeta(2)-\frac{1}{t+1}\right)x^{t+2}\\&-\frac{1}{2}x^{t+1}\sum_{d\leq k}P_2\left(\frac{x}{d}\right)-\frac{1}{2}x^t\sum_{d\leq k}d^2P_2\left(\frac{x}{d}\right)\\&+\left(P_1(x)-\frac{1}{2}\right)x^{t+1}\sum_{d\leq k}\frac{1}{d}P_1\left(\frac{x}{d}\right)+O(x^{t+1}),\end{aligned}$$

$$\int_1^x E_t(u)\,du = -\frac{1}{2(t+1)(t+2)}x^{t+2} - \frac{1}{2}x^{t+1}\sum_{d \leqslant k} P_2\left(\frac{x}{d}\right)$$
$$-\frac{1}{2}x^t \sum_{d \leqslant k} d^2 P_2\left(\frac{x}{d}\right) + O(x^{t+1}).$$

Proof. The proof uses Theorem 3, Lemma 7, Theorem 2, Lemmas 18, 3, and 1, and Theorem 5. The details take about nine pages. It is worth noting that by working directly from Theorem 4 instead of using Theorem 5 the result is obtained much more rapidly, but unfortunately the error term is $O(x^{t+1+a/2})$ instead of $O(x^{t+1/2+a/2})$. It is the cancelling of the two terms for $r = 1$ in Theorem 5 that allows the improvement.

THEOREM 7. *Let t be a non-negative integer. Then*

$$\sum_{n \leqslant x} n^t d(n) = \frac{1}{t+1}x^{t+1}\log x + \frac{1}{t+1}\left(2\gamma - \frac{1}{t+1}\right)x^{t+1}$$
$$-2x^t \sum_{d \leqslant k} P_1\left(\frac{x}{d}\right) + O(x^t).$$

Writing

$$\sum_{n \leqslant x} n^t d(n) = \frac{1}{t+1}x^{t+1}\log x + \frac{1}{t+1}\left(2\gamma - \frac{1}{t+1}\right)x^{t+1} + E_t(x),$$

we have

$$\sum_{n \leqslant x} E_t(n) = \frac{1}{2(t+1)}x^{t+1}\log x$$
$$+\frac{1}{2(t+1)}\left(2\gamma - \frac{1}{t+1} + \frac{1}{2}\left[\frac{1}{t+1}\right]\right)x^{t+1}$$
$$-x^t \sum_{d \leqslant k} dP_2\left(\frac{x}{d}\right) + O(x^{t+1/2})$$

and

$$\int_1^x E_t(u)\,du = \frac{1}{4}\left[\frac{1}{t+1}\right]x^{t+1} - x^t \sum_{d \leqslant k} dP_2\left(\frac{x}{d}\right) + O(x^{t+1/2}).$$

Proof. The details are quite similar to the proof of Theorem 6.

8. Negative Powers

We need somewhat different arguments to handle the case where the integer t is negative. First, a revised form of Theorem 3 enables us to handle sums related to $\sum \sigma_a(n)\, n^{-t}$ for $0 \leqslant t \leqslant a$, and then a "back-track" method can be applied where we have $t > a$.

Lemma 19. *Let c be any numerical function, let $f(n) = \sum_{d/n} c(d)$, and let t be any real number. Then*

$$\sum_{n \leqslant x} n^t f(n) = \sum_{d \leqslant x} d^t \sum_{m \leqslant x/d} m^t c(m).$$

In particular, for $c(m) = m^a$, we have

$$\sum_{n \leqslant x} \frac{\sigma_a(n)}{n^t} = \sum_{d \leqslant x} \frac{1}{d^t} L_{a-t}\left(\frac{x}{d}\right).$$

Proof.

$$\begin{aligned} \sum_{n \leqslant x} n^t f(n) &= \sum_{n \leqslant x} n^t \sum_{m/n} c(m) \\ &= \sum_{md \leqslant x} m^t d^t c(m) \\ &= \sum_{d \leqslant x} d^t \sum_{m \leqslant x/d} m^t c(m). \end{aligned}$$

Corollary. *Let a and t be integers, with $0 \leqslant t \leqslant a$. Then*

$$\begin{aligned} &\sum_{n \leqslant x} \frac{\sigma_a(n)}{n^t} \\ &\quad = \frac{1}{a-t+1} x^{a-t+1} \sum_{d \leqslant x} \frac{1}{d^{a+1}} \\ &\qquad + \frac{1}{a-t+1} \sum_{r=1}^{a-t+1} (-1)^r \binom{a-t+1}{r} x^{a-t+1-r} \\ &\qquad \times \sum_{d \leqslant x} \frac{1}{d^{a+1-r}} P_r\left(\frac{x}{d}\right) - \left(\frac{1}{a-t+1} B_{a-t+1} + \left[\frac{1}{a-t+1}\right]\right) \sum_{d \leqslant x} \frac{1}{d^t}. \end{aligned}$$

Proof. This follows directly from Lemma 19 and Lemma 7(i).

LEMMA 20.

(i) $$x \sum_{n \leqslant x} \frac{\sigma(n)}{n} - \sum_{n \leqslant x} \sigma(n)$$
$$= \frac{1}{2}\zeta(2)\, x^2 - \frac{1}{2} x \log x - \frac{1}{2}(\log 2\pi + \gamma - 1)\, x$$
$$- \frac{1}{2}\frac{1}{x} \sum_{d \leqslant k} d^2 P_2\left(\frac{x}{d}\right) - \frac{1}{2} \sum_{d \leqslant k} P_2\left(\frac{x}{d}\right) + O(1),$$

(ii) $$x \sum_{n \leqslant x} \frac{\sigma_2(n)}{n} - \sum_{n \leqslant x} \sigma_2(n)$$
$$= \frac{1}{6}\zeta(3)\, x^3 - \frac{1}{12} x \log x + \sum_{d \leqslant k} \frac{1}{d} P_2\left(\frac{x}{d}\right)$$
$$- \left(\frac{7}{12} + \frac{1}{12}\gamma + 2A_1\right) x - \frac{1}{2}\frac{1}{x} \sum_{d \leqslant k} d^3 P_2\left(\frac{x}{d}\right)$$
$$+ O(\sqrt{x}),$$

(iii) $$x \sum_{n \leqslant x} \frac{\sigma_a(n)}{n} - \sum_{n \leqslant x} \sigma_a(n)$$
$$= \frac{1}{a(a+1)}\zeta(a+1)\, x^{a+1}$$
$$+ \sum_{m=1}^{a-1} \frac{(-1)^m}{m+1} \binom{a-1}{m-1} x^{a-m} \sum_{d \leqslant k} \frac{1}{d^{a-m}} P_{m+1}\left(\frac{x}{d}\right)$$
$$- \frac{1}{2}\frac{1}{x} \sum_{d \leqslant k} d^{a+1} P_2\left(\frac{x}{d}\right) + O(x^{(a-1)/2}), \qquad a \geqslant 3,$$

(iv) $$x \sum_{n \leqslant x} \frac{\sigma_2(n)}{n^2} - \sum_{n \leqslant x} \frac{\sigma_2(n)}{n}$$
$$= \frac{1}{2}\zeta(3)\, x^2 - \frac{1}{2}\zeta(2)\, x + \frac{1}{12}\log x + \left(A_1 + \frac{1}{12}\gamma\right)$$
$$+ \sum_{d \leqslant k} \frac{1}{d} P_2\left(\frac{x}{d}\right) + \frac{1}{x^2} \sum_{d \leqslant k} d^3 P_2\left(\frac{x}{d}\right) + O(x^{-1/2}),$$

(v) $$x \sum_{n \leqslant x} \frac{\sigma_a(n)}{n^2} - \sum_{n \leqslant x} \frac{\sigma_a(n)}{n}$$
$$= \frac{1}{(a-1)\,a}\zeta(a+1)\, x^a$$

$$+\sum_{m=1}^{a-2}\frac{(-1)^m}{m+1}\binom{a-2}{m-1}x^{a-m-1}\sum_{d\leqslant k}\frac{1}{d^{a-m}}P_{m+1}\left(\frac{x}{d}\right)$$

$$-\frac{1}{2}\frac{1}{x^2}\sum_{d\leqslant k}d^{a+1}P_2\left(\frac{x}{d}\right)-\frac{B_{a-1}}{a-1}\zeta(2)\,x$$

$$+\frac{(-1)^{a+1}}{a}\sum_{d\leqslant k}\frac{1}{d}P_a\left(\frac{x}{d}\right)+O(x^{(a-3)/2}),\qquad a\geqslant 3,$$

(vi) $$x\sum_{n\leqslant x}\frac{\sigma_a(n)}{n^t}-\sum_{n\leqslant x}\frac{\sigma_a(n)}{n^{t-1}}$$

$$=\frac{1}{(a-t+1)(a-t+2)}\zeta(a+1)\,x^{a-t+2}$$

$$+\sum_{m=1}^{a-t}\frac{(-1)^m}{m+1}\binom{a-t}{m-1}x^{a-t+1-m}\sum_{d\leqslant k}\frac{1}{d^{a-m}}P_{m+1}\left(\frac{x}{d}\right)$$

$$-\frac{1}{2}\frac{1}{x^t}\sum_{d\leqslant k}d^{a+1}P_2\left(\frac{x}{d}\right)$$

$$-\left(\frac{B_{a-t+1}}{a-t+1}+\left[\frac{1}{a-t+1}\right]\right)\zeta(t)\,x+\frac{B_{a-t+2}}{a-t+2}\zeta(t-1)$$

$$+\frac{(-1)^{a-t+1}}{a-t+2}\sum_{d\leqslant k}\frac{1}{d^{t-1}}P_{a-t+2}\left(\frac{x}{d}\right)$$

$$+O(x^{a/2-t+1/2}),\qquad 3\leqslant t\leqslant a.$$

Proof. The proof uses the corollary to Lemma 19, together with Lemma 7 and Theorem 2.

THEOREM 8.

(i) $$\sum_{n\leqslant x}\frac{\sigma(n)}{n}=\zeta(2)\,x-\frac{1}{2}\log x-\sum_{d\leqslant k}\frac{1}{d}P_1\left(\frac{x}{d}\right)-\frac{1}{2}(\log 2\pi+\gamma)$$

$$-\frac{1}{x}\sum_{d\leqslant k}dP_1\left(\frac{x}{d}\right)+O(x^{-1/2});$$

(ii) $$\sum_{n\leqslant x}\frac{\sigma_a(n)}{n}=\frac{1}{a}\zeta(a+1)\,x^a$$

$$+\frac{1}{a}\sum_{1\leqslant r\leqslant a}(-1)^r\binom{a}{r}x^{a-r}\sum_{d\leqslant k}\frac{1}{d^{a+1-r}}P_r\left(\frac{x}{d}\right)$$

$$-\frac{1}{x}\sum_{d\leqslant k}d^aP_1\left(\frac{x}{d}\right)-\frac{1}{12}\log x+O(x^{a/2-1}),$$

$$a\geqslant 2;$$

(iii) $$\sum_{n\leqslant x}\frac{\sigma_a(n)}{n^t}=\frac{1}{a-t+1}\zeta(a+1)\,x^{a-t+1}$$
$$+\frac{1}{a-t+1}\sum_{r=1}^{a-t+1}(-1)^r\binom{a-t+1}{r}x^{a-t+1-r}$$
$$\times\sum_{d\leqslant k}\frac{1}{d^{a+1-r}}P_r\left(\frac{x}{d}\right)-\frac{1}{x^t}\sum_{d\leqslant k}d^aP_1\left(\frac{x}{d}\right)$$
$$-\left(\frac{1}{a-t+1}B_{a-t+1}+\left[\frac{1}{a-t+1}\right]\right)\zeta(t)$$
$$+O(x^{a/2-t}),\qquad 2\leqslant t\leqslant a;$$

on defining $E^a_{-t}(x)$ by

$$E^a_{-t}(x)=\sum_{n\leqslant x}\frac{\sigma(n)}{n}-\zeta(2)\,x+\frac{1}{2}\log x,\qquad a=t=1$$
$$=\sum_{n\leqslant x}\frac{\sigma_a(n)}{n^t}-\frac{1}{a-t+1}\zeta(a+1)\,x^{a-t+1}\qquad \text{otherwise}$$

we have

(iv) $$\sum_{n\leqslant x}E^1_{-1}(n)=\frac{1}{2}(\zeta(2)-\log 2\pi-\gamma)\,x$$
$$-\frac{1}{2}\frac{1}{x}\sum_{d\leqslant k}d^2P_2\left(\frac{x}{d}\right)-\frac{1}{2}\sum_{d\leqslant k}P_2\left(\frac{x}{d}\right)$$
$$-\frac{1}{4}\log x$$
$$-\left(\frac{1}{2}-P_1(x)\right)\sum_{d\leqslant k}\frac{1}{d}P_1\left(\frac{x}{d}\right)+O(1);$$

(v) $$\int_1^x E^1_{-1}(u)\,du=-\frac{1}{2}(\log 2\pi+\gamma)\,x-\frac{1}{2}\frac{1}{x}\sum_{d\leqslant k}d^2P_2\left(\frac{x}{d}\right)$$
$$-\frac{1}{2}\sum_{d\leqslant k}P_2\left(\frac{x}{d}\right)+O(1);$$

(vi) $$\sum_{n\leqslant x}E^2_{-1}(n)=\frac{1}{4}\zeta(3)\,x^2-\frac{1}{12}x\log x+x\sum_{d\leqslant k}\frac{1}{d}P_2\left(\frac{x}{d}\right)$$
$$-x\left(\frac{7}{12}+\frac{1}{12}\gamma+2A_1+\zeta(3)\,P_2(x)\right.$$

$$+\left(\frac{1}{2}-P_1(x)\right)\sum_{d\leqslant k}\frac{1}{d^2}P_1\left(\frac{x}{d}\right)\Bigg)$$

$$-\frac{1}{2}\frac{1}{x}\sum_{d\leqslant k}d^3P_2\left(\frac{x}{d}\right)+O(\sqrt{x});$$

(vii) $$\int_1^x E_{-1}^2(u)\,du=-\frac{1}{12}x\log x+x\sum_{d\leqslant k}\frac{1}{d}P_2\left(\frac{x}{d}\right)$$

$$-\left(\frac{7}{12}+\frac{1}{12}\gamma+2A_1\right)x$$

$$-\frac{1}{2}\frac{1}{x}\sum_{d\leqslant k}d^3P_2\left(\frac{x}{d}\right)+O(\sqrt{x});$$

(viii) $$\sum_{n\leqslant x}E_{-1}^a(n)=\frac{1}{2}\frac{1}{a}\zeta(a+1)\,x^a$$

$$+\sum_{m=1}^{a-1}\frac{(-1)^m}{m+1}\binom{a-1}{m-1}x^{a-m}$$

$$\times\sum_{d\leqslant k}\frac{1}{d^{a-m}}P_{m+1}\left(\frac{x}{d}\right)$$

$$+\frac{1}{a}\zeta(a+1)\sum_{m=1}^{a-1}\frac{(-1)^m}{m+1}\binom{a}{m}P_{m+1}(x)\,x^{a-m}$$

$$+\left(\frac{1}{2}-P_1(x)\right)\frac{1}{a}\sum_{m=1}^{a-1}(-1)^m\binom{a}{m}x^{a-m}$$

$$\times\sum_{d\leqslant k}\frac{1}{d^{a+1-m}}P_m\left(\frac{x}{d}\right)$$

$$-\frac{1}{2}\frac{1}{x}\sum_{d\leqslant k}d^{a+1}P_2\left(\frac{x}{d}\right)+O(x^{(a-1)/2}),$$

for $a\geqslant 3$;

(ix) $$\int_1^x E_{-1}^a(u)\,du=\sum_{m=1}^{a-1}\frac{(-1)^m}{m+1}\binom{a-1}{m-1}x^{a-m}$$

$$\times\sum_{d\leqslant k}\frac{1}{d^{a-m}}P_{m+1}\left(\frac{x}{d}\right)$$

$$-\frac{1}{2}\frac{1}{x}\sum_{d\leqslant k}d^{a+1}P_2\left(\frac{x}{d}\right)+O(x^{(a-1)/2});$$

(x) $$\sum_{n\leqslant x} E^2_{-2}(u)\,du = \frac{1}{2}(\zeta(3)-\zeta(2))\,x + \frac{1}{12}\log x + \sum_{d\leqslant k}\frac{1}{d}P_2\left(\frac{x}{d}\right)$$
$$-\left(\frac{1}{2}-P_1(x)\right)\sum_{d\leqslant k}\frac{1}{d^2}P_1\left(\frac{x}{d}\right)$$
$$+\left(A_1+\frac{1}{12}\gamma+\frac{1}{12}\zeta(3)\,P_2(x)\right.$$
$$\left.-\frac{1}{2}\zeta(2)+P_1(x)\,\zeta(2)\right)$$
$$+\frac{1}{x^2}\sum_{d\leqslant k} d^3P_2\left(\frac{x}{d}\right)+O(x^{-1/2});$$

(xi) $$\int_1^x E^2_{-2}(u)\,du = -\frac{1}{2}\zeta(2)\,x + \frac{1}{12}\log x + \sum_{d\leqslant k}\frac{1}{d}P_2\left(\frac{x}{d}\right)$$
$$+\left(A_1+\frac{1}{12}\gamma+\frac{1}{2}\zeta(3)\right)+\frac{1}{x^2}\sum_{d\leqslant k} d^3P_2\left(\frac{x}{d}\right)$$
$$+O(x^{-1/2});$$

(xii) $$\sum_{n\leqslant x} E^a_{-2}(u)\,du = \frac{1}{2}\frac{1}{a-1}\zeta(a+1)\,x^{a-1}$$
$$+\sum_{m=1}^{a-2}\frac{(-1)^m}{m+1}\binom{a-2}{m-1}x^{a-m-1}$$
$$\times\sum_{d\leqslant k}\frac{1}{d^{a-m}}P_{m+1}\left(\frac{x}{d}\right)$$
$$+\frac{1}{a-1}\zeta(a+1)\sum_{m=1}^{a-2}\frac{(-1)^m}{m+1}\binom{a-1}{m}$$
$$\times P_{m+1}(x)\,x^{a-m-1}$$
$$+\left(\frac{1}{2}-P_1(x)\right)\frac{1}{a-1}$$
$$\times\sum_{m=1}^{a-2}(-1)^m\binom{a-1}{m}x^{a-m-1}$$
$$\times\sum_{d\leqslant k}\frac{1}{d^{a+1-m}}P_m\left(\frac{x}{d}\right)$$
$$-\frac{1}{2}\frac{1}{x^2}\sum_{d\leqslant k} d^{a+1}P_2\left(\frac{x}{d}\right)-\frac{B_{a-1}}{a-1}\zeta(2)\,x$$

$$+\frac{(-1)^{a+1}}{a}\sum_{d\leqslant k}\frac{1}{d}P_a\left(\frac{x}{d}\right)+O(x^{(a-3)/2}),$$
$$a\geqslant 3;$$

(xiii) $$\int_1^x E^a_{-2}(u)\,du=\sum_{m=1}^{a-2}\frac{(-1)^m}{m-1}\binom{a-2}{m-1}x^{a-m-1}$$
$$\times\sum_{d\leqslant k}\frac{1}{d^{a-m}}P_{m+1}\left(\frac{x}{d}\right)$$
$$-\frac{1}{2}\frac{1}{x^2}\sum_{d\leqslant k}d^{a+1}P_2\left(\frac{x}{d}\right)-\frac{B_{a-1}}{a-1}\zeta(2)\,x$$
$$+\frac{(-1)^{a+1}}{a}\sum_{d\leqslant k}\frac{1}{d}P_a\left(\frac{x}{d}\right)+O(x^{(a-3)/2}),$$
$$a\geqslant 3;$$

(xiv) $$\sum_{n\leqslant x}E^a_{-t}(n)=\frac{1}{2}\frac{1}{a-t+1}\zeta(a+1)\,x^{a-t+1}$$
$$+\sum_{m=1}^{a-t}\frac{(-1)^m}{m+1}\binom{a-t}{m-1}x^{a-t+1-m}$$
$$\times\sum_{d\leqslant k}\frac{1}{d^{a-m}}P_{m+1}\left(\frac{x}{d}\right)$$

$$+\frac{1}{a-t+1}\zeta(a+1)\sum_{m=1}^{a-t+1}\frac{(-1)^m}{m+1}$$
$$\times\binom{a-t+1}{m}P_{m+1}(x)\,x^{a-t+1-m}$$
$$+\left(\frac{1}{2}-P_1(x)\right)\frac{1}{a-t+1}$$
$$\times\sum_{m=1}^{a+t+1}(-1)^m\binom{a-t+1}{m}x^{a-t+1-m}$$
$$\times\sum_{d\leqslant k}\frac{1}{d^{a+1-m}}P_m\left(\frac{x}{d}\right)$$
$$-\frac{1}{2}\frac{1}{x^t}\sum_{d\leqslant k}d^{a+1}P_2\left(\frac{x}{d}\right)$$
$$-\left(\frac{B_{a-t+1}}{a-t+1}+\left[\frac{1}{a-t+1}\right]\right)\zeta(t)\,x$$

$$+\frac{(-1)^{a-t+1}}{a-t+2}\sum_{d\leqslant k}\frac{1}{d^{t-1}}P_{a-t+2}\left(\frac{x}{d}\right)$$

$$+\frac{B_{a-t+2}}{a-t+2}\zeta(t-1)$$

$$+\frac{B_{a-t+2}}{(a-t+1)(a-t+2)}\zeta(a+1)$$

$$-\left(\frac{1}{2}-P_1(x)\right)\left(\frac{B_{a-t+1}}{a-t+1}\right.$$

$$\left.+\left[\frac{1}{a-t+1}\right]\right)\zeta(t)$$

$$+O(x^{a/2-t+1/2}),\qquad 3\leqslant t\leqslant a;$$

(xv) $$\int_1^x E^a_{-t}(u)\,du=\frac{1}{a-t+1}\sum_{m=1}^{a-t+1}(-1)^m\binom{a-t+1}{m}x^{a-t+1-m}$$

$$\times\sum_{d\leqslant k}\frac{1}{d^{a+1-m}}P_m\left(\frac{x}{d}\right)$$

$$-\frac{1}{2}\frac{1}{x^t}\sum_{d\leqslant k}d^{a+1}P_2\left(\frac{x}{d}\right)$$

$$-\left(\frac{B_{a-t+1}}{a-t+1}+\left[\frac{1}{a-t+1}\right]\right)\zeta(t)\,x$$

$$+\frac{(-1)^{a-t+1}}{a-t+2}\sum_{d\leqslant k}\frac{1}{d^{t-1}}P_{a-t+2}\left(\frac{x}{d}\right)$$

$$+\frac{B_{a-t+2}}{a-t+2}\zeta(t-1)$$

$$+\frac{1}{(a-t+1)(a-t+2)}\zeta(a+1)$$

$$+O(x^{a/2-t+1/2}),\qquad 3\leqslant t\leqslant a.$$

Proof. Parts (i) – (iii) follow easily from the corollary to Lemma 19, Theorem 2, and Lemma 7. Parts (iv)–(xv) follow directly from Theorem 4 and Lemma 20.

The following "back-track" method has proven very useful in dealing with negative integers t.

THEOREM 9. *Let t be an arbitrary real number, and f any numerical function. Let g_t and E_t be such that $\sum_{n\leqslant x} n^t f(n) = g_t(x) + E_t(x)$. Then we have*

$$\sum_{n\leqslant x} n^{t-1}f(n) = \sum_{n\leqslant x} \frac{g_t(n)}{n^2} - \sum_{n\leqslant x} \frac{g_t(n)}{n^2(n+1)} + \sum_{n\leqslant x} \frac{E_t(n)}{n^2} - \sum_{n\leqslant x} \frac{E_t(n)}{n^2(n+1)} + \frac{P_1(x)-\frac{1}{2}}{x([x]+1)}(g_t(x)+E_t(x)) + \frac{1}{x}(g_t(x)+E_t(x)). \tag{16}$$

$$x\sum_{n\leqslant x} n^{t-1}f(n) - \sum_{n\leqslant x} n^t f(n) = x\sum_{n\leqslant x} \frac{g_t(n)}{n^2} - x\sum_{n\leqslant x} \frac{g_t(n)}{n^2(n+1)} + x\sum_{n\leqslant x} \frac{E_t(n)}{n^2} - x\sum_{n\leqslant x} \frac{E_t(n)}{n^2(n+1)} + \frac{P_1(x)-\frac{1}{2}}{[x]+1}(g_t(x)+E_t(x)). \tag{17}$$

If in addition the series

$$\sum_{n=1}^{\infty} \frac{E_t n}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{E_t(n)}{n^2(n+1)}, \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{g_t(n)}{n^2(n+1)}$$

all converge, and if $(1/x^2)\sum_{n\leqslant x} E_t(n)$ approaches 0 as x approaches ∞, then we have

$$\sum_{n\leqslant x} n^{t-1}f(n) = \sum_{n\leqslant x} \frac{g_t(n)}{n^2} + \sum_{n>x} \frac{g_t(n)}{n^2(n+1)} - 2\sum_{n>x} \frac{G_t(n)}{n(n+1)^2} - \sum_{n>x} \frac{G_t(n)}{n^2(n+1)^2} + \frac{G_t(x)}{([x]+1)^2} + \sum_{n\leqslant x} \frac{E_t(n)}{n^2(n+1)} + K_t + \frac{P_1(x)-\frac{1}{2}}{x([x]+1)}(g_t(x)+E_t(x)) + \frac{1}{x}(g_t(x)+E_t(x)), \tag{18}$$

$$x\sum_{n\leqslant x} n^{t-1}f(n) - \sum_{n\leqslant x} n^t f(n) = x\sum_{n\leqslant x} \frac{g_t(n)}{n^2} + x\sum_{n>x} \frac{g_t(n)}{n^2(n+1)} - 2x\sum_{n>x} \frac{G_t(n)}{n(n+1)^2} - x\sum_{n>x} \frac{G_t(n)}{n^2(n+1)^2} + \frac{xG_t(x)}{([x]+1)^2} + x\sum_{n>x} \frac{E_t(n)}{n^2(n+1)} + K_t x + \frac{P_1(x)-\frac{1}{2}}{[x]+1}(g_t(x)+E_t(x)), \tag{19}$$

where K_t is the sum of the three series, and $G_t(x)$ is defined by

$$G_t(x) = \sum_{n \leqslant x} E_t(n).$$

Proof. On putting $g(x) = 1/x$ and $h(x) = x^t f(x)$ in Lemma 17, we have

$$\sum_{n \leqslant x} n^{t-1} f(n) = \sum_{n \leqslant x} \frac{g_t(n) + E_t(n)}{n(n+1)} + \frac{g_t(x) + E_t(x)}{[x]+1}. \tag{20}$$

The result follows on noting that

$$\frac{1}{n(n+1)} - \frac{1}{n^2} = -\frac{1}{n^2(n+1)}, \qquad \frac{1}{[x]+1} - \frac{1}{x} = \frac{P_1(x) - \frac{1}{2}}{x([x]+1)},$$

and using Lemma 17 again, with $g(x) = 1/x^2$, to get

$$\sum_{n \leqslant x} \frac{E_t(n)}{n^2} = \sum_{n \leqslant x} G_t(n) \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) + \frac{G_t(x)}{([x]+1)^2}. \tag{21}$$

THEOREM 10.

$$\sum_{n \leqslant x} \frac{d(n)}{n} = \frac{1}{2} \log^2 x + 2\gamma \log x + c_{-1} - \frac{2}{x} \sum_{d \leqslant k} dP_2\left(\frac{x}{d}\right) + O\left(\frac{1}{x}\right),$$

$$\sum_{n \leqslant x} E_{-1}(n) = \frac{1}{4} \log^2 x + \gamma \log x + d_{-1} - \frac{1}{x} \sum_{d \leqslant k} dP_2\left(\frac{x}{d}\right)$$

$$+ 2x \sum_{n > x} \frac{\sum_{d \leqslant \sqrt{n}} dP_2(n/d)}{n^3} + O(x^{-1/2}),$$

$$\int_1^x E_{-1}(u)\, du = e_{-1} - \frac{1}{x} \sum_{d \leqslant k} dP_2\left(\frac{x}{d}\right)$$

$$+ 2x \sum_{n > x} \frac{\sum_{d \leqslant \sqrt{n}} dP_2(n/d)}{n^3} + O(x^{-1/2}),$$

where c_{-1}, d_{-1}, e_{-1}, and $E_{-1}(x)$ are defined by

$$c_{-1} = 2\gamma^2 + \gamma - 1 + c_1 + k_0,$$

$$d_{-1} = c_{-1} - \frac{1}{2} A' - 2\gamma A_0 - \frac{1}{4},$$

$$e_{-1} = c_{-1} - 2\gamma + \frac{3}{4},$$

$$E_{-1}(x) = \sum_{n \leqslant x} \frac{d(n)}{n} - \left(\frac{1}{2} \log^2 x + 2 \log x + c_{-1}\right),$$

and A' and A_0 are the constants in the Euler–MacLaurin expansions of $\sum_{n\leqslant x}\log^2 n$ *and* $\sum_{n\leqslant x}\log n$.

Proof. The proof is based on Theorems 7 and 9.

Note 1. It is almost certainly true that

$$2x\sum_{n>x}\frac{\sum_{d\leqslant\sqrt{n}} dP_2(n/d)}{n^3}=O(x^{-1/2}).$$

Hence this term would vanish in the statement of Theorem 10.

Note 2. Chowla and Walum [3] have proved

$$\sum_{d\leqslant k} dP_2\left(\frac{x}{d}\right)=O(x^{3/4}). \tag{22}$$

This implies the following corollary.

COROLLARY.

$$\sum_{n\leqslant x} E_{-1}(n)=\tfrac{1}{4}\log^2 x+\gamma\log x+d_{-1}+O(x^{-1/4}),$$

$$\int_1^x E_{-1}(u)=e_{-1}+O(x^{-1/4}),$$

where

$$d_{-1}=c_{-1}-\tfrac{1}{2}A'-2\gamma A_0-\tfrac{1}{4},$$

$$e_{-1}=c_{-1}-2\gamma+\tfrac{3}{4},$$

and A' and A_0 are the constants in the Euler–MacLaurin expansions of $\sum_{n\leqslant x}\log^2 n$ *and* $\sum_{n\leqslant x}\log n$.

THEOREM 11. *Let t be an integer, $t\geqslant 2$. Then*

$$\sum_{n\leqslant x}\frac{d(n)}{n^t}=\zeta^2(t)-\frac{1}{t-1}\frac{\log x}{x^{t-1}}-\frac{1}{t-1}\left(2\gamma+\frac{1}{t-1}\right)\frac{1}{x^{t-1}}-\frac{2}{x^t}\sum_{d\leqslant k}P_1\left(\frac{x}{d}\right)+O\left(\frac{1}{x^t}\right),$$

$$\sum_{n\leqslant x}E_{-t}(n)=T_t-\frac{1}{2(t-1)}\frac{\log x}{x^{t-1}}-\frac{1}{2(t-1)}\left(2\gamma+\frac{1}{t-1}\right)\frac{1}{x^{t-1}}+O\left(\frac{1}{x^{t-3/4}}\right),$$

$$\int_1^x E_{-t}(u)\,du=V_t+O\left(\frac{1}{x^{t-3/4}}\right),$$

where T_t, V_t, and $E_{-t}(x)$ are defined by

$$T_2 = \zeta^2(2) - 2\gamma - k_0,$$

$$V_2 = \zeta^2(2) - 2\gamma^2 - 3\gamma - c_1 - k_0,$$

$$T_t = \zeta^2(t) - \zeta^2(t-1) - \frac{1}{t-1}\left(2\gamma + \frac{1}{t-1}\right)\zeta(t-1) - \frac{1}{t-1}\zeta'(t-1), \qquad t \geqslant 3,$$

$$V_t = \zeta^2(t) - \zeta^2(t-1) + \frac{1}{t(t-1)}\left(2\gamma + \frac{1}{t} + \frac{1}{t-1}\right), \qquad t \geqslant 3,$$

$$E_{-t}(x) = \sum_{n \leqslant x} \frac{d(n)}{n^t} - \left(\zeta^2(t) - \frac{1}{t-1}\frac{\log x}{x^{t-1}} - \frac{1}{t-1}\left(2\gamma + \frac{1}{t-1}\right)\frac{1}{x^{t-1}}\right),$$

where k_0 is the constant defined in Theorem 9 and c_1 is the constant in the Euler–MacLaurin expansion of $\sum_{n \leqslant x} (\log n)/n$.

Proof. The result is obtained by using Theorem 9 to handle the case $t = 2$ and then using induction on t; the details require about eight pages of manipulations.

Note. As with the previous theorem, it is probably true that

$$\sum_{n \leqslant x} E_{-t}(n) = T_t - \frac{1}{2(t-1)}\frac{\log x}{x^{t-1}} - \frac{1}{2(t-1)}\left(2\gamma + \frac{1}{t-1}\right)\frac{1}{x^{t-1}} - \frac{1}{x^t}\sum_{d \leqslant k} dP_2\left(\frac{x}{d}\right) + O\left(\frac{1}{x^{t-1/2}}\right),$$

$$\int_1^x E_{-t}(u)\,du = V_t - \frac{1}{x^t}\sum_{d \leqslant k} dP_2\left(\frac{x}{d}\right) + O\left(\frac{1}{x^{t-1/2}}\right).$$

THEOREM 12. *Let t be an integer, $t \geqslant 3$. Then*

$$\sum_{n \leqslant x} \frac{\sigma(n)}{n^2} = \zeta(2)\log x + (\zeta(2)\gamma + \zeta'(2)) - \frac{1}{x}\sum_{d \leqslant k}\frac{1}{d}P_1\left(\frac{x}{d}\right) + \frac{1}{2}\frac{1}{x} - \frac{1}{x^2}\sum_{d \leqslant k} dP_1\left(\frac{x}{d}\right) + O\left(\frac{1}{x^{3/2}}\right),$$

$$\sum_{n \leqslant x} E^1_{-2}(n) = \frac{1}{2}(\zeta(2) + 1)\log x + W_2 + O\left(\frac{1}{x^{3/4}}\right),$$

$$\int_1^x E^1_{-2}(u)\,du = \frac{1}{2}\log x + Y_2 + O\left(\frac{1}{x^{3/4}}\right),$$

$$\sum_{n\leqslant x}\frac{\sigma(n)}{n^t}=\zeta(t)\,\zeta(t-1)-\frac{1}{t-2}\zeta(2)\frac{1}{x^{t-2}}$$
$$-\frac{1}{x^{t-1}}\sum_{d\leqslant k}\frac{1}{d}P_1\left(\frac{x}{d}\right)+\frac{1}{2(t-1)}\frac{1}{x^{t-1}}$$
$$-\frac{1}{x^t}\sum_{d\leqslant k}dP_1\left(\frac{x}{d}\right)+O\left(\frac{1}{x^{t-1/2}}\right),\qquad t\geqslant 3,$$

$$\sum_{n\leqslant x}E^1_{-t}(n)=W_t-\left(\frac{1}{2(t-1)(t-2)}+\frac{1}{2(t-2)}\zeta(2)\right)\frac{1}{x^{t-2}}$$
$$+O\left(\frac{1}{x^{t-5/4}}\right),$$

$$\int_1^x E^1_{-t}(u)\,du=Y_t-\frac{1}{2(t-1)(t-2)}\frac{1}{x^{t-2}}+O\left(\frac{1}{x^{t-5/4}}\right),$$

where W_k, Y_k, and $E^1_{-k}(x)$ are defined by

$$W_2=\frac{1}{2}(\log 2\pi-\zeta(2)\log 2\pi+\gamma+2\zeta(2)\,\gamma+1+2\zeta'(2)),$$

$$W_3=\zeta(3)\,\zeta(2)-\zeta(2)-\zeta'(2),$$

$$W_t=\zeta(t)\,\zeta(t-1)-\zeta(t-1)\,\zeta(t-2)+\frac{1}{t-2}\zeta(2)\,\zeta(t-2),\qquad t\geqslant 4,$$

$$Y_2=\frac{1}{2}(\log 2\pi+\gamma+2\zeta(2)\,\gamma+1+2\zeta'(2)-2\zeta(2)),$$

$$Y_3=\zeta(3)\,\zeta(2)-\zeta(2)-\zeta(2)\,\gamma-\zeta'(2),$$

$$Y_t=\zeta(t)\,\zeta(t-1)-\zeta(t-1)\,\zeta(t-2)+\frac{1}{(t-2)(t-1)}\zeta(2),\qquad t\geqslant 4,$$

$$E^1_{-2}(x)=\sum_{n\leqslant x}\frac{\sigma(n)}{n^2}-\zeta(2)\log x-(\zeta(2)\,\gamma+\zeta'(2)),$$

$$E^1_{-t}(x)=\sum_{n\leqslant x}\frac{\sigma(n)}{n^t}-\left(\zeta(t)\,\zeta(t-1)-\frac{1}{t-2}\zeta(2)\frac{1}{x^{t-2}}\right),\qquad t\geqslant 3.$$

Proof. The proof is similar to that of Theorem 11, and uses the results

$$\sum_{n=1}^{\infty}\frac{\sigma(n)}{n^t}=\zeta(t)\,\zeta(t-1),\qquad t\geqslant 3,$$

$$\frac{1}{x}\sum_{d\leqslant k}d^2P_2\left(\frac{x}{d}\right)+\sum_{d\leqslant k}P_2\left(\frac{x}{d}\right)=O(x^{1/4}).$$

For the first, see Hardy and Wright [6, Theorem 290]. The second follows from our Theorem 6, together with the result of Segal [8] that

$$\sum_{n \leqslant x} \left(\sigma(n) - \frac{\zeta(2)}{2} n^2\right) = \frac{1}{4} (\zeta(2) - 1)\, x^2 + O(x^{5/4}).$$

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