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A no-go theorem for the consistent quantization of spin-3/2 fields on general curved spacetimes

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ABSTRACT

It is well known that coupling a spin-$\frac{3}{2}$ field to a gravitational or electromagnetic background leads to potential problems both in the classical and in the quantum theory. Various solutions to these problems have been proposed so far, which are all restricted to a limited class of backgrounds. On the other hand, negative results for general gravitational backgrounds have been reported only for a limited set of couplings to the background to date. Hence, to our knowledge, a comprehensive analysis of all possible couplings to the gravitational field and general gravitational backgrounds including off-shell ones has not been performed so far. In this work we analyse whether it is possible to couple a spin-$\frac{3}{2}$ field to a gravitational field in such a way that the resulting quantum theory is consistent on arbitrary gravitational backgrounds. We find that this is impossible as all couplings require the background to be an Einstein spacetime for consistency. This enforces the widespread belief that supergravity theories are the only meaningful models which contain spin-$\frac{3}{2}$ fields as in these models such restrictions of the gravitational background appear naturally as on-shell conditions.

1. Introduction – Problems of spin-$\frac{3}{2}$ fields in non-trivial backgrounds

A free spin-$\frac{3}{2}$ field ψ of mass $m \geqslant 0$ in flat four-dimensional Minkowski spacetime is described by the *Rarita–Schwinger equations* [1]

$$(\mathcal{R}_0\psi)^\alpha := (-i\partial\!\!\!/ + m)\psi^\alpha$$
$$:= \left(-i\gamma^\mu\partial_\mu + m\right)\psi^\alpha = 0, \quad (1)$$
$$\psi\!\!\!/ := \gamma_\mu\psi^\mu = 0. \quad (2)$$

Here and in the following Greek indices denote (co)tangent space indices, γ^μ are the usual γ-matrices, and ψ is a Dirac spinor-valued vector field whose spinor indices we suppress throughout. Buchdahl realised already more than fifty years ago that a minimal coupling of the above equation to a background gravitational field leads to problems [2]: the minimally coupled equations imply $R_{\mu\nu}\gamma^\mu\psi^\nu = 0$, with $R_{\mu\nu}$ denoting the Ricci curvature tensor, and this equation can only be satisfied by $\psi \equiv 0$ unless the spacetime is an Einstein spacetime s.t. $R_{\mu\nu}$ is a constant multiple of the metric $g_{\mu\nu}$.

Later Johnson and Sudharsan found that the quantum theory of a spin-$\frac{3}{2}$ field minimally coupled to an electromagnetic background field fails to satisfy unitarity [3]. This result has been complemented by Velo and Zwanziger who pointed out that the coupling to an electromagnetic field is already problematic at the classical level as it leads to superluminal propagation [4].

This last finding seemed to be the most shocking as it became famous as the *Velo–Zwanziger problem*.

All three problems have been analysed in great detail and various solutions have been proposed. As it is impossible to provide a comprehensive list of earlier works, we only mention a few selected ones. Special, i.e. maximally symmetric or constant gravitational and electromagnetic backgrounds have been studied e.g. in [5–8] where the causality and/or unitarity problems have been proven to be absent for special values of the mass and/or the couplings. In [9,10] it was pointed out that all problems can be solved in Einstein–Maxwell backgrounds at the cost of very small or very large masses m. The most prominent solution of the Buchdahl-problem is arguably supergravity [11,12], where the Einstein condition on the spacetime appears as a natural on-shell condition. The causal behaviour of supergravity was shown in [13], whereas unitarity had mostly been discussed on maximally symmetric Einstein backgrounds such as Minkowski and

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Anti-de Sitter spacetime [5–7]. Recently unitarity has also been proven for general, asymptotically flat and Ricci flat Einstein backgrounds[1] [15]. Other solutions to the Buchdahl-problem, which avoid restrictions on the background, have been proposed and analysed both in the $(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$ representation, e.g. [16–18], and in the $(\frac{3}{2}, 0) \oplus (\frac{1}{2}, 1)$ representation of $SL(2, \mathbb{C})$ [19–23]. While the former suffer either from the causality or the unitarity problem, the latter satisfy causality, but a unitarity proof is lacking to date.

All the above-mentioned analyses have in common that they consider restrictions on the couplings, the mass, or the background fields. Whereas in [24] general non-minimal couplings to the electromagnetic field have been studied with a negative result, it seems that a comprehensive study of general non-minimal couplings to the gravitational field and general gravitational backgrounds has not been available to date. In this Letter, we thus investigate whether it is possible to couple a spin-$\frac{3}{2}$ field to a gravitational field in a way, such that the resulting quantum theory is causal, unitary, and propagates the correct degrees of freedom on arbitrary spacetime backgrounds – including off-shell ones. This generality is motivated by the modern approach to quantum field theory on curved spacetimes [25] (see also [26] for an extensive review) where one tries to quantize a model without using any knowledge on the background spacetime other than its defining properties such as e.g. the Lorentzian metric signature. As this turns out to be possible for spins $\leqslant 1$, see e.g. [27,25,28–32], the question, whether this is the case for higher spins as well, naturally arises. However, we find that a background-independent consistent quantization seems to be impossible for spin-$\frac{3}{2}$ fields in gravitational backgrounds.

We work solely in the $(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$ representation of $SL(2, \mathbb{C})$ and do not consider the $(\frac{3}{2}, 0) \oplus (\frac{1}{2}, 1)$ representation, which is equivalent to the former on flat spacetimes, but not on curved ones. This is motivated by the results of [23] where it has been found that unitarity of a quantum field in this representation is unlikely to hold due to its very structure in curved spacetimes.

Our Letter is organised as follows. In Section 2 we compile four conditions which a consistent quantum theory of a spin-$\frac{3}{2}$ field on an arbitrary curved spacetime should satisfy. While the causality condition and the condition on the degrees of freedom are well known, the very "background-independence" condition has apparently not been discussed so far in this context. Our fourth condition, a certain symmetry condition of the field equations, is shown to be virtually equivalent to unitarity and thus replaces the unitarity condition. Furthermore we point out that, in contrast to statements in the literature, causality and unitarity are not equivalent for spin-$\frac{3}{2}$ fields. In Section 3 we finally prove our no-go theorem and show that no non-minimally coupled spin-$\frac{3}{2}$ field equation satisfies all four conditions. The Letter ends with a discussion of our findings in Section 4.

2. Conditions for a consistent spin-$\frac{3}{2}$ quantum theory in curved spacetimes

We consider a spin-$\frac{3}{2}$ field ψ on a general curved spacetime $(M, g_{\mu\nu})$, i.e. M is a four-dimensional manifold, $g_{\mu\nu}$ a metric with signature $(+, -, -, -)$ and ψ is a four-spinor-valued vector field whose vector index we shall write only if necessary. We shall often denote $(M, g_{\mu\nu})$ by M for simplicity. The field equations for ψ are

$$\mathcal{R}\psi = 0, \tag{3}$$

$$\not{\psi} := \gamma_\mu \psi^\mu = A_\mu \psi^\mu, \tag{4}$$

where $\mathcal{R}$ is an arbitrary first order differential operator constructed out of the metric, the curved-spacetime γ-matrices γ^μ, and the mass m, and A_μ is an arbitrary zeroth order operator of that kind. Thus, with tuples $(\mathcal{R}, A_\mu)$ we parametrise all non-minimal couplings of ψ to the background gravitation field. By $S(\mathcal{R}, M)$ we denote the set of all (infinitely often differentiable) solutions of (3) on the spacetime M, whereas by $S(\mathcal{R}, A_\mu, M)$ we denote the subset of $S(\mathcal{R}, M)$ which satisfies in addition (4). We now list four conditions on $(\mathcal{R}, A_\mu)$ and argue why they sufficient for a spin-$\frac{3}{2}$ quantum theory induced by $(\mathcal{R}, A_\mu)$ to be consistent in arbitrary curved spacetimes.

2.1. Condition 1: Irreducibility

On Minkowski spacetime $\mathbb{M}$, $A_\mu \equiv 0$ and $S(\mathcal{R}, 0, \mathbb{M}) = S(\mathcal{R}_0, 0, \mathbb{M})$.

This condition requires that $(\mathcal{R}, A_\mu)$ define a theory which propagates the correct number of degrees of freedom for a spin-$\frac{3}{2}$ field of mass m. This is here achieved by comparison with the standard theory in Minkowski spacetime, which after all is the very spacetime in which the concepts of "spin" and "mass" are defined via irreducible representations of the Poincaré group. We don't require $\mathcal{R} \equiv \mathcal{R}_0$ on $\mathbb{M}$ because different $\mathcal{R}$ can be equivalent on-shell.

2.2. Condition 2: Causality

$\mathcal{R}$ is hyperbolic and the constraint $\not{\psi} = A_\mu \psi^\mu$ is compatible with time evolution.

Hyperbolic field equations such as the Klein–Gordon or the Dirac equation guarantee causal propagation of the degrees of freedom, see e.g. [33,34,9,35], as they limit the dependence of a solution $\psi(x)$ at a point x to the past lightcone of x. Hyperbolicity is a condition on the coefficient matrix σ^μ of the highest derivative term $\sigma^\mu \nabla_\mu$ in $\mathcal{R}$, the so-called *principal symbol*: for a spacelike/timelike vector k_μ, $k_\mu \sigma^\mu$ must be invertible, while for a lightlike k_μ, it must have vanishing determinant. Additionally, the above compatibility condition is required to avoid that $S(\mathcal{R}, A_\mu, M)$ contains only the trivial solution $\psi \equiv 0$.

2.3. Condition 3: Background independence

The number of degrees of freedom propagated by $(\mathcal{R}, A_\mu)$ is independent of the background spacetime M. Moreover, either $A_\mu \equiv 0$ on all spacetimes, or (4) is automatically satisfied for all solutions of (3).

This condition is required to avoid the Buchdahl-problem mentioned in Section 1, where it happens that the minimally coupled Rarita–Schwinger equations (1) and (2) propagate the correct number of degrees of freedom on Einstein spacetimes, but no degrees of freedom at all otherwise.

Stated in more technical terms this condition requires that $S(\mathcal{R}, A_\mu, M)$ is *locally contravariant* in the sense of [25]: if we consider two spacetimes $M_1 \subset M_2$ where one is a (suitable) subset of the other, then $S(\mathcal{R}, A_\mu, M_1)$ should be equal to the restriction of $S(\mathcal{R}, A_\mu, M_2)$ to M_1.

We impose the additional condition on A_μ because we have not been able to prove that the constraint (4) satisfies our background-independence condition except in these two special cases.

[1] In a previous preprint version of this work we had argued that supergravity fails to satisfy unitarity on the basis of a spin-$\frac{3}{2}$ field equation derived from the original equation of motion in supergravity. In [14] it was pointed out that our argument fails if one considers the original supergravity equations of motion instead.

2.4. Condition 4: Self-adjointness

Although this condition appears to be the most technical one, it is equivalent to demanding that the field equation (3) can be obtained from a quadratic action. We state the condition first and comment on its relation to unitarity afterwards. To this avail, we introduce the notion $\Gamma_0(M)$ for the set of (infinitely often differentiable) vector-spinor-valued functions which vanish outside of a compact subset of M, so-called *test functions*. For two test functions f_1, f_2, we define a product $\langle f_1, f_2\rangle$ by

$$\langle f_1, f_2\rangle := \int_M d^4x \sqrt{-\det g_{\mu\nu}}\, g_{\alpha\beta}\overline{f_1^\alpha(x)} f_2^\beta(x),$$

where the bar denotes the usual Dirac conjugation of a four-spinor. We can define the *adjoint* $\mathcal{R}^\dagger$ of $\mathcal{R}$ with respect to $\langle\cdot,\cdot\rangle$ by $\langle \mathcal{R}^\dagger f_1, f_2\rangle := \langle f_1, \mathcal{R} f_2\rangle$ and finally state the fourth and last condition.

$\mathcal{R}$ is formally self-adjoint: $\mathcal{R}^\dagger = \mathcal{R}$, *i.e.* $\langle \mathcal{R} f_1, f_2\rangle = \langle f_1, \mathcal{R} f_2\rangle$.

To discuss the relation of this condition to unitarity, we briefly recall the unitarity condition for a spin-$\frac{3}{2}$ field, see e.g. [3,7,15] for details. To wit, the *covariant anticommutator* of the quantized field ψ and its adjoint $\overline{\psi}$ is after canonical quantization given by

$$\{\psi(x), \overline{\psi}(y)\} = iG(x, y) \tag{5}$$

where $G(x, y)$ is the so-called *anticommutator function*, a generalisation of the *Pauli–Jordan-function* for scalar fields. $G(x, y)$ is equal to the difference of the advanced and retarded Green's function[2] of the differential operator $\mathcal{R}$ and thus $G(x, y)$ depends on the specific form of $\mathcal{R}$ and satisfies $\mathcal{R}_x G(x, y) = \mathcal{R}^\dagger_y G(x, y) = 0$ for a general hyperbolic $\mathcal{R}$. The operator G defined by

$$[Gf](x) := \int_M d^4y \sqrt{-\det g_{\mu\nu}}\, G(x, y) f(y),$$

maps test functions to solutions which have finite spatial extent at each time, i.e. "wave packets". Accordingly, the quantized field $\psi(x)$ integrated with the Dirac adjoint of a test section f – henceforth denoted by $\psi(\overline{f})$ – can be interpreted as the quantum operator corresponding to the classical wave packet Gf. Physical wave packets should satisfy the constraint $\gamma_\mu (Gf)^\mu = A_\mu (Gf)^\mu$ in addition to the equation $\mathcal{R}(Gf) = 0$ and we denote the corresponding "physical subspace" of the test sections $\Gamma_0(M)$ by $\Gamma_0(\mathcal{R}, A_\mu, M)$. If one now considers the anticommutation relations (5) integrated with a test section $f \in \Gamma_0(\mathcal{R}, A_\mu, M)$ and its Dirac adjoint

$$\{\psi(\overline{f}), \overline{\psi}(f)\} = iG(\overline{f}, f) = i\langle f, Gf\rangle,$$

then the right hand side must be a positive number because the left hand side is of the form $B^\dagger B + BB^\dagger$ with $B = \overline{\psi}(f)$ and thus has positive expectation value in any quantum state $|\Omega\rangle$. Hence, the non-trivial unitarity condition for the tuple $(\mathcal{R}, A_\mu)$ is that the anticommutator function $G(x, y)$ determined by $\mathcal{R}$ must satisfy

$$i\langle f, Gf\rangle \geqslant 0$$

for any physical test function $f \in \Gamma_0(\mathcal{R}, A_\mu, M)$. Note that, for a formally self-adjoint $\mathcal{R}$ the previously discussed *covariant anticommutation* relations are equivalent to *equal-time anticommutation relations*, see e.g. [26,17,15] for details. Basically this follows from the identity

$$\langle f_1, Gf_2\rangle = \int_\Sigma d^3x \sqrt{-\det h_{ij}}\, \overline{Gf_1} n_\mu \sigma^\mu G f_2, \tag{6}$$

where Σ is an arbitrary equal-time surface of M with normal vector n_μ and h_{ij} is the spatial metric on Σ induced by $g_{\mu\nu}$.

We shall now demonstrate the close relation between the self-adjointness condition $\mathcal{R}^\dagger = \mathcal{R}$ and the unitarity condition $i\langle f, Gf\rangle \geqslant 0$ which lead us to replace the latter, which is difficult to check directly on all spacetimes, with the former, which can be checked more easily.

To start with, we shall argue why the self-adjointness condition implies unitarity on any topologically trivial spacetime M if unitarity is known in Minkowski spacetime $\mathbb{M}$. To see this, we consider any topologically trivial spacetime M and deform it in such a way that it becomes Minkowski in the past, see [36] for details. Loosely speaking, we consider a fiducial spacetime M' such that the metric on M' equals the metric on M for large positive times, whereas for large negative times it equals the Minkowski metric. Given such a deformation and a formally self-adjoint $\mathcal{R}$, the identity (6) allows us to compute $\langle f, Gf\rangle$ on any equal-time surface of M', in particular also in the Minkowski region where we know that it is positive by assumption. Moreover, for Eqs. (1) and (2), unitarity can be easily checked by an explicit computation in Fourier space, thus our first condition together with self-adjointness is sufficient to guarantee unitarity on any topologically trivial M.

We now prove that $i\langle f, Gf\rangle \geqslant 0$ for $f \in \Gamma_0(\mathcal{R}, A_\mu, M)$ implies $\langle f_1, \mathcal{R} f_2\rangle = \langle \mathcal{R} f_1, f_2\rangle$ for $f_i \in \Gamma_0(\mathcal{R}, A_\mu, M)$ on arbitrary spacetimes. Defining a product on physical test functions by $(f_1, f_2) := i\langle f_1, Gf_2\rangle$, our assumption $(f, f) \geqslant 0$ implies by polarisation that the complex conjugate of (f_1, f_2) equals (f_2, f_1) from which we can deduce that iG is formally self-adjoint on $\Gamma_0(\mathcal{R}, A_\mu, M)$. As $G^\dagger$ is the operator corresponding to the anticommutator function of $\mathcal{R}^\dagger$, we find that $G^\dagger = G$ on physical test functions and the same is true for the advanced $G_+^{(\dagger)}$ and retarded $G_-^{(\dagger)}$ pieces of G and $G^\dagger$ respectively because these are unique. Using this, $\mathcal{R}G_\pm = G_\pm \mathcal{R} = 1$ and the fact that $\mathcal{R}$ maps $\Gamma_0(\mathcal{R}, A_\mu, M)$ to itself we can compute

$$\mathcal{R}^\dagger f = \mathcal{R}^\dagger G_\pm \mathcal{R} f = \mathcal{R}^\dagger G_\pm^\dagger \mathcal{R} f = \mathcal{R} f.$$

In order for the general self-adjointness condition to be equivalent to the unitarity condition for the purposes of a no-go theorem, it would be necessary to prove that unitarity implies self-adjointness of $\mathcal{R}$ on all test functions and not only on the physical ones. Alternatively, we could also require the latter, weaker self-adjointness condition. However, one could just as well argue that the stronger, general self-adjointness condition is important in its own right irrespective of unitarity because it is equivalent to demand that $\mathcal{R}$ comes from a quadratic action. Thus, we proceed with this stronger condition, because it is easier to verify.

3. A no-go theorem for the consistent quantization of non-minimally coupled spin-$\frac{3}{2}$ fields on general curved spacetimes

We shall prove in the following that a large class of non-minimally coupled field equations $(\mathcal{R}, A_\mu)$ does not satisfy the four conditions compiled in the previous section. In the course of proving this no-go theorem, it will become clear that the proof can be extended to any larger class of operators without much effort, such that the class we shall consider can be safely regarded as effectively exhausting all possible covariant field equations in the $(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$ representation of $SL(2, \mathbb{C})$.

[2] For a hyperbolic $\mathcal{R}$, these Green's functions exist and are unique on any spacetime which fulfills the so-called *global hyperbolicity* condition, see [35,17] for details; this quite natural condition on M shall be tacitly assumed throughout this Letter.

To wit, we consider $\mathcal{R}$ of the form

$$(\mathcal{R}\psi)^\alpha := (-i\nabla\!\!\!/ + m)\psi^\alpha + a_0 m\gamma^\alpha\psi\!\!\!/ + a_1 i\nabla^\alpha\psi\!\!\!/ + a_2 i\gamma^\alpha\nabla_\mu\psi^\mu + a_3 i\gamma^\alpha\nabla\!\!\!/\psi\!\!\!/ + \widetilde{\psi}^\alpha,$$

$$\widetilde{\psi}^\alpha := m\gamma^\alpha B + mC^\alpha + iD^\alpha + i\gamma^\alpha E,$$

$$B := b_1 R_{\mu\nu}\gamma^\mu\psi^\nu + b_2 R\psi\!\!\!/,$$

$$C^\alpha := c_1 R^\alpha{}_\nu\psi^\nu + c_2 R^\alpha{}_\nu\gamma^\nu\psi\!\!\!/ + c_3 R\psi^\alpha + c_4\mathfrak{R}^\alpha{}_\nu\psi^\nu,$$

$$\begin{aligned}D^\alpha := {} & d_1 R^\alpha{}_\nu\psi^\nu + d_2\big(\nabla\!\!\!/ R^\alpha{}_\nu\big)\psi^\nu + d_3 R^\alpha{}_\nu\gamma^\nu\nabla\!\!\!/\psi\!\!\!/ \\ & + d_4\big(\nabla\!\!\!/ R^\alpha{}_\nu\big)\gamma^\nu\psi\!\!\!/ + d_5 R\nabla\!\!\!/\psi^\alpha + d_6(\nabla\!\!\!/ R)\psi^\alpha \\ & + d_7 R^\alpha{}_\nu\nabla^\nu\psi\!\!\!/ + d_8\big(\nabla^\alpha R\big)\psi\!\!\!/ + d_9 R\nabla^\alpha\psi\!\!\!/ \\ & + d_{10}\mathfrak{R}^\alpha{}_\nu\nabla^\nu\psi\!\!\!/ + d_{11}\big(\nabla^\nu\mathfrak{R}^\alpha{}_\nu\big)\psi\!\!\!/ \\ & + d_{12}R_{\mu\nu}\nabla^\alpha\gamma^\mu\psi^\nu + d_{13}\big(\nabla^\alpha R_{\mu\nu}\big)\gamma^\mu\psi^\nu \\ & + d_{14}\big(\nabla_\nu R^\alpha{}_\mu\big)\gamma^\mu\psi^\nu + d_{15}R^\alpha{}_\mu\gamma^\mu\nabla_\nu\psi^\nu,\end{aligned}$$

$$\begin{aligned}E := {} & e_1 R_{\mu\nu}\gamma^\nu\nabla\!\!\!/\psi^\nu + e_2(\nabla\!\!\!/ R_{\mu\nu})\gamma^\mu\psi^\nu + e_3 R\nabla\!\!\!/\psi\!\!\!/ \\ & + e_4(\nabla\!\!\!/ R)\psi\!\!\!/ + e_5(\nabla_\nu R)\psi^\nu + e_6 R\nabla_\nu\psi^\nu \\ & + e_7\big(\nabla^\mu\mathfrak{R}_{\mu\nu}\big)\psi^\nu + e_8\mathfrak{R}_{\mu\nu}\nabla^\mu\psi^\nu + e_9 R_{\mu\nu}\nabla^\mu\psi^\nu,\end{aligned}$$

where ∇_μ is the spin covariant derivative, $a_i \in \mathbb{C}$ are arbitrary constants whereas $\mathfrak{R}_{\alpha\beta} = \frac{1}{4}R_{\alpha\beta\mu\nu}\gamma^\mu\gamma^\nu$ denotes the spin curvature tensor.[3] Moreover, derivatives in parenthesis are meant to act only on the jointly enclosed curvature tensors, and b_i, c_i, d_i, e_i are arbitrary complex-valued functions of curvature invariants and m of mass dimension -2.

We start our proof by checking self-adjointness, since this turns out to be the strongest condition. Indeed, as one can check by direct computation, it is fulfilled on arbitrary curved spacetimes if and only if the following equations are true.

$$a_0^* = a_0, \qquad a_2 = a_1^*, \qquad a_3^* = a_3, \qquad b_1 = c_2^*,$$

$$b_2^* = b_2, \qquad c_1^* = c_1, \qquad c_3^* = c_3, \qquad c_4^* = c_4,$$

$$d_1 = d_3 = d_5 = d_7 = d_9 = d_{10} = d_{12} = d_{15} = 0,$$

$$e_1 = e_3 = e_6 = e_8 = e_9 = 0,$$

$$d_2^* = d_2, \qquad d_4^* = e_2, \qquad d_6^* = d_6, \qquad d_8 = e_5^*,$$

$$d_{11} = e_7^*, \qquad d_{13}^* = d_{14}, \qquad e_4^* = e_4.$$

Here, $*$ denotes complex conjugation. In essence, requiring $\mathcal{R}^\dagger = \mathcal{R}$ rules out terms where a curvature tensor multiplies a derivative of ψ^α, because such terms generate derivatives of curvature tensors by the partial integration involved in the definition of the formal adjoint of $\mathcal{R}^\dagger$. These curvature tensor derivatives cannot be cured by explicitly adding couplings of ψ^α to curvature derivatives, as such terms must be present both in $\mathcal{R}$ and in $\mathcal{R}^\dagger$. Hence, self-adjointness rules out *arbitrary* terms where a curvature tensor multiplies a derivative of ψ^α, extending the validity of this proof to a larger class of $\mathcal{R}$ containing all possible such terms.

We proceed by checking the hyperbolicity bit of our causality condition. Let k_μ be timelike or spacelike and let ψ^α fulfill

$$ik_\mu\sigma^\mu\psi^\alpha = k\!\!\!/\psi^\alpha - a_1 k^\alpha\psi\!\!\!/ - a_2\gamma^\alpha k_\mu\psi^\mu - a_3\gamma^\alpha k\!\!\!/\psi\!\!\!/ = 0,$$

where we have already taken into account that the allowed principal symbols are reduced by self-adjointness. We have to check for which a_i the above equation implies $\psi^\alpha \equiv 0$. By multiplying the above equation with $k\!\!\!/$ and k^α, we can obtain the following derived equations

$$(1 - a_2)k\!\!\!/ k_\mu\psi^\mu = (a_1 + a_3)k^2\psi\!\!\!/,$$

$$(1 - 3a_2)k\!\!\!/ k_\mu\psi^\mu = (1 + 3a_3)k^2\psi\!\!\!/,$$

which can be rewritten as

$$\begin{pmatrix} (1-a_2)\mathbb{1} & -(a_1+a_3)\mathbb{1} \\ (1-3a_2)\mathbb{1} & -(1+3a_3)\mathbb{1} \end{pmatrix}\begin{pmatrix} k\!\!\!/ k_\mu\psi^\mu \\ k^2\psi\!\!\!/ \end{pmatrix} = 0,$$

where $\mathbb{1}$ is the 4×4 identity matrix. As k_μ is timelike or spacelike, this equation together with $ik_\mu\sigma^\mu\psi^\alpha = 0$ implies $\psi^\alpha \equiv 0$ if and only if the determinant of the appearing 8×8 matrix is non-zero; this in turn is the case iff

$$-3a_1a_2 + a_1 + a_2 - 2a_3 - 1 \neq 0. \tag{7}$$

We do not discuss lightlike k_μ, as (7) will be sufficient to prove the theorem.

Finally, we verify the background-independence and irreducibility conditions. To this avail, we contract $(\mathcal{R}\psi)^\alpha = 0$ with both γ_α and ∇_α and combine the results to obtain the following equation for $\psi\!\!\!/$:

$$\begin{aligned}& -\left(\frac{(a_2-1)(1+a_2+4a_3)}{2-4a_2} + a_1 + a_3\right)\nabla_\mu\nabla^\mu\psi\!\!\!/ \\ & + \left(\frac{(a_2-1)(1+4a_0)}{2-4a_2} + \frac{1+a_1+4a_3}{2-4a_2} + a_0\right) im\nabla\!\!\!/\psi\!\!\!/ \\ & + \left(\frac{(a_2-1)(1+a_2+4a_3)}{2-4a_2} + a_3\right)\frac{R}{4}\psi\!\!\!/ \\ & + \frac{1+4a_0}{2-4a_2}m^2\psi\!\!\!/ - \frac{1}{2}R_{\mu\nu}\gamma^\mu\psi^\nu \\ & + \frac{a_2-1}{2-4a_2}i\nabla\!\!\!/\widetilde{\psi\!\!\!/} + i\nabla_\mu\widetilde{\psi}^\mu + \frac{m}{2-4a_2}\widetilde{\psi\!\!\!/} = 0.\end{aligned} \tag{8}$$

Here, our first condition assures that $2 - 4a_2 \neq 0$. To see this, note that contracting $\mathcal{R}\psi^\alpha = 0$ with γ_α yields an equation which can be rewritten as

$$(2 - 4a_2)i\nabla_\mu\psi^\mu = (1 + a_1 + 4a_3)i\nabla\!\!\!/\psi\!\!\!/ + (1 + 4a_0)m\psi\!\!\!/ + \widetilde{\psi\!\!\!/}. \tag{9}$$

If $2 - 4a_2 = 0$, then $\nabla_\mu\psi^\mu = 0$ would not follow from $\mathcal{R}\psi^\alpha = 0$ and $\psi\!\!\!/ = 0$ on Minkowski spacetime, hence $S(\mathcal{R}, 0, \mathbb{M}) = S(\mathcal{R}_0, 0, \mathbb{M})$ would not hold because all elements of $S(\mathcal{R}_0, 0, \mathbb{M})$ satisfy $\nabla_\mu\psi^\mu = 0$.

To assure that our background-independence condition holds, we have to either guarantee that $\psi^\alpha = A_\mu\psi^\mu$ holds automatically for solutions of $\mathcal{R}\psi = 0$ or that $A_\alpha \equiv 0$ on all spacetimes. Let us check if the first of these conditions can be fulfilled. Without specifying A_μ explicitly, we know that, in Minkowski spacetime, $A_\mu \equiv 0$ must hold on account of the irreducibility condition. However, in flat spacetime, (8) is a hyperbolic partial differential equation for $\psi\!\!\!/$, as the coefficient of $\nabla_\mu\nabla^\mu\psi\!\!\!/$ is non-zero if we apply the condition (7) derived from causality and self-adjointness. Such a differential equation has certainly more possible solutions than just $\psi\!\!\!/ \equiv 0$, hence, by combining causality, self-adjointness, and irreducibility, we find that only the optional background-independence condition that A_μ be identically vanishing on all spacetimes can be fulfilled. Inserting this into (8), we are left with

$$-\frac{1}{2}R_{\mu\nu}\gamma^\mu\psi^\nu - \frac{a_2-1}{2-4a_2}i\nabla\!\!\!/\widetilde{\psi\!\!\!/} + i\nabla_\mu\widetilde{\psi}^\mu + \frac{m}{2-4a_2}\widetilde{\psi\!\!\!/} = 0. \tag{10}$$

In Minkowski spacetime, this equation is identically fulfilled and, hence, poses no additional constraints on solutions of $\mathcal{R}\psi = 0$ and

[3] Note that all couplings containing the Riemann tensor $R_{\alpha\beta\mu\nu}$ can be expressed via the spin curvature tensor $\mathfrak{R}_{\alpha\beta}$. Furthermore, we have omitted all couplings which would be linearly dependent by means of Bianchi identities. We follow [37] regarding conventions in the definition of the curvature tensors.

$\not\psi = 0$. To check if our background independence holds, we have to make sure that (10) is identically fulfilled on *all* spacetimes once $\mathcal{R}\psi^\alpha = 0$ and $\not\psi = 0$ hold. To this avail, we insert $\not\psi = 0$ into (9), and both $\not\psi = 0$ and (9) into $\mathcal{R}\psi^\alpha = 0$ to obtain

$$i\nabla_\mu \psi^\mu = \frac{1}{2-4a_2}\widetilde{\not\psi},$$

$$(-i\not\nabla + m)\psi^\alpha + \frac{a_2}{2-4a_2}\gamma^\alpha \widetilde{\not\psi} + \widetilde{\psi}^\alpha = 0.$$

These two equations are the only information on first derivatives of ψ^α one can obtain from $\mathcal{R}\psi = 0$ and $\not\psi = 0$. However, the summand $\nabla_\mu \widetilde{\psi}^\mu$ in (10) contains first derivatives of ψ^α also in terms like e.g. $R_{\mu\nu}\nabla^\mu \psi^\nu$, on which $\mathcal{R}\psi = 0$ and $\not\psi = 0$ give no information in general curved spacetimes. Hence, these terms must identically vanish in $\nabla_\mu \widetilde{\psi}^\mu$, which implies that the coefficients of all terms in $\widetilde{\psi}^\alpha$ surviving the insertion of $\not\psi = 0$ and whose free index $^\alpha$ does not belong to γ^α or ψ^α must vanish. Moreover the coefficients of all terms where γ^α appears followed by other γ-matrices must vanish as well, as these terms also give rise to terms like e.g. $R_{\mu\nu}\nabla^\mu \psi^\nu$ if one considers them in $\nabla_\mu \widetilde{\psi}^\mu$ and commutes the contracted covariant derivative $\not\nabla$ with the additional γ-matrices in order to use the available information on $\not\nabla \psi^\alpha$. Analogously, the terms in $\widetilde{\psi}^\alpha$ where the free index $^\alpha$ belongs to ψ^α but ψ^α is multiplied by γ-matrices are problematic in $\not\nabla \widetilde{\not\psi}$ and have to vanish identically. Altogether, avoiding the appearance of in general undetermined ψ^α-derivatives in (10) enforces

$$b_1 = c_1 = c_4 = d_2 = d_6 = d_{13} = d_{14} = e_2 = e_7 = 0,$$

hence, the remaining terms in $\widetilde{\psi}^\alpha$ not yet ruled out by background independence are

$$\widetilde{\psi}^\alpha = mc_3 R\psi^\alpha + e_5 \gamma^\alpha (\nabla_\nu R)\psi^\nu.$$

We can now explicitly compute the left hand side of (10) by inserting this expression for $\widetilde{\psi}^\alpha$ and the knowledge on $\nabla_\mu \psi^\mu$ and $\not\nabla \psi^\alpha$ obtained from $\mathcal{R}\psi = 0$ and $\not\psi = 0$. The result does not contain any derivatives of ψ^α, but is a sum of various curvature tensors multiplying ψ^α. In general spacetimes, some of these terms are linearly independent and, hence, have to vanish individually in order for (10) to be identically fulfilled on all spacetimes. Particularly, since the only term in the left hand side of (10) containing the Ricci tensor turns out to be the one explicitly visible in (10), we obtain

$$R_{\mu\nu}\gamma^\mu \psi^\nu = 0$$

as a necessary condition for (10) to hold on general spacetimes. However, this is in conflict with background independence, which closes the proof.

One can imagine that the steps taken in the last paragraph of this proof can be generalised to arbitrary couplings of the curvature to ψ^α, and we have argued in the discussion of self-adjointness that the same holds for arbitrary couplings of the curvature to derivatives of ψ^α, hence, we presume that our proof effectively exhausts *all* possible covariant first order differential operators $\mathcal{R}$. Finally, we would like to emphasise that our proof covers both $m > 0$ and $m = 0$.

4. Discussion

The proof of our no-go theorem shows that, even if one allows for a spin-$\frac{3}{2}$ field in a gravitational background to be coupled to the gravitational field in an arbitrary non-minimal way, one is lead to the same Buchdahl-problem present for the minimally coupled equations of motion if one requires in addition that causality and unitarity hold: the model is, at best, only consistent on Einstein spacetimes. Whereas this seems to be a very restrictive condition for the consistent quantization of spin-$\frac{3}{2}$ fields on curved backgrounds, it fits nicely into the widespread picture that supergravity theories are the only consistent models which contain elementary spin-$\frac{3}{2}$ fields, see e.g. [38], as in these models such conditions on the background appear naturally as on-shell conditions [11,12]. One can expect that a generalisation of our no-go theorem to the case where scalar and vector background fields are present in addition to the metric field yields conditions on the background which are compatible with on-shell conditions in supergravity models with $N > 1$ supersymmetries or additional matter multiplets, see e.g. [39–41].

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