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# A Note on an Information Theoretic Form of The Uncertainty Principle

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It is pointed out that an inequality due to Hirschman (1957) makes it possible to express the Heisenberg uncertainty principle in the form of an additive inequality in terms of entropy rather than as a multiplicative inequality in terms of deviations. Application of the new inequality to the spreading of a wave packet is described.

LIST OF SYMBOLS

| | |
|---|---|
| $\psi(r)$ | Wave function in the $r$ representation |
| $\chi(s)$ | Wave function in the $s$ representation |
| $r,s$ | Canonically conjugate dynamical variables |
| $h$ | Planck's constant ($h = 6.6 \times 10^{-27}$ erg sec) |
| $E[\phi]$ | Entropy of the probability distribution $\phi$ |
| $\phi$ | Arbitrary probability distribution |
| $\delta(x)$ | Dirac delta function |
| $m$ | Mass of particle |
| $t$ | Time |

An inequality recently proven by Hirschman (1957) makes it possible to state the uncertainty principle of Heisenberg in the form of an additive inequality involving the entropies or information contents associated with the probability distributions of the pair of canonically conjugate variables. We begin with the statement of the relation between the wave functions for two such variables $r$ and $s$

$$\psi(r) = h^{-1/2} \int e^{2\pi irs/h} \chi(s) \, ds \tag{1}$$

$$\chi(s) = h^{-1/2} \int e^{-2\pi irs/h} \psi(r) \, dr \tag{2}$$

They form a Fourier transform pair. The probability density functions for the two variables are $|\psi(r)|^2$ and $|\chi(s)|^2$. These we take to be normalized to unity over the range of the variables. Changing variables to $r' = h^{-1/2}r$ and $s' = h^{-1/2}s$ formulas (1) and (2) become

$$f(r') = \int e^{2\pi i r' s'} g(s')\, ds' \tag{3}$$

$$g(s') = \int e^{-2\pi i r' s'} f(r')\, dr' \tag{4}$$

where we have defined the new functions by

$$f(r') = h^{1/4}\psi(h^{1/2}r') \tag{5}$$

$$g(s') = h^{1/4}\chi(h^{1/2}s') \tag{6}$$

The factor $h^{1/4}$ assures that these new functions are also normalized to unity. For normalized Fourier-transform pairs of the form (5) and (6) Hirschman has proven the following inequality:

$$E[\,|f(r')|^2] + E[\,|g(s')|^2] \leqq 0 \tag{7}$$

where $E$ is defined by

$$E[\phi] = \int \phi(x) \log \phi(x)\, dx \tag{8}$$

It will be preferable to express this in terms of $H$ where $H = -E$ in order to conform to thermodynamic and information theoretic usage. Thus we write

$$H[\,|f(r')|^2] + H[\,|g(s')|^2] \geqq 0 \tag{9}$$

To obtain the physically interesting form of this theorem we must re-express it in terms of the probability distributions of the original variables $r$ and $s$. To effect this we employ the entropy transformation theorem (Goldman, 1953)

$$H(y) = -\int p(x) \log J(x/y)\, dx + H(x) \tag{10}$$

which gives

$$H(r) = \tfrac{1}{2} \log h + H(r') \tag{11}$$

$$H(s) = \tfrac{1}{2} \log h + H(s') \tag{12}$$

and so Hirschman's inequality becomes

$$H(r) + H(s) \geqq \log h. \tag{13}$$

If the logarithms are taken to the base two we can state this additive form of the uncertainty relation thus: *By no quantum mechanical measurements is it possible to reduce the joint entropy in two canonically conjugate distributions below the level of approximately minus thirty-seven bits.*

Hirschman has made the very plausible conjecture that the inequality (7) might be strengthened to

$$E[\,|f(r')|^2] + E[\,|g(s')|^2] \leqq \log(2/e) \tag{14}$$

for a very general class of functions, but this remains undemonstrated. The additive form of the uncertainty principle would then read

$$H(r) + H(s) \geqq \log \tfrac{1}{2}he. \tag{15}$$

The equality holds in this equation in the case of Gaussian distributions in both $r$ and $s$.

The additive entropic statement of the uncertainty relation given here has a meaning quite different from the conventional statement that

$$\Delta r \Delta s \geqq h/4\pi \tag{16}$$

where $\Delta r$ and $\Delta s$ are the rms deviations of the variables. The difference is due to the fact that the two measures of spread $\Delta r$ and $H(r)$ are left unaltered by redistributions of very different kinds. The entropy measure, quite unlike the deviation measure, is invariant with respect to any redistribution equivalent to cutting up the original distribution into sections under the original curve and mixing or separating them in any fashion. Thus there are possible radical alterations in the dis-

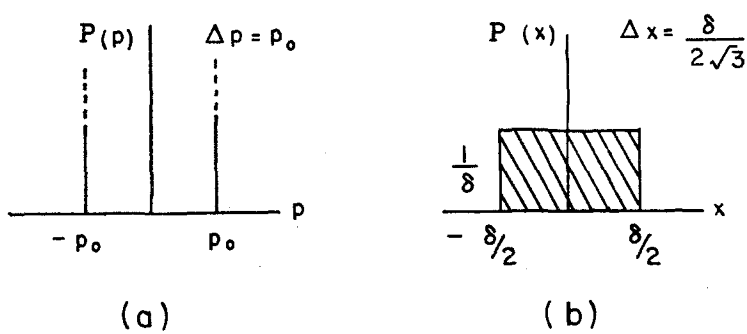


FIG. 1

tribution of $r$ which, even in the limiting case where the equality holds in (13), require no compensatory changes in the entropy of the $s$ distribution. This peculiarity is interesting in connection with the problem of the spreading of a wave packet. Consider the distributions in position and momentum illustrated in Fig. 1. Analytically these are described by

$$P(p) = \tfrac{1}{2}\delta(p - p_0) + \tfrac{1}{2}\delta(p + p_0) \tag{17}$$

$$P(x) = \left.\begin{array}{ll} 1/\delta & -\tfrac{1}{2}\delta < x < \tfrac{1}{2}\delta \\ 0 & \text{otherwise.} \end{array}\right\} \tag{18}$$

Let us take these distributions to be of the optimal kind permitted by the conventional statement of the uncertainty principle. In this case we have, by applying (16) to (17) and (18),

$$\delta = \sqrt{3}h/2\pi p_0 \tag{19}$$

It can readily be seen that the sharp two-valued momentum distribution will cause the distribution in position to divide into two rectangular distributions of half the initial height. The initial coordinate entropy is log $\delta$; after the rectangle has divided it will be log $2\delta$. But once this increase has taken place there will be no further increase of coordinate entropy as the rectangles continue to separate even though their rms deviation in position continues to increase according to

$$\Delta x = p_0 t/m \tag{20}$$

It seems, therefore, that there is no necessity for the net entropy in position and momentum to increase beyond a certain point. This cannot be inferred, however, because although the rms deviations are well defined in the example given, the entropy in the momentum distribution is undefined.

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## REFERENCES

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