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DERIVED CATEGORIES AND STABLE EQUIVALENCE

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Introduction

Happel [6] and Cline, Parshall and Scott [4] showed that the tilting functors of Happel and Ringel [8] can be interpreted in terms of an equivalence of derived categories of the module categories involved. In [10] we generalised this result to give necessary and sufficient conditions for such an equivalence; in this more general case tilting modules must be replaced by 'tilting complexes', which are chain complexes of projective modules that satisfy conditions analogous to those satisfied by tilting modules (see Theorem 1.1 below).

The aim of this paper is to show that this generalisation has interesting applications that do not arise for the more restrictive tilting modules. In particular, if Λ is a self-injective algebra (for example, a modular group algebra for a finite group) then it is easy to see that any tilting module for Λ is projective – in fact, any Λ-module of finite projective dimension is projective. Therefore in this case classical tilting theory reduces to Morita equivalence. In the more general case of tilting complexes we shall show that there are many applications of the theory to self-injective algebras and in Section 2 we shall show that 'derived equivalence' for self-injective algebras is closely connected with stable equivalence.

There has been work connecting tilting theory and self-injective algebras via 'trivial extension algebras'; for example, Tachikawa and Wakamatsu [11] showed that if Γ is a finite-dimensional algebra that is tilted from Λ, then the trivial extension algebras $T\Lambda$ and $T\Gamma$ are stably equivalent. In Section 3 we generalise their result and show that it has a very natural proof in terms of derived equivalence. In fact a tilting complex for Λ with endomorphism ring Γ gives rise, by tensoring with $T\Lambda$, to a tilting complex for $T\Lambda$ with endomorphism ring $T\Gamma$.

We hope that derived equivalence may have useful applications to modular representation theory and in Section 4 we start with the simplest case, blocks with cyclic defect group, and show that the Brauer tree algebras that are stably equivalent are in fact derived equivalent.

One fact that follows from the results of this paper is that derived equivalence is more general than tilting in the sense that there are many examples of derived equivalent algebras Λ and Γ where Γ cannot be obtained from Λ by a sequence of tilting or cotilting steps. Most of the examples in this paper are of self-injective algebras, but in Section 5 we give an example involving algebras of finite global dimension.

1. Preliminaries

Throughout this paper all algebras will be finite-dimensional algebras with 1 over a fixed field k. All modules will be finitely generated unitary right modules unless we specify otherwise, and we shall compose endomorphisms as though written on the left. The category of such modules for an algebra Λ will be denoted by mod-Λ; the full subcategory consisting of projective modules will be denoted by P_Λ.

For basic results on triangulated categories we refer to [9] or [12], but our notation will be that of [2]. In particular, we shall denote by $X[n]$ rather than by T^nX the object obtained from X by applying the 'shift' functor n times. By $D^{\mathrm{b}}(\text{mod-}\Lambda)$ we mean the derived category of bounded complexes over mod-Λ, and by $K^{\mathrm{b}}(P_\Lambda)$ we mean the homotopy category of bounded complexes over P_Λ.

The following theorem summarises the results of [10] that we shall need:

Theorem 1.1. *Let Λ and Γ be two finite-dimensional algebras. The following are equivalent*:

(a) $D^{\mathrm{b}}(\text{mod-}\Lambda)$ *and* $D^{\mathrm{b}}(\text{mod-}\Gamma)$ *are equivalent as triangulated categories.*

(b) $K^{\mathrm{b}}(P_\Lambda)$ *and* $K^{\mathrm{b}}(P_\Gamma)$ *are equivalent as triangulated categories.*

(c) Γ *is isomorphic to the endomorphism ring of an object* P^* *of* $K^{\mathrm{b}}(P_\Lambda)$ *such that*

(i) *For* $n\neq 0$,

$$\mathrm{Hom}(P^*, P^*[n]) = 0.$$

(ii) $\mathrm{add}(P^*)$, *the full subcategory of* $K^{\mathrm{b}}(P_\Lambda)$ *consisting of direct summands of direct sums of copies of* P^*, *generates* $K^{\mathrm{b}}(P_\Lambda)$ *as a triangulated category.*

Moreover, any equivalence as in (a) *restricts to an equivalence between the full subcategories consisting of objects isomorphic to bounded complexes of projectives (which are equivalent to* $K^{\mathrm{b}}(P_\Lambda)$ *and* $K^{\mathrm{b}}(P_\Gamma)$ *respectively).* □

If Λ and Γ satisfy the conditions of the theorem then we say that they are 'derived equivalent' and we call the object P^* of (c) a 'tilting complex' for Λ.

For an algebra Λ of finite global dimension, Happel defined 'Auslander–Reiten triangles' in [6]. He showed that for any indecomposable object X of the derived category there is a sink map $\alpha: Y\to X$; that is, a map α such that

(i) α is not a split epimorphism.

(ii) Any map $\beta: Z\to X$ that is not a split epimorphism factors through α.

(iii) Any endomorphism γ of Y satisfying $\alpha\gamma=\alpha$ is an automorphism.

He also showed that a map with these properties is determined uniquely up to isomorphism by X and fits into a distinguished triangle:

$$Y \to X \to \nu X \to$$

where νX is the complex of injective modules obtained by applying the Nakayama functor

$$\nu = D\,\mathrm{Hom}_\Lambda(-,\Lambda) = - \otimes_\Lambda D\Lambda$$

to a projective resolution of X, $D-$ denoting the duality functor $\mathrm{Hom}_k(-,k)$. The proof carries through word for word even for algebras of infinite global dimension so long as X is isomorphic to a bounded complex of projective modules. Thus the Nakayama functor determines an equivalence ν of triangulated categories between the two full subcategories of $D^{\mathrm{b}}(\text{mod-}\Lambda)$ consisting of objects that are isomorphic to bounded complexes of projectives and injectives respectively, and for any X that is isomorphic to a bounded complex of projectives, νX is characterised by a universal property. The next result now follows easily – recall that an algebra Λ is said to be 'symmetric' if Λ and $D\Lambda$ are isomorphic as Λ-bimodules and that Λ is said to be 'weakly symmetric' if every simple Λ-module has a projective cover isomorphic to its injective hull.

Proposition 1.2. *If Λ and Γ are derived equivalent algebras and if Λ is symmetric, then Γ is weakly symmetric.*

Proof. If Λ is symmetric, then the Nakayama functor $\nu = - \otimes_\Lambda D\Lambda$ is isomorphic to the identity functor, so $X \cong \nu X$ for all objects X of $K^{\mathrm{b}}(P_\Lambda)$.

Γ is weakly symmetric precisely when $P \cong \nu P$ for all projective Γ-modules P. □

Recall from [12] that a full triangulated subcategory E of a triangulated category C is called an épaisse subcategory if the following condition is satisfied:

If $X \to Y$ is a map in C which is contained in a distinguished triangle

$$X \to Y \to Z \to,$$

where Z is in E, and if the map also factors through an object W of E, then X and Y are objects of E.

We refer to [12] for the theory of taking quotients by épaisse subcategories. Our next result gives an alternative formulation of the definition.

Proposition 1.3. *A full triangulated subcategory E of a triangulated category C is épaisse if and only if every object of C that is a direct summand of an object of E is itself an object of E.*

Proof. Suppose first that E is épaisse and that $X \cong X_1 \oplus X_2$ is in E. Then the zero map $X_2[-1] \to X_1$ is contained in a distinguished triangle

$$X_2[-1] \to X_1 \to X \to$$

and any zero map factors through an object of E, so X_1 and X_2 are objects of E.

Suppose now that E is closed under taking direct summands, and let $X \to Y$ be a map in C that factors through an object W of E and that is contained in a distinguished triangle

$$X \to Y \to Z \to$$

with Z in E. By the octahedral axiom we have the following commutative diagram where the rows and columns are distinguished triangles and L and M are objects of C:

$$\begin{array}{ccccccc}
 & & M[-1] & = & M[-1] & & \\
 & & \downarrow & & \downarrow & & \\
X & \longrightarrow & W & \longrightarrow & L & \longrightarrow & X[1] \\
\| & & \downarrow & & \downarrow & & \| \\
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
 & & \downarrow & & \downarrow & & \\
 & & M & = & M & &
\end{array}$$

Consider the composition of maps

$$Z \to X[1] \to W[1]$$

coming from this diagram. It gives us the following octahedral diagram, where N is another object of E:

$$\begin{array}{ccccccc}
 & & L & = & L & & \\
 & & \downarrow & & \downarrow & & \\
Z & \longrightarrow & X[1] & \longrightarrow & Y[1] & \longrightarrow & Z[1] \\
\| & & \downarrow & & \downarrow & & \| \\
Z & \longrightarrow & W[1] & \longrightarrow & N & \longrightarrow & Z[1] \\
 & & \downarrow & & \downarrow & & \\
 & & L[1] & = & L[1] & &
\end{array}$$

The map $L \to Y[1]$ in this diagram factors as

$$L \to X[1] \to W[1] \to Y[1],$$

but the composition

$$L \to X[1] \to W[1]$$

is zero, since it is the composition of two maps in a distinguished triangle. Thus N is isomorphic to $Y[1] \oplus L[1]$, so Y (and therefore also X) is an object of E, and so E is an épaisse subcategory of C. □

2. The stable module category

We shall start by recalling the definition of the stable module category **mod**-Λ of a finite-dimensional algebra Λ. It is the k-additive category whose objects are the same as the objects of mod-Λ and where the morphisms are equivalence classes of module maps under the equivalence relation of differing by a map that factors through a projective Λ-module, so

$$\mathbf{Hom}_\Lambda(X, Y) = \mathrm{Hom}_{\mathbf{mod}\text{-}\Lambda}(X, Y)$$

is a quotient space of $\mathrm{Hom}_\Lambda(X, Y)$ for Λ-modules X and Y. If two algebras have equivalent stable module categories, then we say that they are 'stably equivalent'.

In [6] Happel showed that the stable module category of a self-injective algebra carries the structure of a triangulated category in a natural way. In this section we shall give a new description of this structure, linking it to the triangulated structure on the derived category of the algebra.

Theorem 2.1. *Let Λ be a self-injective algebra. The essential image of the natural embedding*

$$K^{\mathrm{b}}(P_\Lambda) \to D^{\mathrm{b}}(\text{mod-}\Lambda)$$

(*that is, the full subcategory of* $D^{\mathrm{b}}(\text{mod-}\Lambda)$ *consisting of all objects isomorphic to objects of* $K^{\mathrm{b}}(P_\Lambda)$) *is an épaisse subcategory. The quotient category*

$$D^{\mathrm{b}}(\text{mod-}\Lambda)/K^{\mathrm{b}}(P_\Lambda)$$

is equivalent as a triangulated category to the stable module category of Λ.

Proof. The first assertion is easy, especially using the characterisation of épaisse subcategories in Proposition 1.3.

Consider the additive functor

$$F' : \text{mod-}\Lambda \to D^{\mathrm{b}}(\text{mod-}\Lambda)/K^{\mathrm{b}}(P_\Lambda)$$

obtained by composing the natural embedding of mod-Λ into $D^{\mathrm{b}}(\text{mod-}\Lambda)$ with the quotient functor. Clearly $F'(P)$ is zero for any projective module P, so F' factors through the natural functor

$$\text{mod-}\Lambda \to \textbf{mod-}\Lambda$$

to give a functor

$$F: \textbf{mod-}\Lambda \to D^{\mathrm{b}}(\text{mod-}\Lambda)/K^{\mathrm{b}}(P_\Lambda).$$

We shall show that F is in fact an equivalence of categories.

First we shall show that F is an exact functor. A distinguished triangle in **mod**-Λ is a triangle

$$X \to Y \to Z \to X[1] = \Omega^{-1}X$$

coming from a pushout diagram of modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & I & \longrightarrow & \Omega^{-1}X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Omega^{-1}X & \longrightarrow & 0, \end{array}$$

where $X \to I$ is the embedding of X into its injective hull [6]. Since short exact sequences of modules give distinguished triangles in the derived category, and since $F'I = 0$,

$$FX \to FY \to FZ \to FX[1]$$

is a distinguished triangle.

Next we note that F is full, since F' is clearly full, and that for no non-zero object X of **mod**-Λ is $FX \cong 0$, since no non-projective Λ-module is isomorphic in the derived category to an object of $K^{\mathrm{b}}(P_\Lambda)$. These properties of F are enough to prove that F is also faithful, for suppose $\alpha: X \to Y$ is a map in **mod**-Λ for which $F\alpha = 0$, and suppose that α sits in a distinguished triangle

$$X \to Y \to Z \to;$$

then the identity map of FY factors through $FY \to FZ$, so, since F is full, there is a map $\beta: Y \to Y$, factoring through $Y \to Z$, such that $F\beta$ is an isomorphism. But then the mapping cone of β is sent to zero by F, so β is an isomorphism, so $Y \to Z$ is a split monomorphism and α is zero.

To complete the proof that F is an equivalence, we just need to show that every object X of $D^{\mathrm{b}}(\text{mod-}\Lambda)/K^{\mathrm{b}}(P_\Lambda)$ is isomorphic to $F'M$ for some module M. As an object of $D^{\mathrm{b}}(\text{mod-}\Lambda)$, X is isomorphic to a complex of projectives

$$P^* = \cdots \to P^r \to P^{r+1} \to \cdots \to P^s \to 0,$$

where $r < 0$ and P^* has zero homology in degrees less than r.

The natural map from P^* to

$$\tilde{P}^* = \cdots \to P^{r-1} \to P^r \to 0 \to \cdots$$

is an isomorphism in $D^b(\text{mod-}\Lambda)/K^b(P_\Lambda)$, since its mapping cone is a bounded complex of projectives, and there is a complex

$$Q^* = \cdots \to P^{r-1} \to P^r \to Q^{r+1} \to \cdots \to Q^0 \to 0$$

which is the projective resolution of some module M and whose natural map to $\tilde{P}^*$ is an isomorphism in $D^b(\text{mod-}\Lambda)/K^b(P_\Lambda)$. Thus $P^* \cong F'M$. □

Corollary 2.2. *Let Λ and Γ be self-injective algebras. If Λ and Γ are derived equivalent then they are stably equivalent.*

Proof. By the last part of Theorem 1.1, an equivalence

$$D^b(\text{mod-}\Lambda) \to D^b(\text{mod-}\Gamma)$$

induces an equivalence of triangulated categories

$$D^b(\text{mod-}\Lambda)/K^b(P_\Lambda) \to D^b(\text{mod-}\Gamma)/K^b(P_\Gamma),$$

and so, by Theorem 2.1, an equivalence of triangulated categories

$$\underline{\textbf{mod}}\text{-}\Lambda \to \underline{\textbf{mod}}\text{-}\Gamma. \quad \square$$

3. Trivial extension algebras

In this section we shall give our first examples of derived equivalent self-injective algebras. First we must recall some properties of trivial extension algebras; further details can be found in [11].

Let Λ be any finite-dimensional algebra. We can define a new algebra $T\Lambda$ as the vector space $\Lambda \oplus D\Lambda$ with multiplication defined by

$$(x, y) \cdot (x', y') = (xx', xy' + yx')$$

using the Λ-bimodule structure of $D\Lambda$. We call $T\Lambda$ the 'trivial extension algebra' of Λ.

The algebra $T\Lambda$ is a symmetric algebra and the natural map $\Lambda \to T\Lambda$ is an embedding of rings.

There are many results linking tilting theory with trivial extension algebras; in particular it has been shown by Tachikawa and Wakamatsu [11] that if an algebra Γ is obtained by tilting another algebra Λ then the trivial extension algebras $T\Lambda$ and $T\Gamma$ are stably equivalent. Wakamatsu has extended this result to the case of 'generalised' tilting modules (that is, allowing finite projective dimension larger than 1). In the light of Corollary 2.2, the next theorem can be regarded as a generalisation of this result and also provides a new proof.

Theorem 3.1. *Let Λ and Γ be derived equivalent algebras. The trivial extension algebras $T\Lambda$ and $T\Gamma$ are also derived equivalent.*

Proof. By Theorem 1.1, Γ is isomorphic to the endomorphism ring of a tilting complex P^* for Λ. We shall show that $P^* \otimes_\Lambda T\Lambda$ is a tilting complex for $T\Lambda$ with endomorphism ring isomorphic to $T\Gamma$. The theorem will then follow by another application of Theorem 1.1.

We recall from [9] the following way of calculating homomorphism groups in homotopy categories:

Let A be an additive category and let X^* and Y^* be objects of $K(A)$. There is a double complex of abelian groups obtained by applying the bifunctor $\mathrm{Hom}_A(-,-)$, whose (i,j) term is $\mathrm{Hom}_A(X^{-i}, Y^j)$. We obtain a single complex by taking the 'completed' total complex (i.e. by taking the direct product rather than the direct sum of terms along each diagonal). The homology in degree n of this single complex is naturally isomorphic to $\mathrm{Hom}_{K(A)}(X^*, Y^*[n])$. If X^* and Y^* are bounded complexes, then we are only taking finite direct products, so this is just the ordinary total complex.

Since P^* is a tilting complex, we therefore know that the homology of the total complex of $\mathrm{Hom}_\Lambda(P^*, P^*)$ is isomorphic to Γ concentrated in degree zero.

To calculate

$$\mathrm{Hom}(P^* \otimes_\Lambda T\Lambda, P^* \otimes_\Lambda T\Lambda[n])$$

we need to consider the double complex

$$\mathrm{Hom}_{T\Lambda}(P^* \otimes_\Lambda T\Lambda, P^* \otimes_\Lambda T\Lambda).$$

This is naturally isomorphic to the direct sum of double complexes

$$\mathrm{Hom}_\Lambda(P^*, P^*) \oplus \mathrm{Hom}_\Lambda(P^*, \nu P^*),$$

where ν is the Nakayama functor. We know the homology of the first term. The second term is isomorphic to the double complex $D\,\mathrm{Hom}_\Lambda(P^*, P^*)$, since $\mathrm{Hom}_\Lambda(-, \nu -)$ and $D\,\mathrm{Hom}_\Lambda(-,-)$ are isomorphic as bifunctors on P_Λ, and this has homology $D\Gamma$ concentrated in degree zero. Thus

$$\mathrm{Hom}(P^* \otimes_\Lambda T\Lambda, P^* \otimes_\Lambda T\Lambda[n]) = 0$$

for $n \neq 0$, and

$$\mathrm{Hom}(P^* \otimes_\Lambda T\Lambda, P^* \otimes_\Lambda T\Lambda) \cong \Gamma \oplus D\Gamma$$

as a vector space.

We now want to check that $\mathrm{End}(P^* \otimes_\Lambda T\Lambda)$ and $T\Gamma$ are isomorphic as rings. As a complex of Λ-modules $P^* \otimes_\Lambda T\Lambda$ is isomorphic to $P^* \oplus \nu P^*$. Since

$$\mathrm{Hom}(P^*, \nu P^*) \cong D\,\mathrm{Hom}(P^*, P^*),$$

the isomorphism of vector spaces gives a map

$$\theta : T\Gamma = \mathrm{End}(P^*) \oplus \mathrm{Hom}(P^*, \nu P^*) \to \mathrm{End}(P^* \oplus \nu P^*)$$

such that, for α in $\mathrm{End}(P^*)$ and β in $\mathrm{Hom}(P^*, \nu P^*)$, $\theta(\alpha+\beta)$ is given by the following matrix (remember that we are writing endomorphisms on the left):

$$\begin{pmatrix} \alpha & 0 \\ \beta & \nu\alpha \end{pmatrix} : P^* \oplus \nu P^* \to P^* \oplus \nu P^*.$$

We want to check that θ is a map of rings, so let α, α' be elements of $\mathrm{End}(P^*)$ and let β, β' be elements of $\mathrm{Hom}(P^*, \nu P^*)$. Since

$$D\,\mathrm{Hom}(P^*, P^*) \cong \mathrm{Hom}(P^*, \nu P^*)$$

is an isomorphism of $\mathrm{End}(P^*)$-bimodules, $\alpha+\beta$ and $\alpha'+\beta'$ multiply in $T\Gamma$ to give $\alpha\alpha' + (\nu(\alpha)\beta' + \beta\alpha')$, which is sent by θ to

$$\begin{pmatrix} \alpha\alpha' & 0 \\ (\nu\alpha)\beta' + \beta\alpha' & \nu(\alpha\alpha') \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \beta & \nu\alpha \end{pmatrix} \begin{pmatrix} \alpha' & 0 \\ \beta' & \nu\alpha' \end{pmatrix},$$

which is just $\theta(\alpha+\beta) \cdot \theta(\alpha'+\beta')$ as required.

All that remains to be checked is that $\mathrm{add}(P^* \otimes_\Lambda T\Lambda)$ generates $K^{\mathrm{b}}(P_{T\Lambda})$ as a triangulated category. The functor

$$- \otimes_\Lambda T\Lambda : P_\Lambda \to P_{T\Lambda}$$

induces an exact functor

$$F : K^{\mathrm{b}}(P_\Lambda) \to K^{\mathrm{b}}(P_{T\Lambda}).$$

Let C be the full triangulated subcategory of $K^{\mathrm{b}}(P_{T\Lambda})$ generated by $\mathrm{add}(P^* \otimes_\Lambda T\Lambda)$, and let D be the full triangulated subcategory of $K^{\mathrm{b}}(P_\Lambda)$ consisting of those X^* for which FX^* is in C. Then D is a full triangulated subcategory of $K^{\mathrm{b}}(P_\Lambda)$ containing $\mathrm{add}(P^*)$, and so

$$D = K^{\mathrm{b}}(P_\Lambda).$$

Therefore $P_{T\Lambda}$ is contained in C, so

$$C = K^{\mathrm{b}}(P_{T\Lambda}).$$

This completes the proof of the theorem. □

Corollary 3.2. *If Λ and Γ are derived equivalent algebras, for example if Γ is tilted from Λ, then $T\Lambda$ and $T\Gamma$ are stably equivalent.*

Proof. Immediate by Theorem 3.1 and Corollary 2.2. □

4. Brauer tree algebras

We recall that a Brauer tree consists of a finite tree T together with:

(a) a cyclic ordering of the edges adjacent to each vertex, usually described by the anticlockwise ordering given by a fixed planar representation of T.

(b) a specified vertex v of T, called the 'exceptional vertex'.

(c) a positive integer m, called the 'multiplicity of the exceptional vertex'.

A Brauer tree (T, v, m) determines up to Morita equivalence a symmetric algebra called a 'Brauer tree algebra'. This algebra has one isomorphism class of simple modules for each edge of T. Let the edges of T be labelled $1, 2, \ldots, e$ and let $S(1), \ldots, S(e)$ be the corresponding simple modules. The projective cover $P(i)$ of $S(i)$ has

$$\mathrm{soc}(P(i)) \cong P(i)/\mathrm{rad}(P(i)) \cong S(i)$$

and $\mathrm{rad}(P(i))/\mathrm{soc}(P(i))$ is the direct sum of two uniserial modules, one associated with each vertex adjacent to the edge i. Suppose the cyclic ordering of the edges adjacent to such a vertex is

$$(i = i_0, i_1, \ldots, i_n, i_0).$$

When the vertex is not the exceptional vertex, the composition factors of the corresponding uniserial module are given in order, from top to socle, by

$$S(i_1), \ldots, S(i_n).$$

When the vertex is the exceptional vertex v, the composition factors are given in order by

$$S(i_1), \ldots, S(i_n), S(i_0), S(i_1), \ldots, S(i_n),$$

where we read through the cyclic ordering m times. Thus when $m = 1$, this is no different from the case of an ordinary vertex.

These Brauer tree algebras are of interest because any block with cyclic defect group of a finite modular group algebra is of this form. For further details we refer to [1] and [5].

One type of Brauer tree that will be of particular interest is the 'star' with e edges and multiplicity m, where all the edges are adjacent to the exceptional vertex. We shall call the corresponding basic Brauer tree algebra $B(e, m)$. Note that all the projective indecomposable modules for $B(e, m)$ are uniserial, since for each edge i one of the two direct summands of $\mathrm{rad}(P(i))/\mathrm{soc}(P(i))$ is the zero module.

Before we state the main result of this section, we shall point out the following trivial but important facts:

Remark 4.1. *Given a Brauer tree algebra associated with a Brauer tree* (T, v, m),

$$\mathrm{Hom}(P(i), P(j)) = 0$$

unless the edges i and j have a vertex in common. If i and j have a vertex in common and $i \neq j$, *then* $\mathrm{Hom}(P(i), P(j))$ *is one-dimensional unless this is the exceptional vertex, in which case* $\mathrm{Hom}(P(i), P(j))$ *is m-dimensional.*

If the edge i is adjacent to the exceptional vertex, then $\mathrm{End}(P(i))$ *is* $(m+1)$-*dimensional, otherwise* $\mathrm{End}(P(i))$ *is two-dimensional.* □

We are now ready to state our result on derived categories of Brauer tree algebras.

Theorem 4.2. *Up to derived equivalence, a Brauer tree algebra is determined by the number of edges of the Brauer tree and the multiplicity of the exceptional vertex.*

Proof. Let Λ be a Brauer tree algebra associated with the Brauer tree (T, v, m), where T has e edges. We shall construct a tilting complex for Λ with endomorphism ring isomorphic to $B(e, m)$.

For each edge i of T there is a unique path in T from v to the furthest end of i; this defines a sequence

$$i_0, i_1, \dots, i_r = i$$

of edges. By Remark 4.1, there is a unique, up to isomorphism, complex of projective Λ-modules

$$\cdots \to 0 \to P(i_0) \to P(i_1) \to \cdots \to P(i_r) \to 0 \to \cdots$$

where all the maps are non-zero and where $P(i_0)$ is the degree zero term. Let $Q(i)$ denote this complex considered as an object of $K^{\mathrm{b}}(P_\Lambda)$ and let Q be the direct sum of the $Q(i)$, one for each edge i. We shall show that Q is the tilting complex that we require.

First, it is clear that $\mathrm{add}(Q)$ generates $K^{\mathrm{b}}(P_\Lambda)$ as a triangulated category, since $P(i)$ is the mapping cone of the obvious map from $Q(i)[r-1]$ to $Q(i_{r-1})[r-1]$.

By Remark 4.1, it is also clear that $\mathrm{Hom}(Q, Q[n]) = 0$ unless n is -1, 0 or 1.

Consider a map α of complexes from $Q(i)$ to $Q(j)[1]$. This consists of maps

$$\alpha_m : P(i_m) \to P(j_{m+1})$$

making the following diagram commute:

$$\begin{array}{ccccccc}
 & & P(i_0) & \longrightarrow & P(i_1) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & \\
P(j_0) & \longrightarrow & P(j_1) & \longrightarrow & P(j_2) & \longrightarrow & \cdots .
\end{array}$$

If $\alpha \neq 0$, then we can choose s as large as possible so that $\alpha_s \neq 0$. We may assume that we have chosen α from its homotopy class so as to minimise this value of s. By Remark 4.1, $i_s = j_s$ and α_s factors through $P(j_s) \to P(j_{s+1})$. But this factoring map $P(i_s) \to P(j_s)$ gives a homotopy from α to a map β for which $\beta_t = 0$ for $t \geq s$. Thus α must be zero, so

$$\mathrm{Hom}(Q, Q[1]) = 0.$$

Consider next a map α of complexes from $Q(i)$ to $Q(j)[-1]$. This consists of maps

$$\alpha_m : P(i_{m+1}) \to P(j_m)$$

making the following diagram commute:

$$\begin{array}{ccccccc} P(i_0) & \longrightarrow & P(i_1) & \longrightarrow & P(i_2) & \longrightarrow & \cdots \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \alpha_0} & & \Big\downarrow{\scriptstyle \alpha_1} & & \\ 0 & \longrightarrow & P(j_0) & \longrightarrow & P(j_1) & \longrightarrow & \cdots . \end{array}$$

If α is non-zero, then choose s as small as possible so that $\alpha_s \neq 0$. By Remark 4.1, $i_s = j_s$ and the composition

$$P(i_s) \to P(i_{s+1}) \xrightarrow{\alpha_i} P(j_s)$$

has image $\operatorname{soc}(P(j_s))$ and in particular is non-zero, contradicting the fact that the diagram commutes. Thus $\alpha = 0$ and so

$$\operatorname{Hom}(Q, Q[-1]) = 0.$$

We have now proved that Q is a tilting complex and must show that $\operatorname{End}(Q)$ is isomorphic to $B(e, m)$. By Proposition 1.2, $\operatorname{End}(Q)$ is a weakly symmetric algebra, and by Corollary 2.2 it is stably equivalent to Λ. The results of [5] then show that $\operatorname{End}(Q)$ is a Brauer tree algebra for some Brauer tree with e edges and multiplicity m.

It is easy to calculate the Cartan invariants

$$c_{ij} = \dim_k \operatorname{Hom}(Q(i), Q(j))$$

of $\operatorname{End}(Q)$, since for any objects P_a^* and P_b^* of $K^{\mathrm{b}}(P_\Lambda)$ we have the formula

$$\sum_n (-1)^n \dim_k \operatorname{Hom}(P_a^*, P_b^*[n]) = \sum_{r,s} (-1)^{r-s} \dim_k \operatorname{Hom}(P_a^r, P_b^s),$$

and for $Q(i)$ and $Q(j)$ the left-hand side reduces to c_{ij} and the right-hand side is easy to calculate using Remark 4.1.

This calculation tells us that

$$c_{ij} = \begin{cases} m+1 & \text{if } i = j, \\ m & \text{otherwise} \end{cases}$$

and the only basic Brauer tree algebra with these Cartan invariants is $B(e, m)$.

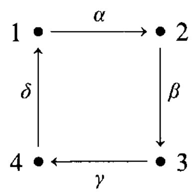
Since $K_0(B(e, m))$ has rank e and the Cartan matrix of $B(e, m)$ has determinant $em + 1$, and since both of these quantities are invariants of the derived category, different values of e and m give algebras $B(e, m)$ that are not derived equivalent. □

One could calculate $\operatorname{End}(Q)$ explicitly without using the results of [5]. This is not very hard but requires more lengthy calculations than the method of proof that we have used.

5. Concluding remarks

In [7], it was proved that if Λ and Γ are derived equivalent algebras and Λ is hereditary, then Γ can be obtained from Λ by a sequence of tilting steps. As was pointed out in the introduction, the examples in Sections 3 and 4 of this paper show that some restriction on Λ is necessary. However, all these examples involve algebras of infinite global dimension. We shall give here an example where the algebras involved have finite global dimension.

Let Λ be the 20-dimensional algebra given by the quiver



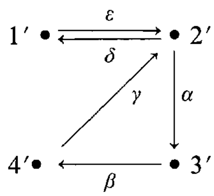
with relations

$$\alpha\beta\gamma\delta\alpha = 0, \qquad \delta\alpha\beta\gamma = 0.$$

The global dimension of Λ is four, and it has a tilting complex

$$0 \to P(2) \oplus P(2) \oplus P(3) \oplus P(4) \to P(1) \to 0,$$

where $P(i)$ denotes the projective module associated with vertex i and the map is given by the unique (up to scalar multiplication) map $P(2) \to P(1)$ on the first factor and by zero on the other three factors. The endomorphism ring Γ of this tilting complex is the 17-dimensional algebra given by the following quiver, where the projective modules at the vertices 1′, 2′, 3′, and 4′ correspond respectively to the direct summands $P(2) \to P(1)$, $P(2) \to 0$, $P(3) \to 0$ and $P(4) \to 0$ of the tilting complex:



with relations

$$\gamma\alpha\beta = 0, \qquad \gamma\delta = 0, \qquad \varepsilon\alpha\beta = 0, \qquad \delta\varepsilon = \alpha\beta\gamma.$$

We shall not give the details of the computation of this endomorphism ring, but as a sample of the kind of calculations needed we shall work out the dimension of the endomorphism ring of $P(2) \to P(1)$.

Both $P(1)$ and $P(2)$ have 2-dimensional endomorphism rings, and we can choose bases $\{1, \theta\}$ and $\{1, \phi\}$, where θ and ϕ are non-zero non-isomorphisms normalised so that, for a map $\psi : P(2) \to P(1)$, $\psi\phi = \theta\psi$. The diagram

$$\begin{array}{ccc} P(2) & \longrightarrow & P(1) \\ {\scriptstyle \lambda_1+\lambda_2\phi}\downarrow & & \downarrow{\scriptstyle \lambda_3+\lambda_4\theta} \\ P(2) & \longrightarrow & P(1) \end{array}$$

commutes if and only if $\lambda_1=\lambda_3$, and there is a map from $P(1)$ to $P(2)$ giving a homotopy to zero if and only if $\lambda_1=\lambda_3=0$ and $\lambda_2=\lambda_4$. Thus the endomorphism ring in the homotopy category is a 3-dimensional ring of endomorphisms in the category of complexes modulo a 1-dimensional ideal of endomorphisms that are homotopic to zero, and so it is 2-dimensional.

Note that Λ has one projective indecomposable module $P(1)$ that is isomorphic to its image under the Nakayama functor, but Γ has none. We shall show that the number of such projectives is an invariant under tilting and cotilting, so Λ and Γ are not tilting-cotilting equivalent.

Every projective-injective indecomposable module P for an algebra must be a direct summand of every tilting or cotilting module for that algebra by a remark of Bongartz [3]. In general, the image of P under the induced equivalence of derived categories is not both projective and injective. However, if $\nu P\cong P$, then by the remarks preceding Proposition 1.2 this is also true of the image of P. In the case of tilting, the image of P is thus a projective that is isomorphic to its image under ν, and so is injective as well. A dual argument works for cotilting.

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