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The FRT-construction via quantum affine algebras and smash products

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A R T I C L E   I N F O

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A B S T R A C T

For every element w in the Weyl group of a simple Lie algebra $\mathfrak{g}$, De Concini, Kac, and Procesi defined a subalgebra $\mathcal{U}_q^w$ of the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$. The algebra $\mathcal{U}_q^w$ is a deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{n}_+ \cap w.\mathfrak{n}_-)$. We construct smash products of certain finite-type De Concini–Kac–Procesi algebras to obtain ones of affine type; we have analogous constructions in types A_n and D_n. We show that the multiplication in the affine type De Concini–Kac–Procesi algebras arising from this smash product construction can be twisted by a cocycle to produce certain subalgebras related to the corresponding Faddeev–Reshetikhin–Takhtajan bialgebras.

1. Introduction

Let k be an infinite field and suppose an algebraic k-torus H acts rationally on a noetherian k-algebra A by k-algebra automorphisms. Goodearl and Letzter [10] showed that spec(A) is partitioned into strata indexed by the H-invariant prime ideals of A. Furthermore, they showed that each stratum is homeomorphic to the prime spectrum of a Laurent polynomial ring. The Goodearl–Letzter stratification results apply to the case when A is a q-skew polynomial ring for q not a root of unity under some assumptions relating the action of H to the structure of A. In this setting Cauchon's deleting derivations algorithm [5] gives an iterative procedure for classifying the H-primes. After several such algebras were studied, such as the algebras of quantum matrices $\mathcal{O}_q(M_{\ell,p}(k))$ [5,9,14], it was noticed that many of these algebras fall into the setting of De Concini–Kac–Procesi algebras [7].

The De Concini–Kac–Procesi algebras are subalgebras of quantized universal enveloping algebras $\mathcal{U}_q(\mathfrak{g})$ associated to the elements of the corresponding Weyl group $W_{\mathfrak{g}}$. They may be viewed as

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deformations of the universal enveloping algebra $\mathcal{U}(\mathfrak{n}_+ \cap w.\mathfrak{n}_-)$, where $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are the positive and negative nilpotent Lie subalgebras of $\mathfrak{g}$, respectively. Yakimov [17] recently proved that the poset of H-primes of a De Concini–Kac–Procesi algebra $\mathcal{U}_q^w$ ordered under inclusion is isomorphic to the poset $W^{\leqslant w}$ of Weyl group elements less than or equal to w under the Bruhat ordering. In [17], Yakimov also gives explicit generating sets for the H-primes in terms of Demazure modules.

Into this setting we introduce a new algebra we refer to as $\mathbb{X}_{n,q}$, closely tied to the FRT bialgebra produced from the R-matrix of type D_n, a q-skew polynomial ring which fits nicely in the circle of the other quantum algebras described above. We moreover argue that $\mathbb{X}_{n,q}$ forms an orthogonal analogue of the algebra $\mathcal{O}_q(M_{2,n}(k))$. Unlike many the previous examples, $\mathbb{X}_{n,q}$ is not realizable as a De Concini–Kac–Procesi algebra of finite type. We do give three descriptions of the algebra, relating it to the FRT construction, a De Concini–Kac–Procesi algebra of affine type, and a smash product of De Concini–Kac–Procesi algebras of finite type.

To demonstrate these connections, we introduce three type-D algebras and demonstrate isomorphisms between them. The first algebra we introduce is obtained from a De Concini–Kac–Procesi algebra of type $\mathfrak{so}_{2n+2}$ and a smash product construction. We show that this algebra is isomorphic to our second algebra, a De Concini–Kac–Procesi algebra associated to the affine Weyl group of type $\widehat{D}_{n+1}$. Finally, we show that twisting the multiplication in these algebras by a certain 2-cocycle produces the algebra $\mathbb{X}_{n,q}$ described above.

We proceed to produce analogous results with type-A algebras to suggest a more general setting for the above results and see explicitly the analogy between $\mathbb{X}_{n,q}$ and $\mathcal{O}_q(M_{2,n})$.

In more detail, Section 3 introduces the first of these algebras, an algebra which resembles a smash product of a De Concini–Kac–Procesi algebra with itself. Suppose $q \in k^\times$ is not a root of unity. Fix an integer $n \geqslant 3$. Let $W(D_{n+1})$ be the Weyl group of type D_{n+1} with standard generating set $\{s_1, \dots, s_{n+1}\}$ and let

$$w_n = (s_{n+1}s_n \cdots s_2 s_1)(s_3 s_4 \cdots s_n s_{n+1}) \in W(D_{n+1}). \tag{1.1}$$

Let $\mathcal{U}_D^{\geqslant 0}$ denote the quantized positive Borel algebra of type D_{n+1} and let $\mathcal{U}_q^{w_n}$ be the De Concini–Kac–Procesi subalgebra of $\mathcal{U}_D^{\geqslant 0}$ corresponding to w_n. In fact, $\mathcal{U}_q^{w_n}$ is isomorphic to $\mathcal{O}_q(\mathfrak{o}k^{2n})$, the algebra of even-dimensional quantum Euclidean space. We define an action λ of $\mathcal{U}_D^{\geqslant 0}$ on $\mathcal{U}_q^{w_n}$, which is a modification of the adjoint action of the Hopf algebra $\mathcal{U}_D^{\geqslant 0}$ on itself. This action equips $\mathcal{U}_q^{w_n}$ with the structure of a left $\mathcal{U}_D^{\geqslant 0}$-module algebra. We then consider the smash product $\mathcal{U}_q^{w_n} \# \mathcal{U}_D^{\geqslant 0}$ with respect to λ and set $(\mathcal{U}_q^{w_n})^\#$ to be the subalgebra of $\mathcal{U}_q^{w_n} \# \mathcal{U}_D^{\geqslant 0}$ generated by $\{u\#1, 1\#u \mid u \in \mathcal{U}_q^{w_n}\}$. This is the first of the three type-D algebras.

In Section 4 we introduce a second type-D algebra; it is a De Concini–Kac–Procesi algebra of affine type. Let $W(\widehat{D}_{n+1})$ denote the affine Weyl group of type $\widehat{D}_{n+1}$ with generating set $\{s_0, s_1, \dots, s_{n+1}\}$ and let

$$\widehat{w}_n = (s_{n+1} \cdots s_1)(s_3 \cdots s_{n+1}) s_0 (s_n \cdots s_3)(s_1 \cdots s_n) s_0 \in W(\widehat{D}_{n+1}). \tag{1.2}$$

The main result of this section is Theorem 4.4, where we prove that the algebras $\mathcal{U}_q^{\widehat{w}_n}$ and $(\mathcal{U}_q^{w_n})^\#$ are isomorphic. The results of this section assume additional hypotheses to be consistent with Beck [2,3]; in particular, we assume k is algebraically closed of characteristic zero and q is transcendental over $\mathbb{Q}$.

In Section 5 we introduce the algebra $\mathbb{X}_{n,q}$. We show that $\mathbb{X}_{n,q}$ is related to the bialgebra $\mathcal{A}(R_{D_n})$ arising from the type-D_n FRT construction. In particular, we label the standard generators of $\mathcal{A}(R_{D_n})$ by Y_{ij}, for $1 \leqslant i, j \leqslant 2n$, and let $T_{2,n} \subseteq \mathcal{A}(R_{D_n})$ be the subalgebra generated by $\{Y_{ij}\colon 1 \leqslant i \leqslant 2,\ 1 \leqslant j \leqslant 2n\}$ and observe that there is a surjective algebra homomorphism $\mathbb{X}_{n,q} \to T_{2,n}$ (see Proposition 5.1). We thus refer to $\mathbb{X}_{n,q}$ as a *parent* of $T_{2,n}$. Finally, in Theorem 5.2 we prove that $\mathbb{X}_{n,q}$ is isomorphic to a cocycle twist (in the sense of [1]) of $\mathcal{U}_q^{\widehat{w}_n}$. From this, it follows that $\mathbb{X}_{n,q}$ is an iterated Ore extension over k.

In Section 6 we proceed to demonstrate analogous results in the type A_m setting. We fix an integer $m > 1$ and let $W(A_m)$ be the Weyl group of type A_m with generating set $\{s_1, \dots, s_m\}$. Let

$$c_m = s_1 \cdots s_m \in W(A_m) \tag{1.3}$$

denote a Coxeter element. Notice that the De Concini–Kac–Procesi algebra $\mathcal{U}_q^{c_m}$ is isomorphic to $\mathcal{O}_q(k^m)$, the algebra known as quantum affine space. Quantum euclidean space, seen in Section 3, can be thought of as a type-D analogue of quantum affine space. Let $\mathcal{U}_A^{\geqslant 0}$ denote the quantized positive Borel algebra of type A_m. We define an action $\lambda_A : \mathcal{U}_A^{\geqslant 0} \otimes \mathcal{U}_q^{c_m} \to \mathcal{U}_q^{c_m}$ endowing $\mathcal{U}_q^{c_m}$ with the structure of a left $\mathcal{U}_A^{\geqslant 0}$-module algebra and define $(\mathcal{U}_q^{c_m})^\#$ to be the subalgebra of $\mathcal{U}_q^{c_m} \# \mathcal{U}_A^{\geqslant 0}$ generated by $\{1\#u, u\#1 \mid u \in \mathcal{U}_q^{c_m}\}$. Finally, we let $W(\widehat{A}_m)$ denote the affine Weyl group of type $\widehat{A}_m$ with generating set $\{s_0, s_1, \dots, s_m\}$ and let

$$\widehat{c}_m = (s_1 \cdots s_m)(s_0 s_1 \cdots s_{m-1}) \in W(\widehat{A}_m). \tag{1.4}$$

In Theorem 6.4, we prove that the corresponding De Concini–Kac–Procesi algebra $\mathcal{U}_q^{\widehat{c}_m}$ is isomorphic to $(\mathcal{U}_q^{c_m})^\#$. We further demonstrate that $\mathcal{U}_q^{\widehat{c}_m}$ is isomorphic to a cocycle twist of $\mathcal{O}_q(M_{2,m})$, the algebra of $2 \times m$ quantum matrices. We see $\mathbb{X}_{m,q}$ as a "type D" analogue of $\mathcal{O}_q(M_{2,m})$ because $\mathcal{O}_q(M_{2,m})$ is a subalgebra of the FRT-bialgebra $\mathcal{A}(R_{A_{m-1}}) \cong \mathcal{O}_q(M_m(k))$. The key distinction is that $\mathcal{O}_q(M_{2,m}(k))$ is a subalgebra of $\mathcal{A}(R_{A_{m-1}})$, whereas $\mathbb{X}_{m,q}$ is a parent of the analogous subalgebra $T_{2,m} \subseteq \mathcal{A}(R_{D_m})$.

2. Preliminaries

Let X be a root system of type A, B, C, D, E, F, G with positive simple roots $\alpha_1, \dots, \alpha_n$. The root lattice is denoted by $Q = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$. Let c_{ij} denote the entries of the Cartan matrix $c_{ij} = 2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_i, \alpha_i \rangle$, and let $d_1, \dots, d_n$ be coprime positive integers so that the matrix $(d_i c_{ij})$ is symmetric. Let k be a field and suppose $q \in k^\times$ is not a root of unity. For $1 \leqslant i \leqslant n$, put $q_i = q^{d_i}$ and define the quantized enveloping algebra $\mathcal{U}$ as the associative k-algebra (or algebra, for brevity) generated by $u_1^\pm, \dots, u_n^\pm$ and $\{v_\mu\colon\ \mu \in Q\}$ and having the defining relations

$$v_0 = 1, \qquad v_\mu v_\rho = v_{\mu+\rho} \qquad (\mu, \rho \in Q), \tag{2.1}$$

$$v_\mu u_i^\pm = q^{\pm\langle \mu, \alpha_i \rangle} u_i^\pm v_\mu \qquad \big(\mu \in Q,\ i \in \{1, \dots, n\}\big), \tag{2.2}$$

$$u_i^+ u_j^- = u_j^- u_i^+ + \delta_{ij} \frac{v_{\alpha_i} - v_{-\alpha_i}}{q_i - q_i^{-1}} \qquad \big(i, j \in \{1, \dots, n\}\big), \tag{2.3}$$

$$\sum_{r=0}^{1-c_{ij}} (-1)^r \begin{bmatrix} 1 - c_{ij} \\ r \end{bmatrix}_{q_i} \big(u_i^\pm\big)^{1-c_{ij}-r} u_j^\pm \big(u_i^\pm\big)^r = 0 \quad (i \neq j). \tag{2.4}$$

Here,

$$[\ell]_{q_i} = \frac{q_i^\ell - q_i^{-\ell}}{q_i - q_i^{-1}}, \qquad [\ell]_{q_i}! = [1]_{q_i} \cdots [\ell]_{q_i}, \qquad \begin{bmatrix} \ell \\ m \end{bmatrix}_{q_i} = \frac{[\ell]_{q_i}!}{[m]_{q_i}![\ell - m]_{q_i}!}. \tag{2.5}$$

The algebra $\mathcal{U}$ is Q-graded with $\deg(u_i^+) = \alpha_i$, $\deg(u_i^-) = -\alpha_i$ and $\deg(K_\mu) = 0$ for every $\mu \in Q$ and $1 \leqslant i \leqslant n$. Furthermore, $\mathcal{U}$ has a Hopf algebra structure with comultiplication Δ, antipode S, and counit ϵ maps given by

$$\Delta\big(u_i^+\big) = v_{-\alpha_i} \otimes u_i^+ + u_i^+ \otimes 1, \qquad \Delta(v_\mu) = v_\mu \otimes v_\mu, \qquad \Delta\big(u_i^-\big) = 1 \otimes u_i^- + u_i^- \otimes v_{\alpha_i}, \tag{2.6}$$

$$S\big(u_i^+\big) = -v_{\alpha_i} u_i^+, \qquad S(v_\mu) = v_{-\mu}, \qquad S\big(u_i^-\big) = -u_i^- v_{-\alpha_i}, \tag{2.7}$$

$$\epsilon\big(u_i^+\big) = 0, \qquad \epsilon(v_\mu) = 1, \qquad \epsilon\big(u_i^-\big) = 0, \tag{2.8}$$

for every $\mu \in Q$ and $1 \leqslant i \leqslant n$.

For every $i \in \{1, \ldots, n\}$, we let $s_i : Q \to Q$ be the simple reflection

$$s_i : \mu \to \mu - \frac{2\langle \mu, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i, \tag{2.9}$$

and let $W = \langle s_1, \ldots, s_n \rangle$ denote the Weyl group. The standard presentation for the braid group B is the generating set $\{T_w\colon\ w \in W\}$ subject to the relations $T_w T_{w'} = T_{ww'}$ for every $w, w' \in W$ satisfying $\ell(w) + \ell(w') = \ell(ww')$, where ℓ is the length function on W. For each $i \in \{1, \ldots, n\}$, we set $T_i := T_{s_i}$. Thus, the braid group B is generated by $T_1, \ldots, T_n$. It is well known that B acts via algebra automorphisms on $\mathcal{U}$ as follows:

$$T_i v_\mu = v_{s_i(\mu)}, \qquad T_i u_i^+ = -u_i^- v_{\alpha_i}, \qquad T_i u_i^- = -v_{-\alpha_i} u_i^+, \tag{2.10}$$

$$T_i u_j^+ = \sum_{r=0}^{-c_{ij}} \frac{(-q_i)^{-r}}{[-c_{ij} - r]_{q_i}! [r]_{q_i}!} \big(u_i^+\big)^{-c_{ij}-r} u_j^+ \big(u_i^+\big)^r \quad (i \neq j), \tag{2.11}$$

$$T_i u_j^- = \sum_{r=0}^{-c_{ij}} \frac{(-q_i)^{r}}{[-c_{ij} - r]_{q_i}! [r]_{q_i}!} \big(u_i^-\big)^{r} u_j^- \big(u_i^-\big)^{-c_{ij}-r} \quad (i \neq j), \tag{2.12}$$

for all $i, j \in \{1, \ldots, n\}$, $\mu \in Q$. These automorphisms were first detailed by Lusztig (e.g. see [15]); our presentation agrees with [4, I.6.7].

Fix $w \in W$. For a reduced expression

$$w = s_{i_1} \cdots s_{i_t} \tag{2.13}$$

define the roots

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1} \alpha_{i_2}, \ldots, \beta_t = s_{i_1} \cdots s_{i_{t-1}} \alpha_{i_t} \tag{2.14}$$

and the root vectors

$$X_{\beta_1} = u_{i_1}^+, X_{\beta_2} = T_{s_{i_1}} u_{i_2}^+, \ldots, X_{\beta_t} = T_{s_{i_1}} \cdots T_{s_{i_{t-1}}} u_{i_t}^+. \tag{2.15}$$

Following [7], let $\mathcal{U}_q^w$ denote the subalgebra of $\mathcal{U}$ generated by the root vectors $X_{\beta_1}, \ldots, X_{\beta_t}$.

While the construction of the subalgebra is dependent upon a choice of reduced expression for w, De Concini, Kac, and Procesi proved the following:

Theorem 2.1. *(See [7, Proposition 2.2].) Up to isomorphism, the algebra $\mathcal{U}_q^w$ does not depend on the reduced expression for w. Furthermore, $\mathcal{U}_q^w$ has the PBW basis*

$$X_{\beta_1}^{n_1} \cdots X_{\beta_t}^{n_t}, \quad n_1, \ldots, n_t \in \mathbb{Z}_{\geqslant 0}.$$

Beck later proved the analogous result for affine root systems (with k algebraically closed of characteristic zero and q transcendental over $\mathbb{Q}$) [3].

3. A smash product of type D_{n+1}

For this section, $q \in k^\times$ is not a root of unity. We assume no other hypotheses on the base field k.

3.1. The algebras $\mathcal{U}_q(\mathfrak{so}_{2n+2})$, $\mathcal{U}_q^{w_n}$, and $\mathcal{O}_q(\mathfrak{o}k^{2n})$

Fix an integer $n \geqslant 3$, and let $Q(D_{n+1})$ denote the root lattice of type D_{n+1}. We denote the simple roots by $\alpha_i = e_i - e_{i-1}$ for $2 \leqslant i \leqslant n+1$ and $\alpha_1 = e_1 + e_2$. With this choice of simple roots, the root lattice is identified with the additive abelian subgroup of $\mathbb{R}^{n+1}$ consisting of vectors having integer-valued coordinates $(a_1, \dots, a_{n+1})$ where the sum $\sum a_i$ is an even number. The inner product on $Q(D_{n+1})$ will be identified with the restriction of the standard inner product on $\mathbb{R}^{n+1}$ (i.e. $\langle e_i, e_j \rangle = \delta_{ij}$) to $Q(D_{n+1})$. For a positive simple root α_i, let s_i denote the corresponding simple reflection and let $W(D_{n+1}) = \langle s_1, \dots, s_{n+1} \rangle$ denote the Weyl group. The associated finite-type Cartan matrix (c_{ij}) is symmetric. Hence, $d_1 = \cdots = d_{n+1} = 1$. Therefore, the parameters $q_1, \dots, q_{n+1}$ are all equal to q. As usual, we put $\hat{q} = q - q^{-1}$. Let $\mathcal{U}_q(\mathfrak{so}_{2n+2})$ denote the corresponding quantized universal enveloping algebra. We label the generators of $\mathcal{U}_q(\mathfrak{so}_{2n+2})$ by $E_1, \dots, E_{n+1}, F_1, \dots, F_{n+1}$ and $\{K_\mu\colon \ \mu \in Q(D_{n+1})\}$ and the defining relations are

$$K_0 = 1, \qquad K_\mu K_\lambda = K_{\mu+\lambda}, \tag{3.1}$$

$$K_\mu E_i = q^{\langle \mu, \alpha_i \rangle} E_i K_\mu, \qquad K_\mu F_i = q^{-\langle \mu, \alpha_i \rangle} F_i K_\mu, \tag{3.2}$$

$$E_i E_j = E_j E_i, \qquad F_i F_j = F_j F_i \qquad \big(\langle \alpha_i, \alpha_j \rangle = 0 \text{ or } 2\big), \tag{3.3}$$

$$E_i[E_i, E_j] = q[E_i, E_j]E_i, \qquad F_i[F_i, F_j] = q[F_i, F_j]F_i \quad \big(\langle \alpha_i, \alpha_j \rangle = -1\big), \tag{3.4}$$

$$E_i F_j = F_j E_i + \frac{\delta_{ij}}{\hat{q}}(K_{\alpha_i} - K_{-\alpha_i}), \tag{3.5}$$

for every $i, j \in \{1, \dots, n+1\}$ and $\mu, \lambda \in Q(D_{n+1})$. Here we use the q^{-1}-commutators, defined by

$$[u, v] := uv - q^{-1} vu$$

for every $u, v \in \mathcal{U}_q(\mathfrak{so}_{2n+2})$. Recall that $\mathcal{U}_q(\mathfrak{so}_{2n+2})$ is $Q(D_{n+1})$-graded with $\deg(E_i) = \alpha_i$, $\deg(F_i) = -\alpha_i$, and $\deg(K_\mu) = 0$ for every $\mu \in Q(D_{n+1})$ and $1 \leqslant i \leqslant n+1$.

Let w_0 denote the longest element of $W(D_{n+1})$ and let w_0^L be the longest element of the parabolic subgroup $\langle s_1, \dots, s_n \rangle \subseteq W(D_{n+1})$. Put $w_n = w_0^L w_0$. We have a reduced expression

$$w_n = (s_{n+1} \cdots s_2 s_1)(s_3 \cdots s_n s_{n+1}) \in W(D_{n+1}) \tag{3.6}$$

and root vectors

$$X_{e_{n+1}-e_n}, X_{e_{n+1}-e_{n-1}}, \dots, X_{e_{n+1}-e_1}, X_{e_{n+1}+e_1} X_{e_{n+1}+e_2}, \dots, X_{e_{n+1}+e_n}. \tag{3.7}$$

For brevity, we put $x_i = X_{e_{n+1}-e_i}$ and $y_i = X_{e_{n+1}+e_i}$ for every $i \in \{1, \dots, n\}$. Let $\mathcal{U}_q^{w_n}$ denote the corresponding De Concini–Kac–Procesi algebra.

The following is suggested by [11], Section 5.6.a.

Theorem 3.1. *The algebra $\mathcal{U}_q^{w_n}$ is isomorphic to the even-dimensional quantum Euclidean space $\mathcal{O}_q(\mathfrak{o}k^{2n})$.*

Proof. We observe that the root vectors $x_1, \dots, x_n, y_1, \dots, y_n$ of $\mathcal{U}_q^{w_n}$ can be written inductively as $x_n = E_{n+1}$, $y_1 = [x_2, E_1]$ and

$$x_i = [x_{i+1}, E_{i+1}], \tag{3.8}$$

$$y_{i+1} = [y_i, E_{i+1}], \tag{3.9}$$

for all $1 \leqslant i < n$. Using these identities, one can readily check that the root vectors satisfy the defining relations of $\mathcal{O}_q(\mathfrak{o}k^{2n})$ (cf. [13, Section 9.3.2]),

$$x_i x_j = q^{-1} x_j x_i, \qquad y_i y_j = q y_j y_i \quad (1 \leqslant i < j \leqslant n), \tag{3.10}$$

$$x_i y_j = q^{1-\delta_{ij}} y_j x_i + \delta_{ij} \hat{q} \sum_{r=1}^{i-1} (-q)^{i-r-1} x_r y_r \quad \bigl(i, j \in \{1, \dots, n\}\bigr). \tag{3.11}$$

Since $\mathcal{U}_q^{w_n}$ has a PBW basis of ordered monomials, Eqs. (3.10) and (3.11) are the defining relations. Hence, $\mathcal{U}_q^{w_n} \cong \mathcal{O}_q(\mathfrak{o}k^{2n})$. □

3.2. $\mathcal{U}_q^{w_n}$ *as a left* $\mathcal{U}_D^{\geqslant 0}$*-module algebra*

Let $\mathcal{U}_D^{\geqslant 0}$ be the sub-Hopf algebra of $\mathcal{U}_q(\mathfrak{so}_{2n+2})$ generated by $E_1, \dots, E_{n+1}$, and K_μ for all $\mu \in Q(D_{n+1})$. We let $\pi : \mathcal{U}_D^{\geqslant 0} \to \mathcal{U}_D^{\geqslant 0}$ be the unique algebra map such that

$$\pi(E_{n+1}) = 0, \tag{3.12}$$

$$\pi(E_i) = E_i \quad (i \leqslant n), \tag{3.13}$$

$$\pi(K_\mu) = K_\mu \quad \bigl(\mu \in Q(D_{n+1})\bigr). \tag{3.14}$$

We define a function $\lambda : \mathcal{U}_D^{\geqslant 0} \otimes \mathcal{U}_q^{w_n} \to \mathcal{U}_D^{\geqslant 0}$ by the following sequence of linear maps:

$$\lambda : \mathcal{U}_D^{\geqslant 0} \otimes \mathcal{U}_q^{w_n} \xrightarrow{\text{incl.}} \bigl(\mathcal{U}_D^{\geqslant 0}\bigr)^{\otimes 2} \xrightarrow{\pi \otimes id} \bigl(\mathcal{U}_D^{\geqslant 0}\bigr)^{\otimes 2} \xrightarrow{\text{adjoint}} \mathcal{U}_D^{\geqslant 0} \tag{3.15}$$

and have the following:

Theorem 3.2. *For the function λ above, we have $\mathrm{Im}(\lambda) \subseteq \mathcal{U}_q^{w_n}$. In particular, λ endows $\mathcal{U}_q^{w_n}$ with the structure of a left $\mathcal{U}_D^{\geqslant 0}$-module algebra.*

Proof. For brevity, we set $u.v = \lambda(u \otimes v)$ for every $u \in \mathcal{U}_D^{\geqslant 0}$ and $v \in \mathcal{U}_q^{w_n}$. One can verify that

$$E_j.x_r = \begin{cases} -q(\delta_{1r} y_2 + \delta_{2r} y_1) & (j = 1), \\ -q\delta_{jr} x_{r-1} & (j \neq 1), \end{cases} \tag{3.16}$$

$$E_j.y_r = \begin{cases} 0 & (j = n+1), \\ -q\delta_{j,r+1} y_{r+1} & (j \neq n+1), \end{cases} \tag{3.17}$$

for all $r \in \{1, \dots, n\}$, $j \in \{1, \dots, n+1\}$. Since $\mathcal{U}_D^{\geqslant 0}$ is a left $\mathcal{U}_D^{\geqslant 0}$-module algebra (with respect to the adjoint action), Eqs. (3.16) and (3.17) above, together with the fact that the K_μ's act diagonally on $\mathcal{U}_q^{w_n}$, prove the desired result. □

We remark that Theorem 3.2 depends heavily on the fact that the set $\Delta_+ \cap w_n.\Delta_-$ is an *upper set* of Δ_+: i.e. if $\mu \in \Delta_+ \cap w_n.\Delta_-$ and $\lambda \in \Delta_+$ with $\lambda - \mu$ a nonnegative linear combination of the simple roots $\alpha_1, \ldots, \alpha_{n+1}$, then $\lambda \in \Delta_+ \cap w_n.\Delta_-$ as well.

Using the action map λ, we form the smash product algebra $\mathcal{U}_q^{w_n} \# \mathcal{U}_D^{\geqslant 0}$ and define the following subalgebra

$$\left(\mathcal{U}_q^{w_n}\right)^{\#} := \left\langle 1\#u, u\#1 \;\middle|\; u \in \mathcal{U}_q^{w_n}\right\rangle \subseteq \mathcal{U}_q^{w_n} \# \mathcal{U}_D^{\geqslant 0}. \tag{3.18}$$

Loosely speaking, we can think of $(\mathcal{U}_q^{w_n})^{\#}$ as being a smash product of $\mathcal{U}_q^{w_n}$ with itself. Observe for example that $(\mathcal{U}_q^{w_n})^{\#}$ is isomorphic as a vector space to $\mathcal{U}_q^{w_n} \otimes \mathcal{U}_q^{w_n}$.

3.3. A presentation of $(\mathcal{U}_q^{w_n})^{\#}$

We will spend the rest of this section giving an explicit presentation for the algebra $(\mathcal{U}_q^{w_n})^{\#}$ because this will be necessary for proving the main result of Section 4 (Theorem 4.4).

The algebra $(\mathcal{U}_q^{w_n})^{\#}$ is generated by $1\#x_i$, $1\#y_i$, $x_i\#1$, $y_i\#1$ for $i \in \{1, \ldots, n\}$. To compute the relations among these generators, we need comultiplication formulas for the root vectors $x_1, \ldots, x_n$, $y_1, \ldots, y_n \in \mathcal{U}_q^{w_n}$. First, we must introduce the elements $\epsilon_{ij}, E_{r\downarrow s}, E_{s\uparrow r} \in \mathcal{U}_D^{\geqslant 0}$ for every $i, j \in \{1, \ldots, n\}$ and $r, s \in \{1, \ldots, n+1\}$ with $r \geqslant s$. They are defined recursively via

$$E_{r\downarrow s} = \begin{cases} E_r & (r = s), \\ [E_{r\downarrow s+1}, E_s] & (r \neq s), \end{cases} \tag{3.19}$$

$$E_{s\uparrow r} = \begin{cases} E_s & (r = s), \\ [E_{s\uparrow r-1}, E_r] & (r \neq s), \end{cases} \tag{3.20}$$

$$\epsilon_{1j} = \begin{cases} 0 & (j = 1), \\ T_j T_{j-1} \cdots T_2 E_1 & (j \neq 1), \end{cases} \tag{3.21}$$

$$\epsilon_{i+1,j} = \begin{cases} [\epsilon_{ij}, E_{i+1}] & (j \neq i, i+1), \\ q\epsilon_{i,i+1} E_{i+1} - q^{-1} E_{i+1} \epsilon_{i,i+1} & (j = i+1), \\ \epsilon_{i,i+1} + q^{-1}(\epsilon_{ii} E_{i+1} - E_{i+1} \epsilon_{ii}) & (j = i). \end{cases} \tag{3.22}$$

We have the following:

Lemma 3.3. *For every $i \in \{1, \ldots, n\}$,*

$$\Delta(x_i) = K_{-\deg(x_i)} \otimes x_i + x_i \otimes 1 + \hat{q} \sum_{j=i+1}^{n} E_{j\downarrow i+1} K_{-\deg(x_j)} \otimes x_j, \tag{3.23}$$

$$\begin{aligned} \Delta(y_i) = {} & K_{-\deg(y_i)} \otimes y_i + y_i \otimes 1 \\ & + \hat{q}\left(\sum_{j=1}^{n} \epsilon_{ij} K_{-\deg(x_j)} \otimes x_j + \sum_{j=1}^{i-1} E_{j+1\uparrow i} K_{-\deg(y_j)} \otimes y_j\right). \end{aligned} \tag{3.24}$$

Proof. Use the induction formulas from Eqs. (3.8) and (3.9) together with the comultiplication formula given in Eq. (2.6). $\square$

From Eqs. (3.16) and (3.17) it follows that for all $r \in \{1, \dots, n\}$, we have

$$E_{j\downarrow i+1}.x_r = -q\delta_{jr}x_i, \qquad E_{j\downarrow i+1}.y_r = (-q)^{j-i}\delta_{ir}y_j \quad (1 \leqslant i < j \leqslant n), \tag{3.25}$$

$$E_{j+1\uparrow i}.x_r = (-q)^{i-j}\delta_{ir}x_j, \qquad E_{j+1\uparrow i}.y_r = -q\delta_{jr}y_i \quad (1 \leqslant j < i \leqslant n), \tag{3.26}$$

$$\epsilon_{ij}.x_r = (-q)^{i+j-2}q^{\delta_{ij}}\delta_{ir}y_j - q\delta_{jr}y_i, \qquad \epsilon_{ij}.y_r = 0 \quad (1 \leqslant i, j \leqslant n). \tag{3.27}$$

Using the identities (3.25)–(3.27) together with the comultiplication formulas, (3.23)–(3.24), we compute the following "cross-relations" in $(\mathcal{U}_q^{w_n})^\#$.

Proposition 3.4. *For every $i, j \in \{1, \dots, n\}$,*

$$(1\#x_i)(x_j\#1) = \begin{cases} q^{-1}x_j\#x_i - q^{-1}\hat{q}x_i\#x_j, & i < j, \\ q^{-2}x_j\#x_i, & i = j, \\ q^{-1}x_j\#x_i, & i > j, \end{cases} \tag{3.28}$$

$$(1\#y_i)(y_j\#1) = \begin{cases} q^{-1}y_j\#y_i - q^{-1}\hat{q}y_i\#y_j, & i > j, \\ q^{-2}y_j\#y_i, & i = j, \\ q^{-1}y_j\#y_i, & i < j, \end{cases} \tag{3.29}$$

$$\begin{aligned}(1\#y_i)(x_j\#1) &= q^{-1+\delta_{ij}}x_j\#y_i - \hat{q}q^{-1}y_i\#x_j \\ &\quad + \hat{q}q^{-1}\delta_{ij}\left(\sum_{m=1}^{n}(-q)^{i+m-2}y_m\#x_m + \sum_{m=1}^{i-1}(-q)^{i-m}x_m\#y_m\right),\end{aligned} \tag{3.30}$$

$$(1\#x_i)(y_j\#1) = q^{-1+\delta_{ij}}y_j\#x_i + \hat{q}q^{-1}\delta_{ij}\sum_{m=i+1}^{n}(-q)^{m-i}y_m\#x_m. \tag{3.31}$$

We have the following presentation for $(\mathcal{U}_q^{w_n})^\#$:

Theorem 3.5. *The algebra $(\mathcal{U}_q^{w_n})^\#$ is generated by $1\#x_i$, $x_i\#1$, $1\#y_i$, $y_i\#1$ for $1 \leqslant i \leqslant n$, and its defining relations are Eqs.* (3.28)–(3.31) *together with the relations*

$$(1\#x_i)(1\#x_j) = q^{-1}(1\#x_j)(1\#x_i) \qquad (1 \leqslant i < j \leqslant n), \tag{3.32}$$

$$(1\#y_i)(1\#y_j) = q(1\#y_j)(1\#y_i) \qquad (1 \leqslant i < j \leqslant n), \tag{3.33}$$

$$(1\#x_i)(1\#y_j) = q^{1-\delta_{ij}}(1\#y_j)(1\#x_i) + \delta_{ij}\hat{q}\sum_{r=1}^{i-1}(-q)^{i-r-1}(1\#x_r)(1\#y_r) \qquad \big(i, j \in \{1, \dots, n\}\big), \tag{3.34}$$

$$(x_i\#1)(x_j\#1) = q^{-1}(x_j\#1)(x_i\#1) \qquad (1 \leqslant i < j \leqslant n), \tag{3.35}$$

$$(y_i\#1)(y_j\#1) = q(y_j\#1)(y_i\#1) \qquad (1 \leqslant i < j \leqslant n), \tag{3.36}$$

$$(x_i\#1)(y_j\#1) = q^{1-\delta_{ij}}(y_j\#1)(x_i\#1) + \delta_{ij}\hat{q}\sum_{r=1}^{i-1}(-q)^{i-r-1}(x_r\#1)(y_r\#1) \qquad \big(i, j \in \{1, \dots, n\}\big). \tag{3.37}$$

Proof. The generators $1\#x_1, \ldots, 1\#x_n, 1\#y_1, \ldots, 1\#y_n$ generate a subalgebra isomorphic to $\mathcal{U}_q^{w_n}$, as do the generators $x_1\#1, \ldots, x_n\#1, y_1\#1, \ldots, y_n\#1$, giving us the relations (3.32)–(3.37). The universal property of smash products (for example, see [12, Section 1.8]) and the PBW basis of De Concini–Kac–Procesi algebras imply that the cross relations of (3.28)–(3.31) together with the above relations are a presentation of $(\mathcal{U}_q^{w_n})^{\#}$. $\square$

4. The quantum affine algebra $\mathcal{U}_q^{\widehat{w}_n}$

For this section, we assume the base field k is algebraically closed of characteristic zero, and $q \in k$ is transcendental over $\mathbb{Q}$. These hypotheses are chosen to be consistent with Beck [3].

Let $Q(\widehat{D}_{n+1}) = Q(D_{n+1}) \oplus \mathbb{Z}$ denote the root lattice of type $\widehat{D}_{n+1}$. As an abelian group, $Q(\widehat{D}_{n+1})$ is generated additively by the positive simple roots $\alpha_0 := -e_{n+1} - e_n + 1$, $\alpha_1 := e_1 + e_2$, and $\alpha_i := e_i - e_{i-1}$ for $2 \leqslant i \leqslant n+1$. We extend the bilinear form $\langle\ ,\ \rangle$ on $Q(D_{n+1})$ to $Q(\widehat{D}_{n+1})$ by setting $1 \in Q(\widehat{D}_{n+1})$ to be isotropic. As before, let s_i denote the corresponding simple reflection $s_i : Q(\widehat{D}_{n+1}) \to Q(\widehat{D}_{n+1})$, for $0 \leqslant i \leqslant n+1$, and $W(\widehat{D}_{n+1}) = \langle s_0, \ldots, s_{n+1}\rangle$ is the Weyl group. The corresponding quantized enveloping algebra $\mathcal{U}_q(\widehat{\mathfrak{so}}_{2n+2})$ is generated by $E_0, \ldots, E_{n+1}, F_0, \ldots, F_{n+1}$ and $\{K_\mu\colon\ \mu \in Q(\widehat{D}_{n+1})\}$ and has defining relations

$$K_0 = 1, \qquad K_\mu K_\lambda = K_{\mu+\lambda}, \tag{4.1}$$

$$K_\mu E_i = q^{\langle\mu,\alpha_i\rangle} E_i K_\mu, \qquad K_\mu F_i = q^{-\langle\mu,\alpha_i\rangle} F_i K_\mu, \tag{4.2}$$

$$E_i E_j = E_j E_i, \qquad F_i F_j = F_j F_i \quad \big(\langle\alpha_i,\alpha_j\rangle = 0 \text{ or } 2\big), \tag{4.3}$$

$$E_i[E_i, E_j] = q[E_i, E_j]E_i, \qquad F_i[F_i, F_j] = q[F_i, F_j]F_i \quad \big(\langle\alpha_i,\alpha_j\rangle = -1\big), \tag{4.4}$$

$$E_i F_j = F_j E_i + \frac{\delta_{ij}}{\hat{q}}(K_{\alpha_i} - K_{-\alpha_i}), \tag{4.5}$$

for every $i, j \in \{0, \ldots, n+1\}$ and $\mu, \lambda \in Q(\widehat{D}_{n+1})$ (cf. Eqs. (3.1)–(3.5)).

Let $\widehat{w}_n \in W(\widehat{D}_{n+1})$ be the Weyl group element given by

$$\widehat{w}_n : v + r \mapsto v + r + 2a_{n+1} \tag{4.6}$$

for every $v = \sum_{i=1}^{n+1} a_i e_i \in Q(D_{n+1})$ and $r \in \mathbb{Z}$. We have the reduced expression

$$\widehat{w}_n := (s_{n+1}\cdots s_1)(s_3\cdots s_{n+1})s_0(s_n\cdots s_3)(s_1\cdots s_n)s_0 \in W(\widehat{D}_{n+1}). \tag{4.7}$$

We let $\widehat{B}_{\mathfrak{so}_{2n+2}} = \langle T_0, \ldots, T_{n+1}\rangle$ denote the corresponding braid group of $\widehat{\mathfrak{so}}_{2n+2}$ and label the corresponding ordered root vectors for $\mathcal{U}_q^{\widehat{w}_n}$ by

$$X_n, \ldots, X_1, Y_1, \ldots, Y_n, \overline{X}_n, \ldots, \overline{X}_1, \overline{Y}_1, \ldots, \overline{Y}_n. \tag{4.8}$$

One can readily verify the following lemmas.

Lemma 4.1. *We have the following recursion formulas in the algebra* $\mathcal{U}_q^{\widehat{w}_n}$:

$$X_n = E_{n+1}, \qquad X_i = [X_{i+1}, E_{i+1}] \quad (i \neq n), \tag{4.9}$$

$$Y_1 = [X_2, E_1], \qquad Y_i = [Y_{i-1}, E_i] \quad (i \neq 1), \tag{4.10}$$

$$\overline{X}_n = [Y_{n-1}, T_{n+1}T_n E_0], \qquad \overline{X}_i = [\overline{X}_{i+1}, E_{i+1}] \quad (i \neq n), \tag{4.11}$$

$$\overline{Y}_1 = [\overline{X}_2, E_1], \qquad \overline{Y}_i = [\overline{Y}_{i-1}, E_i] \quad (i \neq 1), \tag{4.12}$$

$$Y_2 = [X_1, E_1], \qquad \overline{Y}_2 = [\overline{X}_1, E_1]. \tag{4.13}$$

Lemma 4.2. *For all $i, j \in \{1, \dots, n\}$, we have the following*:

$$T_i.X_j = \begin{cases} [E_i, X_j] & (i = j \text{ or } (i, j) = (1, 2)), \\ X_{j+1} & (i = j + 1), \\ X_j, & \text{otherwise}, \end{cases} \tag{4.14}$$

$$T_i.Y_j = \begin{cases} Y_{j-1} & (i = j \text{ and } i \neq 1), \\ X_{3-j} & (i = 1,\ j \in \{1, 2\}), \\ [E_{j+1}, Y_j] & (i = j + 1), \\ Y_j, & \text{otherwise}, \end{cases} \tag{4.15}$$

$$T_i.\overline{X}_j = \begin{cases} [E_i, \overline{X}_j] & (i = j \text{ or } (i, j) = (1, 2)), \\ \overline{X}_{j+1} & (i = j + 1), \\ \overline{X}_j, & \text{otherwise}, \end{cases} \tag{4.16}$$

$$T_i.\overline{Y}_j = \begin{cases} \overline{Y}_{j-1} & (i = j \text{ and } i \neq 1), \\ \overline{X}_{3-j} & (i = 1,\ j \in \{1, 2\}), \\ [E_{j+1}, \overline{Y}_j] & (i = j + 1), \\ \overline{Y}_j, & \text{otherwise}. \end{cases} \tag{4.17}$$

With the help of Lemmas 4.1 and 4.2, we prove the following.

Proposition 4.3. *The defining relations for the algebra $\mathcal{U}_q^{\widehat{w}_n}$ are*

$$X_i X_j = q^{-1} X_j X_i, \qquad Y_j Y_i = q^{-1} Y_i Y_j \quad (i < j), \tag{4.18}$$

$$\overline{X}_i \overline{X}_j = q^{-1} \overline{X}_j \overline{X}_i, \qquad \overline{Y}_j \overline{Y}_i = q^{-1} \overline{Y}_i \overline{Y}_j \quad (i < j), \tag{4.19}$$

$$Y_j X_i = q^{\delta_{ij}-1} X_i Y_j - \delta_{ij} \hat{q} \sum_{r=1}^{i-1} (-q)^{i-r-1} X_r Y_r, \tag{4.20}$$

$$\overline{Y}_j \overline{X}_i = q^{\delta_{ij}-1} \overline{X}_i \overline{Y}_j - \delta_{ij} \hat{q} \sum_{r=1}^{i-1} (-q)^{i-r-1} \overline{X}_r \overline{Y}_r, \tag{4.21}$$

$$\overline{X}_i X_i = q^{-2} X_i \overline{X}_i, \qquad \overline{Y}_i Y_i = q^{-2} Y_i \overline{Y}_i, \tag{4.22}$$

$$\overline{X}_j X_i = q^{-1} X_i \overline{X}_j, \qquad \overline{Y}_i Y_j = q^{-1} Y_j \overline{Y}_i \quad (i < j), \tag{4.23}$$

$$\overline{X}_i X_j = q^{-1} X_j \overline{X}_i - q^{-1} \hat{q} X_i \overline{X}_j, \qquad \overline{Y}_j Y_i = q^{-1} Y_i \overline{Y}_j - q^{-1} \hat{q} Y_j \overline{Y}_i \quad (i < j), \tag{4.24}$$

$$\overline{X}_i Y_j = q^{-1+\delta_{ij}} Y_j \overline{X}_i + \hat{q} q^{-1} \delta_{ij} \sum_{m=i+1}^{n} (-q)^{m-i} Y_m \overline{X}_m, \tag{4.25}$$

$$\overline{Y}_i X_j = q^{-1+\delta_{ij}} X_j \overline{Y}_i - \hat{q}q^{-1} Y_i \overline{X}_j + \hat{q}q^{-1}\delta_{ij}\left[\sum_{m=1}^{n}(-q)^{i+m-2} Y_m \overline{X}_m + \sum_{m=1}^{i-1}(-q)^{i-m} X_m \overline{Y}_m\right], \tag{4.26}$$

for $i, j \in \{1, \ldots, n\}$.

Proof. The first $2n$ letters in the reduced expression for $\widehat{w}_n$ coincide with w_n, as do the last $2n$ letters. This gives us the relations (4.18)–(4.21). Using Lemmas 4.1 and 4.2, one can prove inductively that the remaining relations hold. To illustrate how to obtain the identities in Eq. (4.22) for example, one can first verify the base cases, $\overline{X}_1 X_1 = q^{-2} X_1 \overline{X}_1$ and $\overline{Y}_n Y_n = q^{-2} Y_n \overline{Y}_n$, and then apply appropriate braid group automorphisms (refer to Lemma 4.2) to both sides of the equations. Since $\mathcal{U}_q^{\widehat{w}_n}$ has a PBW basis of ordered monomials, Eqs. (4.18)–(4.26) are the defining relations. □

By comparing Eqs. (3.28)–(3.37) with Eqs. (4.18)–(4.26), we observe the following theorem.

Theorem 4.4. *As* k*-algebras,* $\mathcal{U}_q^{\widehat{w}_n} \cong (\mathcal{U}_q^{w_n})^{\#}$ *via the isomorphism*

$$\begin{array}{lll} X_i \mapsto (x_i\#1), & Y_i \mapsto (y_i\#1), & \\ & & \text{for } i = 1, \ldots, n. \\ \overline{X}_i \mapsto (1\#x_i), & \overline{Y}_i \mapsto (1\#y_i), & \end{array}$$

5. The FRT-construction and the algebra $\mathbb{X}_{n,q}$

We will briefly review the Faddeev–Reshetikhin–Takhtajan (FRT) construction of [8] (see [6, Section 7.2] for more details). We let V be a k-module with basis $\{v_1, \ldots, v_N\}$. For a linear map $R \in \mathrm{End}_k(V \otimes V)$, we write

$$R(v_i \otimes v_j) = \sum_{s,t} R_{ij}^{st} v_s \otimes v_t \quad \text{for all } 1 \leqslant i, j < N, \tag{5.1}$$

with all $R_{ij}^{st} \in k$. The *FRT algebra* $\mathcal{A}(R)$ associated to R is the k-algebra presented by generators X_{ij} for $1 \leqslant i, j \leqslant N$ and has the defining relations

$$\sum_{s,t} R_{st}^{ji} X_{sl} X_{tm} = \sum_{s,t} R_{lm}^{ts} X_{is} X_{jt} \tag{5.2}$$

for every $i, j, l, m \in \{1, \ldots, N\}$. Up to algebra isomorphism, $\mathcal{A}(R)$ is independent of the chosen basis of V.

Let us specialize now to the case when $N = 2n$. Following [13, Section 8.4.2], for each $i, j \in \{1, \ldots, 2n\}$, let E_{ij} denote the linear map on V defined by $E_{ij}.v_\ell = \delta_{j\ell} v_i$. Let $i' := 2n + 1 - i$, and let

$$\begin{aligned} R_{D_n} = q \sum_{i:\, i \neq i'} (E_{ii} \otimes E_{ii}) &+ \sum_{i,j: i \neq j, j'} (E_{ii} \otimes E_{jj}) + q^{-1} \sum_{i: i \neq i'} (E_{i'i'} \otimes E_{ii}) \\ &+ \hat{q}\left(\sum_{i,j:\, i>j} (E_{ij} \otimes E_{ji}) - \sum_{i,j:\, i>j} q^{\rho_i - \rho_j} (E_{ij} \otimes E_{i'j'})\right), \end{aligned} \tag{5.3}$$

where $(\rho_1, \rho_2, \ldots, \rho_{2n})$ is the $2n$-tuple $(n-1, n-2, \ldots, 1, 0, 0, -1, \ldots, -n+1)$.

We define an algebra $\mathbb{X}_{n,q}$ presented by generators X_{ij} with $i \in \{1,2\}$, $j \in \{1,\dots,2n\}$, and having the defining relations

$$X_{rt}X_{rs} = q^{-1}X_{rs}X_{rt} \quad \bigl(r \in \{1,2\},\ s<t,\ t \neq s'\bigr), \tag{5.4}$$

$$X_{rs'}X_{rs} = X_{rs}X_{rs'} + \hat{q}\sum_{l=s+1}^{n} q^{l-s-1}X_{rl}X_{rl'} \quad \bigl(r \in \{1,2\},\ s<s'\bigr), \tag{5.5}$$

$$X_{2s}X_{1s} = q^{-1}X_{1s}X_{2s}, \tag{5.6}$$

$$X_{2s}X_{1t} = X_{1t}X_{2s} \quad \bigl(s<t,\ t \neq s'\bigr), \tag{5.7}$$

$$X_{2t}X_{1s} = X_{1s}X_{2t} - \hat{q}X_{1t}X_{2s} \quad \bigl(s<t,\ t \neq s'\bigr), \tag{5.8}$$

$$X_{2s}X_{1s'} = qX_{1s'}X_{2s} + \hat{q}\sum_{l=1}^{s-1} q^{s-l}X_{1l'}X_{2l} \quad \bigl(s<s'\bigr), \tag{5.9}$$

$$X_{2s'}X_{1s} = qX_{1s}X_{2s'} + \hat{q}\sum_{l=s+1}^{n} q^{l-s}X_{1l}X_{2l'}, + \hat{q}q^{-1}\sum_{l=1}^{n} q^{l'-s}X_{1l'}X_{2l} - \hat{q}X_{1s'}X_{2s} \quad \bigl(s<s'\bigr). \tag{5.10}$$

We label the canonical generators of $\mathcal{A}(R_{D_n})$ by Y_{ij} for $i,j = 1,\dots,2n$, and let $T_{2,n}$ be the subalgebra of $\mathcal{A}(R_{D_n})$ generated by $\{Y_{ij}\colon\ 1 \leqslant i \leqslant 2,\ 1 \leqslant j \leqslant 2n\}$.

Proposition 5.1. *There is a surjective algebra homomorphism $\mathbb{X}_{n,q} \to T_{2,n}$ with kernel $\langle \Omega_1, \Omega_2, \Upsilon \rangle$, where*

$$\Omega_1 := \sum_{r=1}^{n} q^{\rho_{r'}}X_{1,r}X_{1,r'}, \qquad \Omega_2 := \sum_{r=1}^{n} q^{\rho_{r'}}X_{2,r}X_{2,r'}, \qquad \Upsilon := \sum_{r=1}^{2n} q^{\rho_r}X_{1,r'}X_{2,r}. \tag{5.11}$$

Proof. Using the FRT construction (see Eqs. (5.2) and (5.3)), one can readily compute the defining relations for the algebra $\mathcal{A}(R_{D_n})$ and see that they line up appropriately with Eqs. (5.4)–(5.10) together with $\Omega_1 = \Omega_2 = \Upsilon = 0$. □

Notice that the definition of $\mathbb{X}_{n,q}$ makes sense when $n = 2$, and Proposition 5.1 holds in this case as well. However, the rest of the results of this paper require $n \geqslant 3$.

Following [1], we recall the details on twisting algebras by cocycles. Let M be an additive abelian group and $c: M \times M \to k^\times$ a 2-cocycle of M. If Λ is a k-algebra graded by M, we can twist Λ by c to obtain a new M-graded k-algebra Λ' that is canonically isomorphic to Λ as a k-module via $x \leftrightarrow x'$. Multiplication of homogeneous elements in Λ' is given by

$$x'y' = c\bigl(\deg(x), \deg(y)\bigr)(xy)'.$$

For our purposes, we will let $\beta : Q(\widehat{D}_{n+1}) \times Q(\widehat{D}_{n+1}) \to k^\times$ be the bicharacter (hence, also a 2-cocycle) defined by

$$\beta(\alpha_i, \alpha_j) = \begin{cases} q, & (i,j) = (0, n+1), \\ 1, & (i,j) \neq (0, n+1), \end{cases} \tag{5.12}$$

and have the following:

Theorem 5.2. *Suppose k is algebraically closed of characteristic zero, and $q \in k$ is transcendental over $\mathbb{Q}$. The β-twisted algebra $(\mathcal{U}_q^{\widehat{w}_n})'$ is isomorphic to $\mathbb{X}_{n,q}$.*

Proof. We label the corresponding generators of $(\mathcal{U}_q^{\widehat{w}_n})'$ by

$$X_n', \ldots, X_1', Y_1', \ldots, X_n', \overline{X}_n', \ldots, \overline{X}_1', \overline{Y}_1', \ldots, \overline{Y}_n'. \tag{5.13}$$

By comparing Eqs. (4.18)–(4.26) and (5.4)–(5.10), we observe that the algebra map $(\mathcal{U}_q^{\widehat{w}_n})' \to \mathbb{X}_{n,q}$ defined by

$$X_i' \mapsto (-1)^{n+1-i} X_{1,n+1-i}, \qquad \overline{X}_i' \mapsto (-1)^{n+1-i} X_{2,n+1-i}, \tag{5.14}$$

$$Y_i' \mapsto X_{1,n+i}, \qquad \overline{Y}_i' \mapsto X_{2,n+i}, \tag{5.15}$$

for every $i \in \{1, \ldots, n\}$, is an isomorphism. $\square$

From this, we deduce the following:

Theorem 5.3. *Using the same hypothesis as Theorem* 5.2*, the algebra $\mathbb{X}_{n,q}$ is an iterated Ore extension over k,*

$$\mathbb{X}_{n,q} = k[X_{11}][X_{12}; \tau_{12}, \delta_{12}] \cdots [X_{2,2n}; \tau_{2,2n}, \delta_{2,2n}].$$

Proof. It suffices to check that ordered monomials are linearly independent. From Theorem 5.2, we have a canonical vector space isomorphism $\mathcal{U}_q^{\widehat{w}_n} \to \mathbb{X}_{n,q}$ that preserves the ordered generating sets. Since $\mathcal{U}_q^{\widehat{w}_n}$ has a basis of ordered monomials, $\mathbb{X}_{n,q}$ does as well. $\square$

A straightforward computation gives $\delta_{ij}\tau_{ij} = q^2\tau_{ij}\delta_{ij}$ all for $i \in \{1, 2\}$, $j \in \{1, \ldots, 2n\}$. Hence, $\mathbb{X}_{n,q}$ is an iterated q^2-skew polynomial algebra over k.

6. A type A_m analogue

For this section, we assume $q \in k^\times$ is not a root of unity.

6.1. The algebras $\mathcal{U}_q(\mathfrak{sl}_{m+1})$, $\mathcal{U}_q^{c_m}$, and $\mathcal{O}_q(k^m)$

Fix an integer $m > 1$, and let $Q(A_m)$ denote the root lattice of type A_m. The positive simple roots are given by $\alpha_i := e_i - e_{i+1}$ for $i \in \{1, \ldots, m\}$. With this choice, we identify $Q(A_m)$ with the abelian subgroup of $\mathbb{R}^{m+1}$ consisting of integral $(m+1)$-tuples $(a_1, \ldots, a_{m+1})$ with the sum $\sum a_i$ equaling 0. Let $W(A_m)$ and $B_{\mathfrak{sl}_{m+1}}$ denote the corresponding Weyl group and braid group, respectively. Let $\mathcal{U}_q(\mathfrak{sl}_{m+1})$ denote the corresponding quantum enveloping algebra, and let $\mathcal{U}_A^{\geqslant 0}$ be the positive Borel subalgebra of $\mathcal{U}_q(\mathfrak{sl}_{m+1})$. We consider the Coxeter element

$$c_m = s_1 \cdots s_m \in W(A_m) \tag{6.1}$$

and the associated De Concini–Kac–Procesi algebra $\mathcal{U}_q^{c_m}$. We label the root vectors in $\mathcal{U}_q^{c_m}$ by

$$z_1 := X_{e_1 - e_2}, \qquad z_2 := X_{e_1 - e_3}, \ldots, z_m := X_{e_1 - e_{m+1}} \tag{6.2}$$

and have the following

Proposition 6.1. *The root vectors $z_1, \dots, z_m$ satisfy the relations*

$$z_i z_j = q z_j z_i \tag{6.3}$$

for all $i, j \in \{1, \dots, m\}$ with $i < j$.

Since $\mathcal{U}_q^{c_m}$ has a PBW basis of ordered monomials, the relations of Eq. (6.3) are the defining relations for $\mathcal{U}_q^{c_m}$. In particular, we have the following well-known result (cf. for example [16]):

Corollary 6.2. *The algebra $\mathcal{U}_q^{c_m}$ is isomorphic to the algebra of quantum affine space $\mathcal{O}_q(k^m)$.*

Denote by $\pi_A : \mathcal{U}_A^{\geqslant 0} \to \mathcal{U}_A^{\geqslant 0}$ the unique algebra map such that

$$\pi(E_1) = 0, \tag{6.4}$$

$$\pi(E_i) = E_i \quad (1 < i \leqslant m), \tag{6.5}$$

$$\pi(K_\mu) = K_\mu \quad \big(\mu \in Q(A_m)\big). \tag{6.6}$$

Let $\lambda_A : \mathcal{U}_A^{\geqslant 0} \otimes \mathcal{U}_q^{c_m} \to \mathcal{U}_A^{\geqslant 0}$ be defined by the following sequence of linear maps:

$$\lambda_A : \mathcal{U}_A^{\geqslant 0} \otimes \mathcal{U}_q^{c_m} \xrightarrow{\text{incl.}} \big(\mathcal{U}_A^{\geqslant 0}\big)^{\otimes 2} \xrightarrow{\pi_A \otimes id} \big(\mathcal{U}_A^{\geqslant 0}\big)^{\otimes 2} \xrightarrow{\text{adjoint}} \mathcal{U}_A^{\geqslant 0}\ . \tag{6.7}$$

The identities in Eq. (3.16) imply the following

Corollary 6.3. *The linear map λ_A satisfies $\mathrm{Im}(\lambda_A) \subseteq \mathcal{U}_q^{c_m}$. In particular, λ_A endows the algebra $\mathcal{U}_q^{c_m}$ with the structure of a left $\mathcal{U}_A^{\geqslant 0}$-module algebra.*

As before (see (3.18)), we use the action map λ_A to construct the smash product $\mathcal{U}_q^{c_m} \# \mathcal{U}_A^{\geqslant 0}$ and let $(\mathcal{U}_q^{c_m})^\#$ denote the subalgebra

$$\big(\mathcal{U}_q^{c_m}\big)^\# := \big\langle 1\#u, u\#1 \ \big|\ u \in \mathcal{U}_q^{c_m} \big\rangle \subseteq \mathcal{U}_q^{c_m} \# \mathcal{U}_A^{\geqslant 0}. \tag{6.8}$$

6.2. *The quantum affine algebra $\mathcal{U}_q^{\widehat{c}_m}$*

Let $Q(\widehat{A}_m) = Q(A_m) \oplus \mathbb{Z}$ denote the root lattice of type $\widehat{A}_m$. As an abelian group, $Q(\widehat{A}_m)$ is generated additively by the positive simple roots $\alpha_0 := e_m - e_1 + 1$, and $\alpha_i := e_i - e_{i+1}$ for $i \in \{1, \dots, m\}$. We extend the inner product $\langle\ ,\ \rangle$ on $Q(A_m)$ to an inner product on $Q(\widehat{A}_m)$ by setting $1 \in Q(\widehat{A}_m)$ to be isotropic. We let s_i denote the corresponding simple reflection $s_i : Q(\widehat{A}_m) \to Q(\widehat{A}_m)$, for $0 \leqslant i \leqslant m$, and let $W(\widehat{A}_m) = \langle s_0, \dots, s_m \rangle$ denote the corresponding affine Weyl group. We let $\mathcal{U}_q(\widehat{\mathfrak{sl}}_{m+1})$ denote the corresponding quantized enveloping algebra.

We set

$$\widehat{c}_m := (s_1 \cdots s_m)(s_0 s_1 \cdots s_{m-1}) \in W(\widehat{A}_m) \tag{6.9}$$

and note the following analogue of Theorem 4.4.

Theorem 6.4. *Suppose k is an algebraically closed field of characteristic zero, and $q \in k$ is transcendental over $\mathbb{Q}$. As k-algebras, $\mathcal{U}_q^{\widehat{c}_m} \cong (\mathcal{U}_q^{c_m})^{\#}$.*

Proof. Compute. One can use an analogous isomorphism of Theorem 4.4. □

Now let V be a k-module with basis $\{v_1, \ldots, v_m\}$, and for all $i, j, \ell \in \{1, \ldots, m\}$, define linear maps e_{ij} by the rule $e_{ij}.v_\ell = \delta_{j\ell} v_i$.

Set

$$R_{A_{m-1}} = q \sum_{i=1}^{m} (e_{ii} \otimes e_{ii}) + \sum_{i \neq j} (e_{ii} \otimes e_{jj}) + \hat{q} \sum_{i>j} (e_{ij} \otimes e_{ji}). \tag{6.10}$$

This is the standard R-matrix of type A_{m-1} (see [13, Section 8.4.2]).

The algebra of $m \times m$ quantum matrices, denoted $\mathcal{O}_q(M_m(k))$, is the algebra $\mathcal{A}(R_{A_{m-1}})$ and was defined in [8]. More generally, one considers $\ell \times p$ quantum matrices, denoted $\mathcal{O}_q(M_{\ell,p}(k))$, by looking at appropriate subalgebras of square quantum matrices.

We let $\gamma : Q(\widehat{A}_m) \times Q(\widehat{A}_m) \to k^\times$ be the bicharacter defined by

$$\gamma(\alpha_i, \alpha_j) = \begin{cases} q, & (i, j) = (0, 1), \\ 1, & (i, j) \neq (0, 1), \end{cases} \tag{6.11}$$

and have the following analogue of Theorem 5.2.

Theorem 6.5. *Assuming the same hypotheses on k and q from Theorem 6.4, the γ-twisted algebra $(\mathcal{U}_q^{\widehat{c}_m})'$ is isomorphic to $\mathcal{O}_q(M_{2,m})$.*

Proof. Compute (cf. Theorem 5.2). □

Theorem 6.5, together with Proposition 5.1, allows us to view $\mathbb{X}_{n,q}$ as an orthogonal analogue of $2 \times n$ quantum matrices. The key distinction is that $\mathcal{O}_q(M_{2,n}(k))$ is a subalgebra of $\mathcal{A}(R_{A_{n-1}})$, whereas $\mathbb{X}_{n,q}$ is a parent of the analogous subalgebra $T_{2,n} \subseteq \mathcal{A}(R_{D_n})$.

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