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Compositions with superlinear deterministic top-down tree transformations

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Abstract

We denote the class of deterministic top-down tree transformations by DT and the class of homomorphism tree transformations by HOM. The sign of a class with the prefix l- (sl-, nd-) denotes the *linear* (*superlinear*, *nondeleting*) subclass of that class. We fix the set $M = \{HOM, sl\text{-}DT, l\text{-}DT, nd\text{-}DT, DT\}$ of tree transformation classes. Then consider the monoid $[M]$ of all tree transformation classes of the form $X_1 \circ \cdots \circ X_m$, where $\circ$ is the operation composition, $m \geqslant 0$ and the X_i's are elements of M. As the main result of the paper, we give an effective description of the monoid $[M]$ with respect to inclusion. This means that we present an algorithm which can decide, given arbitrary two elements of the monoid, whether some inclusion, equality or incomparability holds between them.

1. Introduction

Top-down tree transducers were introduced in [18] with the motivation of studying abstract properties of syntax-directed compilers. A top-down tree transducer induces a tree transformation, which consists of pairs of terms over ranked alphabets. Hence, a tree transformation is a binary relation over terms. Terms are called trees in this area and a tree transformation is the abstract model of the translation realized by a syntax-directed compiler.

A top-down tree transducer translates an input tree by applying so-called *rules* at nodes of the input tree processing the tree from the root to the leaves. Each rule has the form $q(\sigma(x_1,\ldots,x_m)) \to \xi$, where σ is an input symbol of rank m that labels a

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node of the input tree, q is a state of the tree transducer and ξ is a term consisting of output symbols and terms of the form $p(x_i)$, where $1 \leqslant i \leqslant m$ and x_i refers to the ith descendant and p is a state of the tree transducer. A top-down tree transducer is said to be *deterministic* if, for any state q and symbol σ, there is at most one rule of the above form. In this paper we consider only deterministic top-down tree transducers. A top-down tree transducer is called *linear* (*nondeleting*) if, for every rule of the above form, for each $1 \leqslant i \leqslant m$, a term of the form $p(x_i)$ appears at most once (at least once) in ξ. As a result of this condition, the translation of any direct subtree t_i of a tree $\sigma(t_1,\ldots,t_m)$ appears at most once (at least once) in the translation of $\sigma(t_1,\ldots,t_m)$. The linear and the nondeleting subclasses of the class of top-down tree transformations were intensively studied in the papers [5] and [1]. Recently, an even more special subclass of deterministic top-down tree transducers was considered in [4]. It is called *superlinear* and defined as follows. A top-down tree transducer is superlinear if it is linear and, for any input symbol σ of rank m and integer i with $1 \leqslant i \leqslant m$, each term of the form $p(x_i)$ may appear in the right-hand side ξ of at most one rule of the form $q(\sigma(x_1,\ldots,x_m)) \to \xi$.

In our sense, a tree transformation class is a class consisting of tree transformations having a certain property. Thus, one can speak about the class of top-down tree transformations and its subclasses such as linear top-down tree transformations, nondeleting top-down tree transformations, etc. The operation composition, denoted by $\circ$, is defined for tree transformations and models applying two translation devices to a language after each other in such a way that the output of the first device is the input of the second one. The concept of composition is extended for tree transformation classes. The composition of two classes C_1 and C_2 is denoted by $C_1 \circ C_2$ and is defined as the class of all tree transformations of the form $\tau_1 \circ \tau_2$, where $\tau_1 \in C_1$ and $\tau_2 \in C_2$.

The main task of this paper is studying compositions and decompositions of top-down tree transformation classes.

The motivation of considering composition comes from the fact that applying tree transducers in succession can yield an extra transformational power in the sense that the composition of the tree transformations induced by them cannot be induced in general by a single tree transducer. Hence, some tree transformation classes of the form $C_1 \circ \cdots \circ C_m$, where $C_1,\ldots,C_m$ are tree transformation classes of some types, cannot be characterized as the class of tree transformations induced by tree transducers of a certain type. At the same time one may want to know, given some tree transducers, whether the consecutive application of them can still be substituted by a single one of some type. In the language of tree transformation classes, this problem can also be expressed as whether $C_1 \circ \cdots \circ C_m \subseteq C$ holds, where $C_1,\ldots,C_m$ and C are tree transformation classes of some types. In particular, if $C \circ C \subseteq C$ holds for some class C, then we say that C is closed under composition.

Decomposition is motivated as follows. Given a top-down tree transducer of a certain type, one would like to know whether the tree transformation defined by it could also be obtained by applying two or more tree transducers of some simpler types in succession. If this is the case, then the working mechanism of the original tree transducer can be understood more clearly. In terms of tree transformation classes, this can be expressed as whether $C \subseteq C_1 \circ \cdots \circ C_m$, where C is the class of tree transformations induced by tree transducers of the original type, while $C_1, \ldots, C_m$ are proper subclasses of C. In this case we say that C can be decomposed into $C_1, \ldots, C_m$. Of course, decomposition equations like $C = C_1 \circ \cdots \circ C_m$ are also of potential interest.

We denote the class of deterministic top-down tree transformations by DT and its subclass induced by one-state deterministic top-down tree transducers by HOM. The sign of a class with the prefix *l-* (*sl-*, *nd-*) denote the *linear* (*superlinear, nondeleting*) subclass of that class.

Up to now, several decomposition equations and inclusions have been obtained for different subclasses of deterministic top-down tree transformations. For example $DT = nd\text{-}HOM \circ l\text{-}DT$ and $DT \circ nd\text{-}HOM = DT$ were proved in [5] and [1]. For superlinear top-down tree transducers $DT = nd\text{-}HOM \circ sl\text{-}DT$ was obtained in [4]. If one has some decomposition equations like the above ones, then further ones can be derived from them. For example, we also obtain $DT^2 = nd\text{-}HOM \circ sl\text{-}DT^2$, because

$$\begin{aligned} DT^2 &= DT \circ DT \\ &= DT \circ nd\text{-}HOM \circ sl\text{-}DT && (\text{by } DT = nd\text{-}HOM \circ sl\text{-}DT) \\ &= DT \circ sl\text{-}DT && (\text{by } DT \circ nd\text{-}HOM = DT) \\ &= nd\text{-}HOM \circ sl\text{-}DT^2 && (\text{by } DT = nd\text{-}HOM \circ sl\text{-}DT). \end{aligned}$$

Then it is natural to raise the question whether the set of decomposition equations and inclusions we already know are sufficient or not to derive all other ones.

The above problem can be formed more generally as follows. Given a finite set M of tree transformation classes and two tree transformation classes of the form

$$X_1 \circ X_2 \circ \cdots \circ X_m \quad \text{and} \quad Y_1 \circ Y_2 \circ \cdots \circ Y_n$$

such that X_i and Y_j are in M for every i, j with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, we would like to know whether inclusion of some direction or equality or incomparability holds between them. Note the tree transformation classes of the form $X_1 \circ X_2 \circ \cdots \circ X_m$ forms a monoid, called the monoid generated by M with composition. Hence the problem can alternatively be stated as follows. Given M, present an algorithm that decides the inclusion between the elements of the monoid generated by M. (Of course, by standard arguments, a slight modification of this algorithm can also decide whether equality holds or not.) This question was already investigated and a general method was proposed for developing such an algorithm in [12]. A description of that general method is presented also in [14]. In the works [10], [12], and [13] the general method was implemented for a set M consisting of DT and six of its subclasses. Moreover, the method was also applied for a set consisting of deterministic bottom-up tree transformation classes in [8]; for a set of deterministic top-down tree transformation classes with regular look-ahead (see [6]) in [19]; and recently for a set which consists of deterministic top-down tree transformation classes both with look-ahead and without look-ahead and of deterministic bottom-up tree transformation classes in [15]. So it seems that the general method successfully applies to different choices of M. The significance of this method lies in the fact that an exhaustive answer can be given to all questions concerning inclusion or equality between any two elements obtainable from a fixed set of tree transformation classes by composition. Especially, all problems concerning composition and decomposition of the elements of that fixed set can be solved.

In the present paper we slightly modify the general method and apply it to the set $M = \{HOM, \textit{sl-DT}, \textit{l-DT}, \textit{nd-DT}, DT\}$. We chose this particular M because we want to examine how the new class *sl-DT* behaves when composing it with known deterministic top-down tree transformation classes. The implementation of the method for this choice of M involves the following activities. We collect some known decomposition equations, especially from the recent ones appearing in [4], and show that all other equalities can be derived from them. We represent the tree transformation classes obtainable by composition from elements of M as strings over the free monoid M^*. With this we achieve that we can exploit the reduction techniques developed in the theory of string rewriting systems for compositions of tree transformation classes. We give a direction to each equality so that a terminating and confluent rewriting system R over M is obtained. We construct the inclusion diagram of the tree transformation classes represented by the normal forms of R. Then the inclusion can be decided between the tree transformation classes $X_1 \circ X_2 \circ \cdots \circ X_m$ and $Y_1 \circ Y_2 \circ \cdots \circ Y_n$ obtained from elements of M by composition in the following way. We take the strings representing them and reduce these strings by rules of R to normal forms u and v, respectively. Then, inclusion holds between $X_1 \circ X_2 \circ \cdots \circ X_m$ and $Y_1 \circ Y_2 \circ \cdots \circ Y_n$ if and only if the corresponding inclusion holds between tree transformation classes represented by the normal forms u and v. This latter one, however, can be read from the mentioned inclusion diagram immediately.

The paper is organized as follows. In Section 2, we have collected the notions and preliminary results that are necessary for understanding the paper. In Section 3, we describe the problem of deciding the inclusion in a monoid generated by tree transformation classes and present a general method for the solution. In Section 4, we apply this method to the monoid generated by the set $M = \{HOM, \textit{sl-DT}, \textit{l-DT}, \textit{nd-DT}, DT\}$ of tree transformation classes. However, the proof of the correctness of the inclusion diagram of the tree transformation classes represented by the normal forms is postponed to Section 5.

2. Preliminaries

In this section we introduce the notions and notations which are necessary for understanding the paper. Moreover, we recall some preliminary results referred to in our proofs.

Sets, strings: For arbitrary sets A and B, we denote by $A \subseteq B$ that A is a subset of B, by $A \subset B$ that A is a proper subset of B and by $A \bowtie B$ that A and B are incomparable.

Let H be a set of sets partially ordered by the inclusion relation $\subseteq$. By the *inclusion diagram of* H we mean the Hasse diagram [3] of H with respect to the partial order $\subseteq$.

An *alphabet* A is a finite nonempty set. The *set of strings over* A is denoted by A^*. The *empty string* is denoted by e and the *length of a string* $w \in A^*$ is denoted by *length*(w). Recall that A^* is the free monoid generated by A with the operation concatenation.

Relations. Given two sets A and B, any subset θ of the Cartesian product $A \times B$ is called a *relation* from A to B. For $a \in A$ and $b \in B$, we write $a\theta b$ to mean that $(a,b) \in \theta$ and we put $\theta(a) = \{b \mid a\theta b\}$. The set $\{a \in A \mid a\theta b$, for some $b \in B\}$ is called the *domain* of θ and is denoted by $\mathrm{dom}(\theta)$. We say that θ is *total*, if $\mathrm{dom}(\theta) = A$.

A relation from A to A is also called a *relation over* A. The identity relation $\{(a,a) \mid a \in A\}$ over A is denoted by $Id(A)$.

Let θ be a relation from A to B and let μ be a relation from B to C. The *composition of* θ *and* μ is the relation $\theta \circ \mu$ from A to C defined by $\theta \circ \mu = \{(a,c) \mid a\theta b$ and $b\mu c$, for some $b \in B\}$. Moreover, let θ be a relation over A. The n-fold composition θ^n of θ is defined by induction as follows: $\theta^0 = Id(A)$ and $\theta^n = \theta \circ \theta^{n-1}$, where $n > 0$. The *reflexive, transitive closure of* θ is the relation $\theta^* = \bigcup_{n \geqslant 0} \theta^n$.

We extend the concepts of domain and composition for classes of relations. Let Y and Z be classes of relations, the domain of Y is defined by $\mathrm{dom}(Y) = \{\mathrm{dom}(\theta) \mid \theta \in Y\}$ and the composition of Y and Z is the relation class $Y \circ Z = \{\theta \circ \sigma \mid \theta \in Y$ and $\sigma \in Z\}$. Moreover, let $Y^1 = Y$ and $Y^n = Y \circ Y^{n-1}$, where $n > 1$.

A class Y of relations is said to be *closed under the composition* if $Y^2 \subseteq Y$ holds. The *closure of* Y *under the composition* is the class $Y^+ = \bigcup_{n \geqslant 1} Y^n$.

Finally, we introduce the concept of a hierarchy. A family of classes $\{C_k \mid k \geqslant 1\}$ is called a *hierarchy* if $C_k \subseteq C_{k+1}$ holds for each $k \geqslant 1$. A hierarchy is said to be *proper* if each inclusion is proper, i.e. $C_k \subset C_{k+1}$ holds for every $k \geqslant 1$.

Trees: A *ranked alphabet* Σ is a finite alphabet in which every symbol has unique rank in the set of nonnegative integers. For each $m \geqslant 0$, the set of symbols in Σ having rank m is denoted by Σ_m. We write $\Sigma = \{\sigma_1^{(m_1)}, \ldots, \sigma_n^{(m_n)}\}$ meaning that $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ is a ranked alphabet and the symbol σ_i has rank m_i, for each $1 \leqslant i \leqslant n$.

For a set H, disjoint with Σ, the set of *terms* (or rather *trees*) over Σ indexed by H is denoted by $T_\Sigma(H)$ and it is defined as the smallest set U satisfying the following two conditions:

(i) $H \cup \Sigma_0 \subseteq U$,

(ii) $\sigma(t_1, \ldots, t_m) \in U$ whenever $m > 0$, $\sigma \in \Sigma_m$ and $t_1, \ldots, t_m \in U$.

The set $T_\Sigma(\emptyset)$ is written as T_Σ and it is called the set of *ground trees* over Σ.

The trees can be written as expressions with parentheses, e.g. if $\Sigma = \{\delta^{(2)}, \sigma^{(1)}, \#^{(0)}\}$ then $\delta(\sigma(\#), \#) \in T_\Sigma$. A "chain" tree, like $\sigma(\ldots\sigma(\#)\ldots)$, where σ occurs i times, is abbreviated by $\sigma^i(\#)$. For example, $\sigma^3(\#)$ denotes the tree $\sigma(\sigma(\sigma(\#)))$.

Let Σ be a ranked alphabet and let $\sigma \in \Sigma_n$, where $n \geqslant 1$. Moreover, let $L_1, \ldots, L_n \subseteq T_\Sigma$. The set defined as $\{\sigma(t_1, \ldots, t_n) \mid t_i \in L_i$, for $1 \leqslant i \leqslant n\}$ is denoted by the expression of $\sigma(L_1, \ldots, L_n)$.

Tree transformations: Let Σ and Δ be ranked alphabets. A *tree transformation from T_Σ to T_Δ* is a relation from T_Σ to T_Δ. Since tree transformations are relations, the concept of their composition and domain should be clear.

We put $I = \{Id(T_\Sigma) \mid \Sigma \text{ is a ranked alphabet}\}$. Now, it is possible to define the reflexive closure of a tree transformation class Y as $Y^* = \bigcup_{n \geqslant 0} Y^n$, where $Y^0 = I$. If, for a class Y of tree transformations, $I \subseteq Y$ holds, then $Y^2 \subseteq Y$ if and only if $Y^2 = Y$.

We specify a countable set $X = \{x_1, x_2, \ldots\}$ of symbols called *variables* and we set $X_m = \{x_1, \ldots, x_m\}$, for every $m \geqslant 0$. The expression $T_\Sigma(X_m)$ will be abbreviated by $T_{\Sigma,m}$.

We distinguish a subset $\hat{T}_{\Sigma,m}$ of $T_{\Sigma,m}$ as follows: a tree $t \in T_{\Sigma,m}$ is in $\hat{T}_{\Sigma,m}$ if and only if each variable in X_m appears in t exactly once and the order of the variables in t is just $x_1, \ldots, x_m$.

The set $\mathrm{Var}(t)$ of *variables occurring in a tree* $t \in T_{\Sigma,m}$ is defined by induction as follows:

(i) if $t = x_i \in X_m$, then $\mathrm{Var}(t) = \{x_i\}$,

(ii) if $t = \sigma \in \Sigma_0$, then $\mathrm{Var}(t) = \emptyset$,

(iii) if $t = \sigma(t_1, \ldots, t_m)$, where $m > 0$, $\sigma \in \Sigma_m$ and $t_1, \ldots, T_\Sigma$, then $\mathrm{Var}(t) = \bigcup_{1 \leqslant i \leqslant n} \mathrm{Var}(t_i)$.

Finally, we introduce the concept of *tree substitution.* Let $m \geqslant 0$, $t \in T_{\Sigma,m}$ and $h_1, \ldots, h_m \in H$ where H is an arbitrary set of trees. We denote by $t[h_1, \ldots, h_m]$ the tree which is obtained from t by replacing every occurrence of x_i in t by h_i, for every $1 \leqslant i \leqslant m$. Clearly, $t[h_1, \ldots, h_m] \in T_\Sigma(H)$ holds.

Tree transducers: A *top-down tree transducers* is a 5-tuple $T = (Q, \Sigma, \Delta, q_0, R)$, where:

- Q is an unary ranked alphabet, meaning that $Q = Q_1$, called the set of *states*, such that $Q \cap (\Sigma \cup \Delta \cup X) = \emptyset$,
- Σ and Δ are ranked alphabets, called the *input and the output ranked alphabet*, respectively,
- q_0 is a distinguished element of Q, called the *initial state*,
- R is a finite set of *rules* of the form $q(\sigma(x_1, \ldots, x_m)) \to t[q_1(x_{i_1}), \ldots, q_n(x_{i_n})]$ where $m, n \geqslant 0$, $\sigma \in \Sigma_m$, $1 \leqslant i_j \leqslant m$ for every $1 \leqslant j \leqslant n$, $q, q_1, \ldots, q_n \in Q$ and $t \in \hat{T}_{\Delta,n}$.

A rule as above will be referred to as a q-rule for σ or shortly as a (q,σ)-rule. We say that T is *deterministic* if, for every $q \in Q$ and $\sigma \in \Sigma$, there is at most one (q,σ)-rule in R. The expression "deterministic top-down tree transducer" will be abbreviated by "dt transducer" in the rest of the paper.

Consider the above (q,σ)-rule. The term $t[q_1(x_{i_1}), \ldots, q_n(x_{i_n})]$ will be called the *right-hand side* of the rule. We note that the order of the variables from left to right occurring in the right-hand side of that rule is $x_{i_1}, \ldots, x_{i_n}$, because the order of the ones from left to right occurring in t is $x_1, \ldots, x_n$. When we speak about dt transducers and the details are uninteresting, we just write $\mathrm{rhs}(q,\sigma)$ to specify the right-hand side of a (q,σ)-rule.

The rules in R induce a relation $\Rightarrow_T$, called *derivation*, over the set $T_\Delta(Q(T_\Sigma))$. It is defined as follows. For $r, s \in T_\Delta(Q(T_\Sigma))$, $r \Rightarrow_T s$ holds if and only if there is a rule $q(\sigma(x_1, \ldots, x_m)) \to t[q_1(x_{i_1}), \ldots, q_n(x_{i_n})]$ in R such that the tree s is obtained from r by replacing an occurrence of a subtree $q(\sigma(t_1,\dots,t_m))$ of r by $t[q_1(t_{i_1}),\dots,q_n(t_{i_n})]$, for some $t_1,\dots,t_m \in T_\Sigma$.

Now we define the *tree transformation τ_T induced by T as*

$$\tau_T = \{(r,s) \in T_\Sigma \times T_\Delta \mid q_0(r) \overset{*}{\underset{T}{\Rightarrow}} s\}.$$

A tree transformation τ is called a *dt tree transformation* if there exists a dt transducer T so that $\tau = \tau_T$ holds. The class of dt tree transformations is denoted by *DT*.

Special types of deterministic top-down tree transducers: We introduce some special types of dt transducers. The types (a), (b), (d) and (e) are well known from the theory of tree transducers, while (c) is investigated in [4] recently.

Let $T = (Q, \Sigma, \Delta, q_0, R)$ be a dt transducer. We say that T is

(a) *Total* (t) if for every $m \geqslant 0$, $\sigma \in \Sigma_m$ and $q \in Q$, there is a (and hence exactly one) (q,σ)-rule in R. Note that in this case τ_T is a total tree transformation.

(b) *Linear* (l) if, for every rule $q(\sigma(x_1,\dots,x_m)) \to t[q_1(x_{i_1}),\dots,q_n(x_{i_n})]$ in R, each of the variables $x_1,\dots,x_m$ appears at most once in the right-hand side. Note that in this case $m \geqslant n$.

(c) *Superlinear* (sl) if it is linear and, for every $\sigma \in \Sigma_m$ with $m \geqslant 0$ and two different states $q, q' \in Q$, $\mathrm{Var}(\mathrm{rhs}(q,\sigma)) \cap \mathrm{Var}(\mathrm{rhs}(q',\sigma)) = \emptyset$ holds. Equivalently, T is sl-dt if it is linear and, for every $\sigma \in \Sigma_m$ with $m \geqslant 0$ and $1 \leqslant i \leqslant m$, there is at most one $q \in Q$ such that x_i occurs in $\mathrm{rhs}(q,\sigma)$.

(d) *Nondeleting* (nd) if, for every rule $q(\sigma(x_1,\dots,x_m)) \to t[q_1(x_{i_1}),\dots,q_n(x_{i_n})]$ in R, each of the variables $x_1,\dots,x_m$ appears at least once in the right-hand side. Note that in this case $m \leqslant n$.

(e) *A homomorphism tree transducer* (hom) if it is total and Q is singleton set.

These attributes can be combined. For example, by an l-nd-dt transducer, we mean a linear and nondeleting deterministic top-down tree transducer.

Let x be a combination of some of the modifiers in $\{\mathrm{t}, \mathrm{l}, \mathrm{sl}, \mathrm{nd}\}$, such as l-nd, etc. A dt tree transformation is said to be an x-dt transformation if it can be induced by an x-dt transducer. The class of x-dt tree transformations is denoted by *x-DT*. (Note that if x and y are combinations such that y is a permutation of x, then *x-DT* = *y-DT*.)

We simply write hom instead of hom-dt. For example, the l-nd-hom transducer means a linear and nondeleting homomorphism dt transducer. The class of x-hom tree transformations is denoted by *x-HOM*, respectively.

We note that, for any combination x of modifiers, $I \subseteq x\text{-}DT$ and $I \subseteq x\text{-}HOM$ hold. Observe that if C and D are tree transformation classes and $I \subseteq D$, then $C \subseteq C \circ D$. This is because every tree transformation τ in C can be decomposed as $\tau = \tau \circ \iota$, where ι is a suitable identity in D. Specially, $x\text{-}DT^n \subseteq x\text{-}DT^{n+1}$ and $x\text{-}HOM^n \subseteq x\text{-}HOM^{n+1}$ hold for every $n \geqslant 0$.

Recognizable tree languages: A top-down tree transducer $T = (Q, \Sigma, \Delta, q_0, R)$ is called *tree recognizer* if $\Sigma = \Delta$ and each rule in R is of the form

$$q(\sigma(x_1,\dots,x_m)) \to \sigma(q_1(x_1),\dots,q_m(x_m)),$$

where $m \geqslant 0$, $\sigma \in \Sigma_m$ and $q, q_1, \ldots, q_m \in Q$. In addition, if T is deterministic, then it is called *deterministic tree recognizer*. Obviously, if T is a tree recognizer, then τ_T is a partial identity over T_Σ. The *tree language recognized by* T is $\mathrm{dom}(\tau_T)$.

We say that a tree language $L \subseteq T_\Sigma$ is *recognizable* (*deterministic recognizable*) if there is a tree recognizer (deterministic tree recognizer) T such that $L = \mathrm{dom}(\tau_T)$. The class of all recognizable (deterministic recognizable) tree languages is denoted by *REC*, (*DREC*).

It is an well-known result that $DREC \subset REC$. Moreover, we recall from [11] that $DREC = \mathrm{dom}(DT^n) = \mathrm{dom}(l\text{-}DT^n) = \mathrm{dom}(nd\text{-}DT)$ hold for every $n \geqslant 1$.

String rewriting systems: A *string rewriting system* R over an alphabet A is a finite relation over A^*. The elements of R are called *rewriting rules*. The *reduction relation* over A^* induced by R, denoted by $\Rightarrow_R$, is defined as follows: for $w, w' \in A^*$, $w \Rightarrow_R w'$ holds if and only if there exist $u, v \in A^*$ and a rule $x \to y$ in R such that $w = uxv$ and $w' = uyv$. We write $\Leftarrow_R$ to denote $\Rightarrow_R^{-1}$.

The symmetric, reflexive and transitive closure of $\Rightarrow_R$, denoted by $\Leftrightarrow_R^*$, is a *congruence* over A^* [3]. Informally speaking, $w \Leftrightarrow_R^* w'$ holds if and only if there is a chain $w_0, \ldots, w_n$, for some $n \geqslant 0$, such that $w_0 = w$, $w_n = w'$ and, for every $1 \leqslant i \leqslant n$, $w_{i-1} \Rightarrow_R w_i$ or $w_i \Rightarrow_R w_{i-1}$ holds.

We say that a string rewriting system R is *terminating* if there is no infinite chain of the form $w_1 \Rightarrow_R w_2 \Rightarrow_R \cdots$ Moreover, R is said to be *confluent* if for all $v, w, w' \in A^*$, $v \Rightarrow_R^* w$ and $v \Rightarrow_R^* w'$ imply that there exists a string $x \in A^*$ such that $w \Rightarrow_R^* x$ and $w' \Rightarrow_R^* x$ hold.

A string $w \in A^*$ is called an *R-normal form* (or simply *normal form*, if R is understood) if there is no $w' \in A^*$ such that $w \Rightarrow_R w'$. The set of R-normal forms is denoted by $NF(R)$. A string w' is a normal form of w if $w \Rightarrow_R^* w'$ and w' is a normal form.

We recall that a terminating term rewriting system R is confluent if and only if each word of A^* has exactly one normal form. In other words, considering the partitions of A^* by the congruence $\Leftrightarrow_R^*$, a terminating R is confluent if and only if each partition contains exactly one normal form (see [16]).

We now mention a sufficient condition for a string rewriting system R to be terminating. A *weight function* is a mapping $\rho : A \to \{1, 2, \ldots\}$, where $\rho(a)$ is the weight of $a \in A$. It can be extended to a mapping $\rho : A^* \to \{1, 2, \ldots\}$ by letting $\rho(c) = 0$ and $\rho(wa) = \rho(w) + \rho(a)$, for every $w \in A^*$ and $a \in A$. We say that R is *weight reducing* if there exists a weight function ρ such that, for each rule $x \to y$, $\rho(x) > \rho(y)$ holds. It should be clear that a weight reducing string rewriting system is necessarily terminating.

3. The problem and the outline of the solution

In this section we specify the problem that will be solved in the rest of the paper. Then we present the outline of the solution.

In [4] we already stated several inclusions and decomposition equations concerning *sl-DT*. Such an inclusion and decomposition are $sl\text{-}DT \subset l\text{-}DT$ and $HOM \circ sl\text{-}DT = DT$, respectively. On the other hand, several other inclusions and equations concerning tree transformation classes are known from the literature. For example, we recall the inclusion $DT \subset DT^2$ from [18] and $HOM \circ l\text{-}DT = DT$ from [7]. We observe that we can obtain some new inclusions and equations from the above ones, for example $HOM \circ sl\text{-}DT \subset DT^2$ and $HOM \circ sl\text{-}DT = HOM \circ l\text{-}DT$, by "substituting equal for equal" in the corresponding expressions. This observation motivates us to determine an abstract method, with which all inclusions and equations are derivable that are valid among tree transformation classes obtained from some fundamental tree transformation classes by composition. Since we are interested in the compositions of *sl-DT* with the well-known subclasses of *DT*, we shall choose *HOM*, *sl-DT*, *l-DT*, *nd-DT* and *DT* as fundamental classes.

We now describe the problem in a more exact way. Let us fix the set

$$M = \{HOM, sl\text{-}DT, l\text{-}DT, nd\text{-}DT, DT\}.$$

We generalize the problem of inclusion and equality as follows. Whenever given two tree transformation classes

$$X_1 \circ X_2 \circ \cdots \circ X_m \quad \text{and} \quad Y_1 \circ Y_2 \circ \cdots \circ Y_n$$

such that $X_i, Y_j \in M$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, we would like to know whether proper inclusion of some direction, equality or incomparability holds between them. We observe that we can answer the question if we can decide whether the inclusion $X_1 \circ X_2 \circ \cdots \circ X_m \subseteq Y_1 \circ Y_2 \circ \cdots \circ Y_n$ holds or not. (Really, if we can decide this inclusion, then we can also decide whether $Y_1 \circ Y_2 \circ \cdots \circ Y_n \subseteq X_1 \circ X_2 \circ \cdots \circ X_m$. Then, for example, $X_1 \circ X_2 \circ \cdots \circ X_m \subset Y_1 \circ Y_2 \circ \cdots \circ Y_n$, if and only if $X_1 \circ X_2 \circ \cdots \circ X_m \subseteq Y_1 \circ Y_2 \circ \cdots \circ Y_n$ and not $Y_1 \circ Y_2 \circ \cdots \circ Y_n \subseteq X_1 \circ X_2 \circ \cdots \circ X_m$.)

So we conclude that we have solved the problem if we present an algorithm which decides for any two tree transformation classes as above, whether $X_1 \circ X_2 \circ \cdots \circ X_m \subseteq Y_1 \circ Y_2 \circ \cdots \circ Y_n$ holds or not.

We now describe how such an algorithm can be developed. We observe that the tree transformation classes of the form $X_1 \circ X_2 \circ \cdots \circ X_m$, where $m \geqslant 0$ and $X_i \in M$, for $1 \leqslant i \leqslant m$, form a monoid with the operation composition. The identity element of the monoid is I resulting by the empty composition in case $m = 0$. We denote this monoid by $[M]$. Hence our problem is to find an algorithm that decides the inclusion in $[M]$.

We also consider the free monoid M^* generated by M with the operation concatenation, which will be denoted by $\cdot$ in this paper. The identity relation over M can uniquely be extended to a homomorphism $|\,| : M^* \to [M]$. Then $|\,|$ has the property that, for every element $X_1 \cdot X_2 \cdot \ldots \cdot X_m$ of M^*, it holds

$$|X_1 \cdot X_2 \cdot \ldots \cdot X_m| = X_1 \circ X_2 \circ \cdots \circ X_m,$$

in particular, $|\varepsilon| = I$. Let us denote the kernel of $|\,|$ by θ. Then certainly, for any two elements $X_1 \circ X_2 \circ \cdots \circ X_m$ and $Y_1 \circ Y_2 \circ \cdots \circ Y_n$ of $[M]$, we have

$$X_1 \circ X_2 \circ \cdots \circ X_m = Y_1 \circ Y_2 \circ \cdots \circ Y_n$$

in $[M]$, if and only if

$$|X_1 \cdot X_2 \cdot \ldots \cdot X_m| = |Y_1 \cdot Y_2 \cdot \ldots \cdot Y_n|$$

or equivalently

$$X_1 \cdot X_2 \cdot \ldots \cdot X_m \theta Y_1 \cdot Y_2 \cdot \ldots \cdot Y_n$$

in M^*.

Our algorithm rests on the following two corner-stones.

1. We present a confluent and terminating rewriting system $R \subseteq M^* \times M^*$ such that $\Leftrightarrow_R^* = \theta$.
2. Moreover, we present the inclusion diagram of the set

$$|NF(R)| = \{|u| \mid u \in NF(R)\}.$$

Recall that $NF(R)$ denotes the set of normal forms of R.
Having the inclusion diagram for $|NF(R)|$, we can decide for any two normal forms $u, v \in NF(R)$ whether $|u| \subseteq |v|$ holds or not.

Then the algorithm works as follows. Let us be given two elements

$$X_1 \circ X_2 \circ \cdots \circ X_m \quad \text{and} \quad Y_1 \circ Y_2 \circ \cdots \circ Y_n$$

of $[M]$. Take the corresponding elements

$$X_1 \cdot X_2 \cdot \ldots \cdot X_m \quad \text{and} \quad Y_1 \cdot Y_2 \cdot \ldots \cdot Y_n$$

of M^*, and compute the normal forms $u, v \in NF(R)$ such that

$$X_1 \cdot X_2 \cdot \ldots \cdot X_m \overset{*}{\underset{R}{\Rightarrow}} u \quad \text{and} \quad Y_1 \cdot Y_2 \cdot \ldots \cdot Y_n \overset{*}{\underset{R}{\Rightarrow}} v,$$

respectively. Since R is terminating and confluent, u and v exist and unique. Moreover, $|X_1 \cdot X_2 \cdot \ldots \cdot X_m| = |u|$ and $|Y_1 \cdot Y_2 \cdot \ldots \cdot Y_n| = |v|$ because we have $\Rightarrow_R^* \subseteq \Leftrightarrow_R^* = \theta$ and θ is the kernel of $|\,|$. Then, by the definition of $|\,|$, the inclusion

$$X_1 \circ X_2 \circ \cdots \circ X_m \subseteq Y_1 \circ Y_2 \circ \cdots \circ Y_n \tag{$*$}$$

holds, if and only if,

$$|X_1 \cdot X_2 \cdot \ldots \cdot X_m| \subseteq |Y_1 \cdot Y_2 \cdot \ldots \cdot Y_n|.$$

On the other hand, this latter inclusion is equivalent to $|u| \subseteq |v|$. However, we can decide by a direct inspection of the inclusion diagram whether $|u| \subseteq |v|$ or not. Hence, we can also decide whether $(*)$ holds or not.

4. The decidability of inclusions in the composition monoid

Let $M = \{HOM, sl\text{-}DT, l\text{-}DT, nd\text{-}DT, DT\}$. In this section we present an algorithm that decides the inclusion in $[M]$ in the way described in the previous section. We start by giving a rewriting system R over M^*. Later on, we will show that R is terminating, confluent and that $\Leftrightarrow_R^* = \theta$.

Let R consist of the following 22 rewriting rules:

$$\begin{array}{lrcl}
(1) & \textit{l-DT}^2 \cdot \textit{HOM} & \to & \textit{l-DT} \cdot \textit{HOM} \\
(2) & \textit{HOM} \cdot \textit{HOM} & \to & \textit{HOM} \\
(3) & \textit{DT} \cdot \textit{HOM} & \to & \textit{DT}^2 \\
(4) & \textit{sl-DT} \cdot \textit{l-DT} \cdot \textit{HOM} & \to & \textit{l-DT} \cdot \textit{HOM} \\
(5) & \textit{l-DT}^3 & \to & \textit{l-DT}^2 \\
(6) & \textit{l-DT} \cdot \textit{sl-DT} & \to & \textit{l-DT}^2 \\
(7) & \textit{l-DT} \cdot \textit{DT} & \to & \textit{DT}^2 \\
(8) & \textit{HOM} \cdot \textit{l-DT} & \to & \textit{DT} \\
(9) & \textit{HOM} \cdot \textit{sl-DT} & \to & \textit{DT} \\
(10) & \textit{HOM} \cdot \textit{DT} & \to & \textit{DT} \\
(11) & \textit{DT} \cdot \textit{l-DT} & \to & \textit{DT}^2 \\
(12) & \textit{DT} \cdot \textit{sl-DT} & \to & \textit{DT}^2 \\
(13) & \textit{DT}^3 & \to & \textit{DT}^2 \\
(14) & \textit{sl-DT} \cdot \textit{l-DT}^2 & \to & \textit{l-DT}^2 \\
(15) & \textit{sl-DT} \cdot \textit{DT}^2 & \to & \textit{DT}^2 \\
(16) & \textit{nd-DT} \cdot \textit{HOM} & \to & \textit{DT}^2 \\
(17) & \textit{nd-DT} \cdot \textit{sl-DT} & \to & \textit{DT}^2 \\
(18) & \textit{nd-DT} \cdot \textit{l-DT} & \to & \textit{DT}^2 \\
(19) & \textit{nd-DT} \cdot \textit{nd-DT} & \to & \textit{nd-DT} \\
(20) & \textit{nd-DT} \cdot \textit{DT} & \to & \textit{DT}^2 \\
(21) & \textit{l-DT} \cdot \textit{HOM} \cdot \textit{nd-DT} & \to & \textit{l-DT}^2 \cdot \textit{nd-DT} \\
(22) & \textit{DT} \cdot \textit{nd-DT} & \to & \textit{DT}
\end{array}$$

First we prove the following.

Lemma 4.1. *The inclusion $\Leftrightarrow_R^* \subseteq \theta$ holds.*

Proof. To prove the lemma it is sufficient to show that, for every $u \to v \in R$, we have $|u| = |v|$, or equivalently $u \theta v$. In words, we say that the elements of R are valid in $[M]$. Indeed, if elements of R are valid in $[M]$, then $\Rightarrow_R \subseteq \theta$ which can be seen as follows. Let $w, z \in R^*$ such that $w \Rightarrow_R z$. Then, by the definition of $\Rightarrow_R$, there are strings x and y in M^* and there is a rule $u \to v \in R$ so that $w = x \cdot u \cdot y$ and $z = x \cdot v \cdot y$. Then we can compute as follows:

$$\begin{aligned}
|w| &= |x \cdot u \cdot y| \\
&= |x| \circ |u| \circ |y| \quad \text{(because } |\,| \text{ is a homomorphism)} \\
&= |x| \circ |v| \circ |y| \quad \text{(because } |u| = |v|\text{)} \\
&= |x \cdot v \cdot y| \quad \text{(because } |\,| \text{ is a homomorphism)} \\
&= |z|,
\end{aligned}$$

proving that $w\theta z$. Analogously, we can prove that $\Rightarrow_R^{-1} \subseteq \theta$, which yields that also $\Leftrightarrow_R \subseteq \theta$. Finally, we get $\Leftrightarrow_R^* \subseteq \theta^* = \theta$.

So we should prove that all elements of R are valid in $[M]$. As a matter of fact, most of them were already proved in earlier works. For example, the validity of (8), i.e., that $HOM \circ l\text{-}DT = DT$ was proved in [1] and [5]. A lot of the others also follow implicitly from the results of [1] and [5], but we refer the reader to [9] for the proofs because in this latter paper the proofs are explicit. Thus, the validity of (2), (10), (19) and (22) are consequences of Lemma 3 in [9]. Moreover, (5) and (13) are proved in Consequence 7, while (3), (7), (11), (16), (18) and (20) in Lemma 11 of the same paper.

Rules (1) and (21) can be proved using Table 1 of [9] as follows. From Table 1, it turns out that $l\text{-}DT^2 \circ HOM = l\text{-}nd\text{-}DT \circ HOM$ and also that $l\text{-}DT \circ HOM = l\text{-}nd\text{-}DT \circ HOM$, hence $l\text{-}DT^2 \circ HOM = l\text{-}DT \circ HOM$. The validity of (21) can be shown similarly.

Note that the rest of the rules, namely (4), (6), (9), (12), (14), (15), and (17) all contain $sl\text{-}DT$.

We now can prove (4) as follows. Obviously, $l\text{-}DT \circ HOM \subseteq sl\text{-}DT \circ l\text{-}DT \circ HOM$. As for the conversed inclusion, we have $sl\text{-}DT \circ l\text{-}DT \circ HOM \subseteq l\text{-}DT^2 \circ HOM$, because $sl\text{-}DT \subseteq l\text{-}DT$. On the other hand (1) is valid, hence $l\text{-}DT^2 \circ HOM = l\text{-}DT \circ HOM$, which proves the validity of (4).

For (6), we use $l\text{-}DT^2 = l\text{-}DT \circ l\text{-}HOM$ (see Lemma 11 in [9]). Then $l\text{-}DT^2 = l\text{-}DT \circ l\text{-}HOM \subseteq l\text{-}DT \circ sl\text{-}DT$, because $l\text{-}HOM = sl\text{-}HOM \subseteq sl\text{-}DT$, by Observation 3.1 of [4]. Finally, $l\text{-}DT \circ sl\text{-}DT \subseteq l\text{-}DT^2$. These altogether prove the validity of (6).

We note that the validity of (9) was proved in Theorem 3.6 of [4].

Next, (12) can be shown quite similarly to the proof of (6); however here we should use the equation $DT^2 = DT \circ l\text{-}HOM$, which comes again from Lemma 11 of [9]. Also (14) and (15) are obvious, because (5) and (13) are valid. Finally, (17) can be proved using $DT^2 = nd\text{-}DT \circ l\text{-}HOM$, which was verified in Lemma 11 of [9]. □

Next we prove that R is terminating.

Lemma 4.2. *The rewriting system R is terminating.*

Proof. A weight function can easily be defined for R so that R is weight reducing. In fact, let $\rho : M \to \{1,2,\ldots\}$ be such that

$$\begin{aligned} \rho(HOM) &= 3 \\ \rho(sl\text{-}DT) &= 3 \\ \rho(l\text{-}DT) &= 2 \\ \rho(nd\text{-}DT) &= 2 \\ \rho(DT) &= 1. \end{aligned}$$

It is a routine to check that R is weight reducing, hence it is terminating as well.

□

We now want to give $NF(R)$. This happens in three steps. First we present a set $NF \subseteq M^*$, second we present the inclusion diagram of the tree transformation classes represented by the elements of NF and finally we show that $NF(R) = NF$.

So let NF be defined in the following way:

$$\begin{aligned} NF = &\{l\text{-}DT^2, l\text{-}DT \cdot HOM, l\text{-}DT^2 \cdot nd\text{-}DT, DT^2\} \\ &\cup \{sl\text{-}DT^n \mid n \geqslant 0\} \\ &\cup \{sl\text{-}DT^n \cdot HOM \mid n \geqslant 0\} \\ &\cup \{sl\text{-}DT^n \cdot l\text{-}DT \mid n \geqslant 0\} \\ &\cup \{sl\text{-}DT^n \cdot nd\text{-}DT \mid n \geqslant 0\} \\ &\cup \{sl\text{-}DT^n \cdot l\text{-}DT \circ nd\text{-}DT \mid n \geqslant 0\} \\ &\cup \{sl\text{-}DT^n \cdot DT \mid n \geqslant 0\} \end{aligned}$$

Recall that $sl\text{-}DT^0 = e$, the empty string.

Lemma 4.3. *The diagram in Fig.* 1 *is the inclusion diagram of the set* $|NF| = \{|u| \mid u \in NF\}$

Proof. (Actually, the involved diagram is a bit more than the inclusion diagram of $|NF|$ because it also contains the suprema of the six hierarchies appearing in $|NF|$. We inserted these suprema into the diagram because it becomes more complete in this a way.)

The proof is rather technical and long, hence we have separated it into Section 5. □

Corollary 4.4. *For any* $u, w \in NF$, *we have* $|u| = |w|$ *if and only if* $u = w$.

Proof. The statement is shown by the diagram. □

Lemma 4.5. $NF(R) = NF$.

Proof. It is easy, although tedious, to show that $NF \subseteq NF(R)$. If we consider the elements of NF one-by-one, then we realize that there is no element such that a rule is applicable to it.

The proof of $NF(R) \subseteq NF$ is strongly based on Fig. 2, which is organized as follows. Although the table is divided into two parts because of space limitations, it should be considered as one with 10 rows and 5 columns. The 10 rows are labeled, on the one hand, by the 4 elements of the first union member forming NF and, on the other hand, by typical elements from any of the remaining 6 members forming NF. The columns of the table are labeled by the elements of M. We now describe how an entry determined by a row u and a column C is defined. If $u \cdot C$ is also an element of NF, then the entry is $u \cdot C$ and nothing else. However, if $u \cdot C$ is not in NF, then the entry consists

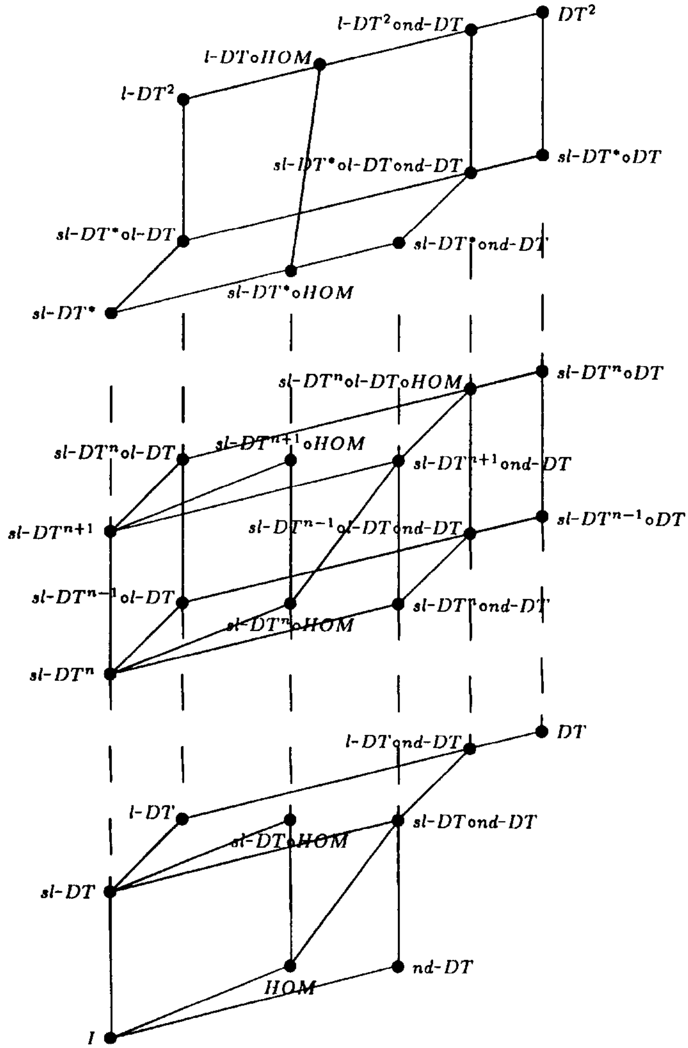

Fig. 1.

of an element v of NF and some number denoting rules in R such that $u \cdot C \Rightarrow_R^* v$ by applying the rules appearing in the entry.

For example, if $u = sl\text{-}DT^n \cdot HOM$ and $C = nd\text{-}DT$, then the corresponding entry is $sl\text{-}DT^n \cdot HOM \cdot nd\text{-}DT$, because this latter is in NF itself. However, if $u = sl\text{-}DT^n \cdot l\text{-}DT$ and $C = sl\text{-}DT$, then the entry consists of $l\text{-}DT^2$ and the numebrs (6) and (14), because $sl\text{-}DT^n \cdot l\text{-}DT \cdot sl\text{-}DT \Rightarrow_R sl\text{-}DT^n \cdot l\text{-}DT^2$ by Eq. (6) and $sl\text{-}DT^n \cdot l\text{-}DT^2 \Rightarrow_R^* l\text{-}DT^2$ by applying n times the Eq. (14).

We now prove that, for every $x \in M^*$, the inclusion $x \in NF(R)$ implies that $x \in NF$. The proof is performed by an induction on $length(x)$.

	HOM	$sl\text{-}DT$	$l\text{-}DT$
$l\text{-}DT^2$	$l\text{-}DT \cdot HOM$ (1)	DT^2 (6), (5)	DT^2 (5)
$l\text{-}DT \cdot HOM$	$l\text{-}DT \cdot HOM$ (2)	$l\text{-}DT^2$ (9), (7)	$l\text{-}DT^2$ (8), (7)
$l\text{-}DT^2 \cdot nd\text{-}DT$	DT^2 (16), (7), (13)	DT^2 (17), (7), (13)	DT^2 (18), (7), (13)
DT^2	DT^2 (3), (13)	DT^2 (12), (13)	DT^2 (11), (13)
$sl\text{-}DT^n$	$sl\text{-}DT^n \cdot HOM$	$sl\text{-}DT^{n+1}$	$sl\text{-}DT^n \cdot l\text{-}DT$
$sl\text{-}DT^n \cdot HOM$	$sl\text{-}DT^n \cdot HOM$ (2)	$sl\text{-}DT^n \cdot DT$ (9)	$sl\text{-}DT^n \cdot DT$ (8)
$sl\text{-}DT^n \cdot l\text{-}DT$	$l\text{-}DT \cdot HOM$ (4)	$l\text{-}DT^2$ (6), (14)	$l\text{-}DT^2$ (14)
$sl\text{-}DT^n \cdot nd\text{-}DT$	DT^2 (16), (15)	DT^2 (17), (15)	DT^2 (18), (15)
$sl\text{-}DT^n \cdot l\text{-}DT \cdot nd\text{-}DT$	DT^2 (16), (7), (15)	DT^2 (17), (7), (15)	DT^2 (18), (7), (15)
$sl\text{-}DT^n \cdot DT$	DT^2 (3), (15)	DT^2 (12), (15)	DT^2 (11), (15)

	$nd\text{-}DT$	DT
$l\text{-}DT^2$	$l\text{-}DT^2 \cdot nd\text{-}DT$	DT^2 (7), (13)
$l\text{-}DT \cdot HOM$	$l\text{-}DT^2 \cdot nd\text{-}DT$ (21)	DT^2 (10), (7)
$l\text{-}DT^2 \cdot nd\text{-}DT$	$l\text{-}DT^2 \cdot nd\text{-}DT$ (19)	DT^2 (20), (7), (13)
DT^2	$l\text{-}DT^2 \cdot nd\text{-}DT$ (22)	DT^2 (13)
$sl\text{-}DT^n$	$sl\text{-}DT^n \cdot nd\text{-}DT$	$sl\text{-}DT^n \cdot DT$
$sl\text{-}DT^n \cdot HOM$	$sl\text{-}DT^n \cdot HOM \cdot nd\text{-}DT$	$sl\text{-}DT^n \cdot DT$ (10)
$sl\text{-}DT^n \cdot l\text{-}DT$	$sl\text{-}DT^n \cdot l\text{-}DT \cdot nd\text{-}DT$	DT^2 (7), (15)
$sl\text{-}DT^n \cdot nd\text{-}DT$	$sl\text{-}DT^n \cdot nd\text{-}DT$ (19)	DT^2 (20), (15)
$sl\text{-}DT^n \circ l\text{-}DT \cdot nd\text{-}DT$	$sl\text{-}DT^n \cdot l\text{-}DT \cdot nd\text{-}DT$ (19)	DT^2 (20), (7), (15)
$sl\text{-}DT^n \cdot DT$	$sl\text{-}DT^n \cdot DT$ (22)	DT^2 (15)

Fig. 2.

If $length(x) = 0$, then certainly $x = e$. Since $e \in NF$, we have nothing to prove.

Now let $x \in M^*$ be such that $length(x) = n + 1$ and suppose that the statement is true for every word in M^* with length at most n. Then $x = y \cdot C$, for some $y \in M^*$ with length n and $C \in M$. Assume now that $x \in NF(R)$. Then certainly $y \in NF(R)$ and thus, by the induction hypothesis, $y \in NF$, too.

Considering the definition of NF, 10 cases are possible, each of which corresponds to a row of the table. These are as follows.

Case 1: $y = l\text{-}DT^2$ and $n = 2$. Then we can see from the table that C can only be $nd\text{-}DT$ because in any other cases $x = y \cdot C = l\text{-}DT^2 \cdot C$ can be reduced with some rule(s) of R, hence cannot be in $NF(R)$. (For example, in case $C = HOM$, $y \cdot C = l\text{-}DT^2 \cdot HOM$ reduces to $l\text{-}DT \cdot HOM$ with rule (1), hence is not irreducible.) However, if $C = nd\text{-}DT$, then $x = y \cdot C = l\text{-}DT^2 \cdot nd\text{-}DT$ is in NF, what we wanted to prove.

Cases 2–4: Here $y = l\text{-}DT \cdot HOM$ and $n = 2$, $y = l\text{-}DT^2 \cdot nd\text{-}DT$ and $n = 3$, finally $y = DT^2$ and $n = 2$, respectively. We can see easily from Fig. 2 that, for every such y and every $C \in M$, $x = y \cdot C$ cannot be in $NF(R)$, hence we have nothing to prove.

Case 5: $y = sl\text{-}DT^n$. We see from Fig. 2 that, for every $C \in M$, $x = sl\text{-}DT^n \cdot C$ is in NF.

Since Cases 6–10 can be handled similarly, we left the rest part of the proof for an exercise. □

Lemma 4.6. *The equality* $\theta = \Leftrightarrow_R^*$ *holds.*

Proof. Since we have already proved in Lemma 4.1 that $\Leftrightarrow_R^* \subseteq \theta$, it is sufficient to show the conversed inclusion $\theta \subseteq \Leftrightarrow_R^*$.

Thus, let $x, y \in M^*$ be such that $x\theta y$. Since R is terminating, see Lemma 4.2, there are $u, v \in NF(R)$ with $x \Rightarrow_R^* u$ and $y \Rightarrow_R^* v$. Then, again by Lemma 4.1, $x\theta u$ and $y\theta v$ and thus $u\theta v$. On the other hand, by Lemma 4.5, u and v are also in NF. Hence, by Corollary 4.4, we have $u = v$. We have obtained $x \Rightarrow_R^* u = v \Leftarrow_R^* y$ meaning that $x \Leftrightarrow_R^* y$. This completes the proof. □

Lemma 4.7. *R is confluent.*

Proof. By Lemma 4.2, R is terminating. Thus, by Proposition 1.1.25 in [12], it is sufficient to show that every $\Leftrightarrow_R^*$-class contains exactly one R-normal form. (For the proof of this fact, see also [2].) Since R is terminating, there is at least one normal form in every $\Leftrightarrow_R^*$-class. Assume now, that $u, v \in NF$ such that $u \Leftrightarrow_R^* v$. Then, by $\Leftrightarrow_R^* = \theta$, we have $u\theta v$. By Corollary 4.4, this implies $u = v$. □

Theorem 4.8. *For any two tree transformation classes* $X_1 \circ X_2 \circ \cdots \circ X_m$ *and* $Y_1 \circ Y_2 \circ \cdots \circ Y_n$ *in* $[M]$, *it is decidable whether the inclusion* $X_1 \circ X_2 \circ \cdots \circ X_m \subseteq Y_1 \circ Y_2 \circ \cdots \circ Y_n$ *holds.*

Proof. Take the words $x = X_1 \cdot X_2 \cdot \ldots \cdot X_m$ and $y = Y_1 \cdot Y_1 \cdot \ldots \cdot Y_n$. Then $|x| = X_1 \circ X_2 \circ \cdots \circ X_m$ and $|y| = Y_1 \circ Y_2 \circ \cdots \circ Y_n$.

Then, let u and v be R-normal forms of x and y, respectively. Since R is terminating and confluent, u and v can be computed in linear time, with respect to $length(x)$ and $length(y)$, just reducing x and y, respectively, as long as possible. Then, by Lemma 4.6, also $x\theta u$ and $y\theta v$. Hence $|x| \subseteq |y|$, if and only if $|u| \subseteq |v|$. However, it is decidable by considering the diagram whether $|u| \subseteq |v|$ holds. □

We finish the section with an example. We would like to know the inclusion relation between the classes $sl\text{-}DT^3 \circ HOM \circ l\text{-}DT \circ nd\text{-}DT$ and $sl\text{-}DT^2 \circ HOM \circ sl\text{-}DT \circ HOM \circ nd\text{-}DT$. Then we compute as follows:

$$\begin{aligned} sl\text{-}DT^3 \cdot HOM \cdot l\text{-}DT \cdot nd\text{-}DT &\Rightarrow_{R,8} sl\text{-}DT^3 \cdot DT \cdot nd\text{-}DT \\ &\Rightarrow_{R,22} sl\text{-}DT^3 \cdot DT, \end{aligned}$$

where we wrote $\Rightarrow_{R,i}$ to denote that we applied the ith rule in that step of the computation. On the other hand,

$$\begin{aligned} sl\text{-}DT^2 \cdot HOM \cdot sl\text{-}DT \cdot HOM \cdot nd\text{-}DT &\Rightarrow_{R,9} sl\text{-}DT^2 \cdot DT \cdot HOM \cdot nd\text{-}DT \\ &\Rightarrow_{R,3} sl\text{-}DT^2 \cdot DT^2 \cdot nd\text{-}DT \\ &\Rightarrow_{R,22} sl\text{-}DT^2 \cdot DT^2 \\ &\Rightarrow^2_{R,15} DT^2. \end{aligned}$$

Since both $sl\text{-}DT^3 \cdot DT$ and DT^2 are in NF, we can see from the inclusion diagram that $sl\text{-}DT^3 \circ DT \subset DT^2$. Hence, $\subset$ holds between the two classes we started from.

5. The inclusion diagram of normal forms

In this section we show that the inclusion diagram of $|NF|$ is the diagram appearing in Fig. 1. We recall that $|NF|$ is the set of tree transformation classes represented by NF, with NF being the set of normal forms of R defined at the beginning of Section 4. However, to present the inclusion diagram we need some preparations.

First, we define some further special types of dt transducers. A dt transducer $T = (Q, \Sigma, \Delta, q_0, R)$ is said to be

(a) *Order-preserving* (op) if, for every rule $q(\sigma(x_1,\dots,x_m)) \to t[q_1(x_{i_1}),\dots,q_n(x_{i_n})]$ in R, the order $i_1 \leqslant \cdots \leqslant i_n$ holds.

(b) *Nonreducing* (nr) if there is no rule of the form $q(\sigma(x_1,\dots,x_m)) \to q'(x_i)$ in R. (A rule being of the mentioned form is called *reducing rule.*)

(c) *Nonincreasing* (ni) if, for every rule $q(\sigma(x_1,\dots,x_m)) \to t[q_1(x_{i_1}),\dots,q_n(x_{i_n})]$ in R, either $t = x_1$ or $t = \delta(x_1,\dots,x_n)$ holds, for some $\delta \in \Delta_n$.

Certain composition and decomposition results on these types have been studied in [4]. Moreover, others can easily be derived from the results of [9]. In the following proposition, we summarize some results we need in the rest of this section.

Proposition 5.1. *The following equations hold*:

(1) $sl\text{-}DT = op\text{-}ni\text{-}sl\text{-}DT \circ nr\text{-}l\text{-}nd\text{-}HOM$

(2) $nr\text{-}l\text{-}nd\text{-}HOM \circ sl\text{-}DT = sl\text{-}DT$

(3) $nr\text{-}l\text{-}nd\text{-}HOM \circ DT = DT$

(4) $nd\text{-}HOM \circ nd\text{-}DT = nd\text{-}DT$

(5) $l\text{-}HOM = sl\text{-}HOM$

(6) $t\text{-}nr\text{-}sl\text{-}DT \circ sl\text{-}DT = sl\text{-}DT$

(7) $t\text{-}nr\text{-}op\text{-}ni\text{-}sl\text{-}DT \circ op\text{-}ni\text{-}sl\text{-}DT = op\text{-}ni\text{-}sl\text{-}DT$.

Proof. (1) This is (4) of Corollary 3.12 in [4].

(2) Since $nr\text{-}l\text{-}nd\text{-}HOM \subseteq sl\text{-}DT$, the statement follows from Lemma 3.11 in [4] and Lemma 3 in [9].

(3) We observe that all elements of $nr\text{-}l\text{-}nd\text{-}HOM$ are total. Hence, we are done by Lemma 3 in [9].

(4) See also Lemma 3 in [9].

(5) See Observation 3.1 in [4].

(6) This is (3) of Corollary 3.12 in [4].

(7) Letting $C = t\text{-}nr\text{-}op\text{-}ni\text{-}sl\text{-}DT \circ op\text{-}ni\text{-}sl\text{-}DT$, we have $op\text{-}ni\text{-}sl\text{-}DT \subseteq C$. Moreover, $C \subseteq sl\text{-}DT$ holds by Lemma 3.11 of [4]. On the other hand, by the proof of that lemma, it is easy to see that $C \subseteq op\text{-}ni\text{-}sl\text{-}DT$ holds as well. □

Combining (1)–(4) we can easily conclude the following statements.

Corollary 5.2. *Let $n \geqslant 0$. Then*

(1) $sl\text{-}DT^n \circ DT = op\text{-}ni\text{-}sl\text{-}DT^n \circ DT$ *and*

(2) $sl\text{-}DT^n \circ nd\text{-}DT = op\text{-}ni\text{-}sl\text{-}DT^n \circ nd\text{-}DT$.

Moreover, we need the main results of [4].

Proposition 5.3. (1) *The hierarchy $\{dom(sl\text{-}DT^n) \mid n \geqslant 0\}$ is proper.*

(2) *For each $n \geqslant 0$, $dom(sl\text{-}DT^n) \subset DREC$ holds.*

(3) $l\text{-}DT - sl\text{-}DT^* \neq \emptyset$.

(4) $sl\text{-}DT^* \subset l\text{-}DT^2$.

Proof. The statements (1) and (2) are in Theorem 4.3 of [4]. Moreover, (3) is exactly Theorem 3.13 and (4) in (2) of Corollary 3.14 in the same paper. □

We shall also need the following stronger version of (2) of Proposition 5.3.

Theorem 5.4. $dom(sl\text{-}DT^*) \subset DREC$.

Proof. Let $\Sigma = \{\sigma^{(1)}, \#^{(0)}\}$. Define the tree language

$$L = \{\sigma^i(\#) \mid i \geqslant 0 \text{ is an even integer}\}$$

over Σ. Informally speaking, L is the set of even-length chains over Σ. Note that obviously $L \in DREC$. We prove $L \notin \mathrm{dom}(sl\text{-}DT^*)$.

To see this, suppose the contrary, i.e., that $L \in \mathrm{dom}(sl\text{-}DT^n)$, for some $n \geqslant 1$. Then there are sl-dt transducers $T_1, \ldots, T_n$, such that $L = \mathrm{dom}(\tau_{T_1} \circ \cdots \circ \tau_{T_n})$. Without loss of generality, we may assume n to be minimal.

Let $T_1 = (Q, \Sigma, \Delta, q, R)$. We investigate the rules in R. Since $\# \in L$, there must be a rule of the form

$$q(\#) \to t_{\#}$$

in R, for some $t_{\#} \in T_\Delta$.

The tree $\sigma(\sigma(\#))$ is also in L, hence there should be a (q,σ)-rule in R. It is easy to see that $\mathrm{rhs}(q,\sigma)$ cannot be a ground tree. On the other hand, T_1 is linear, hence the (q,σ)-rule is of the form $q(\sigma(x_1)) \to t[q'(x_1)]$ in R_1, for some $q' \in Q$ and $t \in \hat{T}_{\Sigma,1}$.

Similar to the previous argumentation, it is easy to show that there must be a rule of the form $q'(\sigma(x_1)) \to t'[q''(x_1)]$ in R, where $q'' \in Q$ and $t' \in \hat{T}_{\Sigma,1}$. However, since T_1 is superlinear, this is possible if and only if $q = q' = q''$, meaning that

$$q(\sigma(x_1)) \to t[q(x_1)] \in R.$$

Since $\Sigma = \{\sigma, \#\}$, there cannot be other useful rules in R. We obtained that T_1 is total, which implies $\mathrm{dom}(T_1) = T_\Sigma$. Hence, $n > 1$ must hold.

Consider the above (q,σ)-rule of R. It is easy to see that $t = x_1$ would imply $\tau_{T_1}(\sigma^i(\#)) = t_{\#}$ for every $i \geqslant 0$. Hence $t \neq x_1$, meaning that T_1 is nonreducing.

We now have that T_1 is t-nr-sl-dt, that is $L \in \mathrm{dom}(\textit{t-nr-sl-DT} \circ \textit{sl-DT}^{n-1})$, where $n > 1$. Hence, by (6) of Proposition 5.1, $L \in \mathrm{dom}(\textit{sl-DT}^{n-1})$ holds, which contradicts that n is minimal. □

Moreover, we prove two technical, but very useful lemmas before considering the inclusion diagram of $|NF|$. We know from [4] that any sequence of sl-dt transducers has "low" computational power. Roughly speaking, the first lemma shows that the computational power does not increase significantly, even if such a sequence is followed by a dt transducer.

Lemma 5.5. *Let $L \subset T_\Sigma$ for some ranked alphabet Σ and let $\sigma^{(2)}, \#^{(0)}$ and $\$^{(0)}$ be new ranked symbols. Put $\Sigma' = \Sigma \cup \{\sigma^{(2)}, \#^{(0)}, \$^{(0)}\}$ and $\Delta = \{\#^{(0)}, \$^{(0)}\}$. Define the tree transformation $\tau \subseteq T_{\Sigma'} \times T_\Delta$ as follows:*

$$\tau = \{(\sigma(t,s), s) \mid t \in L,\ s \in T_\Delta\}.$$

If $\tau \in \textit{sl-DT}^n \circ \textit{DT}$ for some $n \geqslant 1$, then $L \in \mathrm{dom}(\textit{sl-DT}^m)$ holds for some m such that $1 \leqslant m \leqslant n$.

Proof. To be short, we put $K = T_\Delta = \{\#, \$\}$. Observe that $\dim(\tau) = \sigma(L, K)$.

Since $\tau \in \textit{sl-DT}^n \circ \textit{DT}$, the inclusion $\tau \in \textit{op-ni-sl-DT}^n \circ \textit{DT}$ holds by (1) of Corollary 5.2. That is, there exist op-ni-sl-dt transducers $T_1, \ldots, T_n$ and a dt transducer T_{n+1} such that $\tau = \tau_{T_1} \circ \cdots \circ \tau_{T_n} \circ \tau_{T_{n+1}}$.

Put $T_i = (Q_i, \Delta^{(i-1)}, \Delta^{(i)}, q, R_i)$, where $1 \leqslant i \leqslant n+1$. Observe that $\Delta^{(0)} = \Sigma'$ and $\Delta^{(n+1)} = \Delta$. Moreover, we may assume that the initial state of all T_i's is q.

Consider the dt transducer T_{n+1}. Its output alphabet is Δ, which consists of symbols having rank 0. Therefore, each rule of T_{n+1} either must be reducing or it has \$

or # on its right-hand side. That is, T_{n+1} should be an op-ni-l-dt transducer. Hence $\tau \in op\text{-}ni\text{-}sl\text{-}DT^n \circ op\text{-}ni\text{-}l\text{-}DT$.

For every $1 \leqslant i \leqslant n+1$, we define the type of the sequence $T_1, \ldots, T_i$ by induction on i. This type can be (1), (2) or *undefined*.

(i) The type of T_1

- is (1), if there is a rule of the form

$$q(\sigma(x_1,x_2)) \to \sigma_1(p_1(x_1),q_1(x_2))$$

in R_1, where $p_1, q_1 \in Q_1$ and $\sigma_1 \in \Delta_2^{(1)}$,

- is (2), if there is a rule of the form

$$q(\sigma(x_1,x_2)) \to \sigma_1(q_1(x_2))$$

in R_1, where $q_1 \in Q_1$ and $\sigma_1 \in \Delta_1^{(1)} \cup \{x_1\}$, and

- is *undefined* otherwise.

(Note that q is the initial state of T_1.)

(ii) Let $i \geqslant 2$. Assume that the type of the sequence $T_1, \ldots, T_{i-1}$ has already been defined. The type of $T_1, \ldots, T_i$

- is (1), if the type of $T_1, \ldots, T_{i-1}$ is (1) and there is a rule of the form

$$q(\sigma_{i-1}(x_1,x_2)) \to \sigma_i(p_i(x_1),\ q_i(x_2))$$

in R_i, where $p_i, q_i \in Q_i$ and $\sigma_i \in \Delta_2^{(i)}$,

- is (2), if the type of $T_1, \ldots, T_{i-1}$ is (1) and there is a rule of the form

$$q(\sigma_{i-1}(x_1,x_2)) \to \sigma_i(q_i(x_2))$$

in R_i, where $q_i \in Q_i$ and $\sigma_i \in \Delta_1^{(i)} \cup \{x_1\}$, and

- is *undefined* otherwise.

(Here q is the initial state of T_i.)

We finish the proof as follows. First we make two observations. Observation 1 is on the domains of translations generated by sequences. Then, in Observation 2, we characterize the translations generated by sequences of types (1) and (2). Following this, in Step 1 we show that if there is an integer i with $1 < i \leqslant n+1$ such that $T_1, \ldots, T_i$ is of type (2), then $L = \mathrm{dom}(\tau_{M_1} \circ \cdots \circ \tau_{M_{i-1}})$ holds for some sl-dt transducers $M_1, \ldots, M_{i-1}$. Finally, in Step 2, we prove that actually there is a sequence $T_1, \ldots, T_i$ of type (2), for some $1 < i \leqslant n+1$.

Observation 1. Consider the transducer T_1. Since the root of each input tree in the translation τ is σ, there must be a (q,σ)-rule in R_1. On the other hand, since σ appears only as root in the input trees of τ, we may suppose that this is the only rule containing the state q. (Otherwise, we would take a new initial state for T_1.) Then we can also suppose that $\mathrm{dom}(\tau_{T_1}) = \sigma(L_1, K_1)$ holds for some $L_1, K_1 \subseteq T_{\Sigma'}$.

Clearly, $\mathrm{dom}(\tau_{T_1}\circ\cdots\circ\tau_{T_{i+1}})\subseteq\mathrm{dom}(\tau_{T_1}\circ\cdots\circ\tau_{T_i})$, for every $1\leqslant i\leqslant n$. Hence, we get $\mathrm{dom}(\tau_{T_1}\circ\cdots\circ\tau_{T_i})=\sigma(L_i,K_i)$, for each i such that $1\leqslant i\leqslant n+1$. Moreover, $L_{i+1}\subseteq L_i$, $K_{i+1}\subseteq K_i$, for every $1\leqslant i\leqslant n$. Specially, $L_{n+1}=L$ and $K_{n+1}=K$.

Observation 2. Let $1\leqslant i\leqslant n+1$. If the sequence $T_1,\ldots,T_i$ is of type (1), then $\tau_{T_1}\circ\cdots\circ\tau_{T_i}$ consists of all pairs of the form $(\sigma(t,s),\sigma_i(t',s'))$, where there exist trees $t_0,s_0\in T_{\Delta^{(0)}},\ldots,t_i,\ s_i\in T_{\Delta^{(i)}}$ such that

– $t_0=t,\ s_0=s,\ t_i=t',\ s_i=s'$ and

– for every $1\leqslant j\leqslant i,\ p_j(t_{j-1})\Rightarrow^*_{T_j} t_j$ and $q_j(s_{j-1})\Rightarrow^*_{T_j} s_j$

hold. (Note that σ_i and the states $p_1,q_1,\ldots,p_i,q_i$ are defined in the definition of the sequence of type (1).)

Now suppose that the sequence $T_1,\ldots,T_i$ is of type (2). Then $\tau_{T_1}\circ\cdots\circ\tau_{T_i}$ is the set of all pairs $(\sigma(t,s),\sigma_i(s'))$, where there exist trees $t_0,s_0\in T_{\Delta^{(0)}},\ldots,t_{i-1},s_{i-1}\in T_{\Delta^{(i-1)}},s_i\in T_{\Delta^{(i)}}$ such that

– $t_0=t,\ s_0=s,\ s_i=s'$ and

– for every $1\leqslant j\leqslant i-1,\ p_j(t_{j-1})\Rightarrow^*_{T_j} t_j$ and $q_j(s_{j-1})\Rightarrow^*_{T_j} s_j$, and

– $q_i(s_{i-1})\Rightarrow^*_{T_i} s_i$

hold.

Step 1: By the above observations, it is easy to see that if $T_1,\ldots,T_i$ is of type (2), for some $1\leqslant i\leqslant n+1$, then $\mathrm{dom}(\tau_{T_1}\circ\cdots\circ\tau_{T_i})=\sigma(L_{i-1},K_i)$. Moreover, for every j, such that $i\leqslant j\leqslant n+1$, $\mathrm{dom}(\tau_{T_1}\circ\cdots\circ\tau_{T_j})=\sigma(L_{i-1},K_j)$ should hold. On the other hand, $\mathrm{dom}(\tau_{T_1}\circ\cdots\circ\tau_{T_{n+1}})=\mathrm{dom}(\tau)=\sigma(L,K)$, hence we have obtained that $L_{i-1}=L$. (This note provides that T_1 cannot be of type (2). If it were, then $L=T_{\Sigma'}$ would follow, which is a contradiction.)

Now, for each $1\leqslant j\leqslant i-1$, let $M_j=(Q_j,\Delta^{(j-1)},\Delta^{(j)},p_j,R_j)$ be constructed from T_j such that we let p_j be the initial state instead of q. (Recall that, since the sequence $T_1,\ldots,T_{i-1}$ is of type (1), there must be a rule of the form $q(\sigma_{j-1}(x_1,x_2))\to\sigma_j(p_j(x_1),q_j(x_2))$ in R_j.) By the result of the previous paragraph, in this case $\mathrm{dom}(\tau_{M_1}\circ\cdots\circ\tau_{M_{i-1}})=L_{i-1}=L$ holds. Note that T_j is superlinear transducer, hence so M_j is.

Step 2: Consider the transducer T_1. By the shape of τ, it is obvious that T_1, as a sequence, cannot be of type *undefined.* We have seen that it cannot be of type (2) either, hence it is of type (1).

Let $1\leqslant i\leqslant n$. Suppose that $T_1,\ldots,T_i$ is of type (1). By Observation 2, it can be seen that the transducer T_{i+1} should have a (q,σ_i)-rule. Moreover, by the definition of τ, $\mathrm{rhs}(q,\sigma_i)$ should contain x_2. (Otherwise, T_{i+1} would loose information about the second direct subtree of the input tree.) Recall that T_{i+1} is order-preserving, nonincreasing and linear, and hence $T_1,\ldots,T_{i+1}$ must be of type (1) or (2).

Finally, we show that the whole sequence $T_1,\ldots,T_{n+1}$ cannot be of type (1). This follows from the fact that the output alphabet Δ^{n+1} of T_{n+1} consists of symbols having rank 0 only. Hence there is a sequence $T_1,\ldots,T_i$ type (2), for some $1\leqslant i\leqslant n+1$. □

Now let us apply the previous lemma. To present the inclusion diagram of $|NF|$, we shall need the following results.

Corollary 5.6. *Let $n \geqslant 0$ integer. Then*
(1) $sl\text{-}DT^{n+1} \circ l\text{-}HOM \nsubseteq sl\text{-}DT^n \circ DT$ *and*
(2) $sl\text{-}DT^{n+2} \nsubseteq sl\text{-}DT^n \circ DT$.

Proof. (1) Let $L \in \mathrm{dom}(sl\text{-}DT^{n+1}) - \mathrm{dom}(sl\text{-}DT^n)$ (such an L exists by (1) of Proposition 5.3). Define τ as in Lemma 5.5. It is an easy exercise to show that $\tau \in sl\text{-}DT^{n+1} \circ l\text{-}HOM$. Suppose $\tau \in sl\text{-}DT^n \circ DT$. Then, by Lemma 5.5, $L \in \mathrm{dom}(sl\text{-}DT^m)$ holds for some $1 \leqslant m \leqslant n$, which is a contradiction. We have that $\tau \notin sl\text{-}DT^n \circ DT$.

Since $l\text{-}HOM \subseteq sl\text{-}DT$ (see (5) of Proposition 5.1), statement (2) follows from (1) immediately. □

The next technical lemma shows that there exist an l-dt transformation which cannot be induced by a sequence of sl-dt transducers followed by an nd-dt transducer.

Lemma 5.7. $l\text{-}DT \nsubseteq sl\text{-}DT^* \circ nd\text{-}DT$.

Proof. Let $\Sigma = \{\sigma^{(2)}, \#^{(0)}\}$. Define the l-dt transducer

$$T = (\{q_0, q_1, q_2\}, \Sigma, \Sigma - \{\sigma\}, q_0, R),$$

where

$$R = \{q_0(\sigma(x_1, x_2)) \to q_1(x_1),\ q_1(\sigma(x_1, x_2)) \to q_2(x_1),$$

$$q_2(\sigma(x_1, x_2)) \to q_0(x_2),\ q_0(\#) \to \#\}.$$

Let us investigate the set $\mathrm{dom}(\tau_T)$. (Since the output ranked alphabet of T is $\{\#^{(0)}\}$, one can guess that the proof is actually concerned with domains.) Define the set $H \subseteq \hat{T}_{\Sigma,1}$ of environments as $H = \{\sigma(\sigma(\sigma(t_1, x_1), t_2), t_3) \mid t_1, t_2, t_3 \in T_\Sigma\}$. It is easy to check that $\mathrm{dom}(\tau_T) = \{h_1[\ldots h_n[\#]\ldots] \mid n \geqslant 0, h_1, \ldots, h_n \in H\}$. Informally speaking, starting from q_0, T steps to the left twice and to the right once on σs, and reaches q_0 again. Moreover, T accepts # also starting in state q_0. The transducer rejects every other tree not in H.

We show that $\tau_T \notin sl\text{-}DT^* \circ nd\text{-}DT$, which implies the lemma immediately. To prove this, suppose the contrary, i.e., that there exist sl-dt transducers $T_1, \ldots, T_n$ and an nd-dt transducer T_{n+1} such that $\tau_T = \tau_{T_1} \circ \cdots \circ \tau_{T_n} \circ \tau_{T_{n+1}}$. We abbreviate the right-hand side of the previous equation by τ. Suppose n to be minimal. By (2) of Corollary 5.2, it can be assumed that the transducers $T_1, \ldots, T_n$ are op-ni-sl-dt.

We note that T_{n+1} is nondeleting and that, obviously, the transformation τ_T cannot be induced without deleting capacity. Hence $n \geqslant 1$ holds.

Let us assume that $T_1 = (Q_1, \Sigma, \Delta, p, R_1)$. (The input alphabet of T_1 can be supposed to be Σ without loss of generality.)

Consider the trees given in Fig. 3.

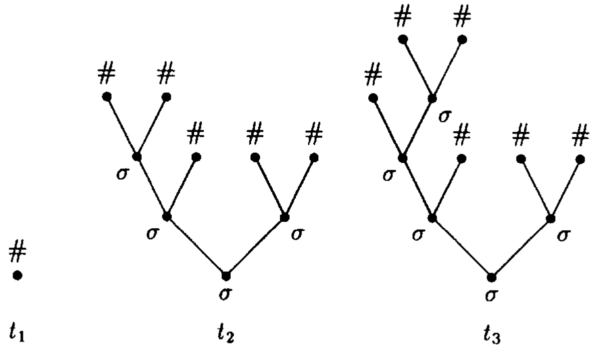

Fig. 3.

Since $\tau = \tau_T$ is supposed, it is easy to check that the following statement holds.

Statement. $t_1, t_2 \in \mathrm{dom}(\tau)$, $t_3 \notin \mathrm{dom}(\tau)$.

We investigate the rules of T_1. Considering the Statement and that T_1 is nonincreasing, we have that there should be a rule of the form

$$p(\#) \to \#_1 \tag{1}$$

in R_1, where $\#_1 \in \Delta_0$.

By the Statement, there should be a (p,σ)-rule in R_1. The transducer T_1 is op-ni-sl-dt, hence this rule is of one of the following forms:

(a) $p(\sigma(x_1,x_2)) \to \sigma_1(p'(x_1))$, where $\sigma_1 \in \Delta_1 \cup \{x_1\}$ and $p' \in Q_1$. In this case, by the Statement, it is easy to see that there should be (p',σ) rule in R_1. Moreover, $\mathrm{rhs}(p',\sigma)$ must contain x_1, otherwise $t_2 \in \mathrm{dom}(\tau)$ would imply $t_3 \in \mathrm{dom}(\tau)$, which contradicts the Statement. By the sl property of T_1, it is possible if and only if $p' = p$. However, in this case $t_2 \in \mathrm{dom}(\tau)$ also implies $t_3 \in \mathrm{dom}(\tau)$. We have that this form is not acceptable for the (p,σ)-rule.

(b) $p(\sigma(x_1,x_2)) \to \sigma_1(p'(x_2))$, where $\sigma_1 \in \Delta_1 \cup \{x_1\}$ and $p' \in Q_1$. In this case $t_2 \in \mathrm{dom}(\tau)$ implies $t_3 \in \mathrm{dom}(\tau)$, hence this form contradicts the Statement as well.

(c) $p(\sigma(x_1,x_2)) \to \sigma_1(p'(x_1), p''(x_2))$, where $\sigma_1 \in \Delta_2$ and $p', p'' \in Q_1$. We have that this form is the only possible form of (p,σ).

Suppose that $p' \neq p$ in (c). Then, by the Statement, there should be a (p',σ)-rule in R_1. By the sl property of T_1, $\mathrm{rhs}(p',\sigma)$ must be a ground tree. However, in this case $t_2 \in \mathrm{dom}(\tau)$ implies $t_3 \in \mathrm{dom}(\tau)$, which contradicts the Statement. We have obtained $p' = p$.

Now suppose $p' = p$ and $p'' \neq p$. Similar to the previous observations, one can easily conclude that a (p'',σ)-rule should be in R_1 and $\mathrm{rhs}(p'',\sigma)$ must be a ground tree. But in this case $t_3 \in \mathrm{dom}(\tau)$ follows contradicting the Statement.

Summarizing up, we have obtained that

$$p(\sigma(x_1,x_2)) \to \sigma_1(p(x_1), p(x_2)) \in R_1. \tag{2}$$

By the rules (1) and (2), it can be supposed that there are no other rules in R_1. Moreover, $Q_1 = \{p\}$ and $\Delta = \{\sigma_1^{(2)}, \#_1^{(0)}\}$ should hold. We have that T_1 is total and nonreducing, that is t-nr-op-ni-sl-dt transducer. Hence, $\tau_T = \tau_{T_1} \circ \cdots \circ \tau_{T_n} \circ \tau_{T_{n+1}}$ implies $\tau_T \in$ *t-nr-op-ni-nd-sl-DT* $\circ$ *op-ni-sl-DT*$^{n-1}$ $\circ$ *nd-DT*, for some $n \geqslant 1$.

Assume $n = 1$, then $\tau_T \in$ *t-op-ni-nr-nd-sl-DT* $\circ$ *nd-DT* $=$ *nd-DT* holds, which is obviously not true.

Assume $n > 1$, then, by (7) of Proposition 5.1, $\tau_T \in$ *op-ni-sl-DT*$^{n-1}$ $\circ$ *nd-DT* follows, which contradicts the minimality of n.

We have that suitable transducers $T_1, \ldots, T_{n+1}$ cannot exist. $\square$

Corollary 5.8. *l-DT* $\circ$ *nd-DT* $\not\subseteq$ *sl-DT** $\circ$ *nd-DT*.

We now begin to prove Lemma 4.3, which states that the diagram depicted in Fig. 1 is the inclusion diagram of $|NF|$. First we show that all the six hierarchies appearing in $|NF|$ are proper.

Let H be a set of tree transformation classes defined as

$$H = \{I, l\text{-}DT, nd\text{-}DT, HOM, l\text{-}DT \circ nd\text{-}DT, DT\}.$$

Observe that the hierarchies in $|NF|$ are of the form $\{sl\text{-}DT^n \circ X \mid n \geqslant 0\}$, where $X \in H$. We prove the following.

Lemma 5.9. *Let $X \in H$ be arbitrary. Then $\{sl\text{-}DT^n \circ X \mid n \geqslant 0\}$ is a proper hierarchy.*

Proof. Let $n \geqslant 0$ and $X \in H$. Recall $sl\text{-}DT^{n+2} \not\subseteq sl\text{-}DT^n \circ DT$ from (2) of Corollary 5.6. Since $X \subseteq DT$, we get $sl\text{-}DT^{n+2} \circ X \not\subseteq sl\text{-}DT^n \circ X$. On the other hand, $sl\text{-}DT^n \circ X \subseteq sl\text{-}DT^{n+2} \circ X$ should be clear. Hence $sl\text{-}DT^n \circ X \subset sl\text{-}DT^{n+2} \circ X$ holds.

Now suppose that $sl\text{-}DT^n \circ X = sl\text{-}DT^{n+1} \circ X$. Then $sl\text{-}DT^{n+1} \circ X = sl\text{-}DT^{n+2} \circ X$ also holds, which implies $sl\text{-}DT^n \circ X = sl\text{-}DT^{n+2} \circ X$. However, this contradicts the result of the previous paragraph.

We have $sl\text{-}DT^n \circ X \subset sl\text{-}DT^{n+1} \circ X$, for every $n \geqslant 0$ and $X \in H$. $\square$

Let $X \in H$ and consider the classes $sl\text{-}DT^* \circ X = \bigcup_{n \geqslant 0}(sl\text{-}DT^n \circ X)$, which are the suprema of the corresponding hierarchies. Note that, for every $n \geqslant 0$, $sl\text{-}DT^n \circ X \subset sl\text{-}DT^* \circ X$ holds by Lemma 5.9. Although the suprema are not elements of $|NF|$, we found them very useful to prove certain inclusions in $|NF|$. Moreover, they make the inclusion diagram of $|NF|$ more complete and clear. Therefore, we represented them in the diagram.

In the following lemma we prove the inclusions relations between the suprema of the hierarchies.

Lemma 5.10. *The diagram in Fig. 4 is the inclusion diagram of the set $\{sl\text{-}DT^* \circ X \mid X \in H\}$, i.e., of the set of suprema of the hierarchies in $|NF|$.*

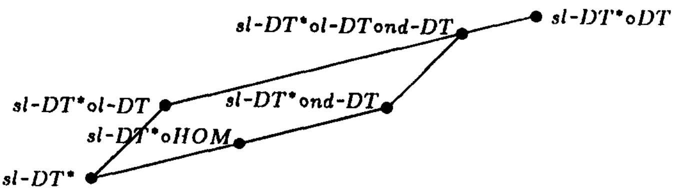

Fig. 4.

Proof. Observe that all inclusions depicted in Fig. 4 are obvious, except $sl\text{-}DT^* \circ HOM \subseteq sl\text{-}DT^* \circ nd\text{-}DT$ and $sl\text{-}DT^* \circ l\text{-}DT \circ nd\text{-}DT \subseteq sl\text{-}DT^* \circ DT$. Hence, to prove the lemma, it is enough to show that the following statements hold:

(1) $sl\text{-}DT^* \circ l\text{-}DT \not\subseteq sl\text{-}DT^* \circ nd\text{-}DT$
(2) $sl\text{-}DT^* \circ HOM \not\subseteq sl\text{-}DT^* \circ l\text{-}DT$
(3) $sl\text{-}DT^* \circ HOM \subset sl\text{-}DT^* \circ nd\text{-}DT$
(4) $sl\text{-}DT^* \circ l\text{-}DT \circ nd\text{-}DT \subset sl\text{-}DT^* \circ DT$.

(1) This follows from Lemma 5.7 immediately.

(2) Recall $HOM \not\subseteq l\text{-}DT^2$ from Fig. 2 of [2], hence $sl\text{-}DT^* \circ HOM \not\subseteq l\text{-}DT^2$. Since $sl\text{-}DT^* \circ l\text{-}DT \subseteq l\text{-}DT^2 \circ l\text{-}DT = l\text{-}DT^2$ (see (4) of Proposition 5.3 and Table 2 of [9]), the statement holds.

(3) Recall the decomposition $HOM = l\text{-}HOM \circ nd\text{-}HOM$ (see (29) of [9]). Since $l\text{-}HOM \subseteq sl\text{-}DT$ holds by (5) of Proposition 5.1 and $nd\text{-}HOM \subset nd\text{-}DT$ is obvious, we have $sl\text{-}DT^* \circ HOM \subseteq sl\text{-}DT^* \circ sl\text{-}DT \circ nd\text{-}DT = sl\text{-}DT^* \circ nd\text{-}DT$. The hom transducers are total, which implies $\mathrm{dom}(sl\text{-}DT^* \circ HOM) = \mathrm{dom}(sl\text{-}DT^*) \subset DREC$ (see Theorem 5.4). On the other hand, $\mathrm{dom}(nd\text{-}DT) = DREC$ implies $\mathrm{dom}(sl\text{-}DT^* \circ nd\text{-}DT) = DREC$, hence the proper inclusion holds.

(4) Since $l\text{-}DT \circ nd\text{-}DT \subseteq DT$ (see Lemma 3 in [9]), $sl\text{-}DT^* \circ l\text{-}DT \circ nd\text{-}DT \subseteq sl\text{-}DT^* \circ DT$ holds. Moreover, $sl\text{-}DT^* \circ l\text{-}DT^* \circ nd\text{-}DT \subseteq l\text{-}DT^2 \circ l\text{-}DT \circ nd\text{-}DT \subseteq l\text{-}DT^2 \circ nd\text{-}DT$ (see (4) of Proposition 5.3 and Table 2 of [9]), and $DT \not\subseteq l\text{-}DT^2 \circ nd\text{-}DT$ (see Fig. 2 of [14]), hence the inclusion is proper.

Observe that the inclusion relation between any two elements depicted in Fig. 4 can be determined using the statements (1)–(4).

For example, we show $sl\text{-}DT^* \subset sl\text{-}DT^* \circ l\text{-}DT$. The inclusions $sl\text{-}DT^* \subseteq sl\text{-}DT^* \circ l\text{-}DT$ and $sl\text{-}DT^* \subseteq sl\text{-}DT^* \circ nd\text{-}DT$ should be obvious. Then considering (1) we have the desired result immediately. □

Besides the hierarchies, there are the classes $l\text{-}DT^2, l\text{-}DT \circ HOM, l\text{-}DT^2 \circ nd\text{-}DT$ and DT^2 in $|NF|$. In the following lemma we attach them to the inclusion diagram of the suprema of the hierarchies.

Lemma 5.11. *The diagram in Fig. 5 is the inclusion diagram of the set consisting of* $l\text{-}DT^2, l\text{-}DT \circ HOM$, $l\text{-}DT^2 \circ nd\text{-}DT$, DT^2 *and the suprema of the six hierarchies.*

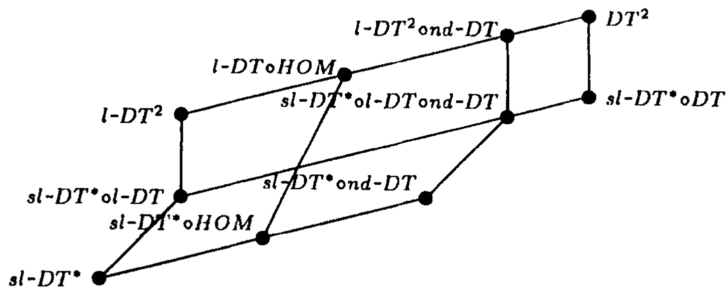

Fig. 5.

Proof. The inclusion relations between the suprema of the hierarchies are clear by Lemma 5.10.

In [14] it has been proved that the new four classes obey the following inclusion relations (see Fig. 2 in that paper):

$$l\text{-}DT^2 \subset l\text{-}DT \circ HOM \subset l\text{-}DT^2 \circ nd\text{-}DT \subset DT^2.$$

First we show that none of the suprema (hence none of the elements of the hierarchies) includes any of the new classes. To prove this, it is enough to show that the least new element $l\text{-}DT^2$ is not included in the largest supremum, i.e., that

$$l\text{-}DT^2 \nsubseteq sl\text{-}DT^* \circ DT$$

holds. By Theorem 5.4, there exists a tree language L such that $L \in DREC\text{–}\mathrm{dom}(sl\text{-}DT^*)$. Construct τ as defined in Lemma 5.5. Then it should be clear that $\tau \notin sl\text{-}DT^n \circ DT$ for every $n \geqslant 0$, hence $\tau \notin sl\text{-}DT^* \circ DT$. On the other hand, $L \in DREC = \mathrm{dom}(l\text{-}DT)$, therefore $\tau \in l\text{-}DT^2$ should be obvious.

Now we prove that the new elements are the topmost elements in the inclusion diagram of $|NF|$. We define the set G of transformation classes as

$$G = \{l\text{-}DT, HOM, l\text{-}DT \circ nd\text{-}DT, DT\}.$$

Note that the set of new four elements of $|NF|$ is exactly $\{l\text{-}DT \circ X \mid X \in G\}$ (for $l\text{-}DT \circ DT = DT^2$ see Table 2 of [14]). Let $X \in G$ be arbitrary. Observe that $l\text{-}DT^2 \circ X = l\text{-}DT \circ X$ holds (see Table 2 of [14]). This implies the inclusion $sl\text{-}DT^* \circ X \subseteq l\text{-}DT \circ X$, because $sl\text{-}DT^* \circ X \subseteq l\text{-}DT^2 \circ X = l\text{-}DT \circ X$ (see (4) of Proposition 5.3). Moreover, the inclusion should be proper by the result of the previous paragraph, that is

$$sl\text{-}DT^* \circ X \subset l\text{-}DT \circ X$$

holds, for each $X \in G$.

Finally, we state that there are no other edges corresponding to the topmost elements in the inclusion diagram of $|NF|$, besides the ones depicted in Fig. 5. To show this, it is enough to prove the following statements:

$$HOM \not\subseteq l\text{-}DT^2, \quad (1)$$

$$nd\text{-}DT \not\subseteq l\text{-}DT \circ HOM \quad \text{and} \quad (2)$$

$$DT \not\subseteq l\text{-}DT^2 \circ nd\text{-}DT. \quad (3)$$

For example, we can show that $sl\text{-}DT^* \circ nd\text{-}DT \not\subseteq l\text{-}DT^2$. For, if $sl\text{-}DT^* \circ nd\text{-}DT \subseteq l\text{-}DT^2$, then by $HOM \subseteq sl\text{-}DT^* \circ HOM$ and $sl\text{-}DT^* \circ HOM \subseteq sl\text{-}DT^* \circ nd\text{-}DT$, we get $HOM \subseteq sl\text{-}DT^2$, which contradicts (1).

However, (1)–(2) have already been proved in [14] (see Fig. 2 in that paper). With this, we have proved Lemma 5.11. □

We should still prove the inclusions between the elements of the hierarchies. The following corollary shows that, roughly speaking, there can be edges descending only from right to left in the inclusion diagram of $|NF|$.

Corollary 5.12. *Denote the bottom elements of the hierarchies as $X_1 = 1$, $X_2 = l\text{-}DT$, $X_3 = HOM, X_4 = nd\text{-}DT$, $X_5 = l\text{-}DT \circ nd\text{-}DT$ and $X_6 = DT$. Let i, j be arbitrary integers such that $1 \leqslant i < j \leqslant 6$. Then $X_j \not\subseteq sl\text{-}DT^* \circ X_i$ holds.*

Proof. By statements (1)–(3) in the proof of Lemma 5.11, we have most of these results immediately. Only the following two cases should be checked:

(1) $l\text{-}DT \not\subseteq sl\text{-}DT^*$ follows from (3) of Proposition 5.3 and

(2) $l\text{-}DT \circ nd\text{-}DT \not\subseteq sl\text{-}DT^* \circ nd\text{-}DT$ holds by Corollary 5.8. □

We now can finish the proof of Lemma 4.3.

Proof of Lemma 4.3. Recall that the inclusion relations between the topmost elements and the suprema of the six hierarchies have been clarified in Lemma 5.10.

First we prove that the inclusions depicted in Fig. 1 hold. Let $n \geqslant 1$. All inclusions should be clear, except the following ones:

(1) $sl\text{-}DT^{n-1} \circ HOM \subseteq sl\text{-}DT^n \circ nd\text{-}DT$,

(2) $sl\text{-}DT^{n-1} \circ l\text{-}DT \circ nd\text{-}DT \subseteq sl\text{-}DT^{n-1} \circ DT$.

(1) Recall the decomposition $HOM = l\text{-}HOM \circ nd\text{-}HOM$ (see (29) of [9]). Since $l\text{-}HOM \subseteq sl\text{-}DT$ holds by (5) of Proposition 5.1 and $nd\text{-}HOM \subseteq nd\text{-}DT$ is obvious, we have $sl\text{-}DT^{n-1} \circ HOM \subseteq sl\text{-}DT^{n-1} \circ sl\text{-}DT \circ nd\text{-}DT = sl\text{-}DT^n \circ nd\text{-}DT$.

(2) The inclusion $l\text{-}DT \circ nd\text{-}DT \subseteq DT$ follows from (a) of Lemma 3 in [9].

Observe that, by Lemma 5.9 and Corollary 5.12, the inclusions depicted in Fig. 1 are necessarily proper.

Finally, we show that there cannot be other inclusions. To prove this, it is enough to consider Corollary 5.12 and the following statements:

(3) $l\text{-}DT \not\subseteq sl\text{-}DT^* \circ nd\text{-}DT$ (by Lemma 5.7),

(4) $sl\text{-}DT^n \circ HOM \not\subseteq sl\text{-}DT^{n-1} \circ DT$ (by (1) of Corollary 5.6),

(5) $sl\text{-}DT^{n+1} \not\subseteq sl\text{-}DT^{n-1} \circ DT$ (by (2) of Corollary 5.6).

We have now obtained that the relations between any two elements depicted in Fig. 1 can be determined using Corollary 5.12 and the statements (1)–(5).

For example, we show $sl\text{-}DT^3 \circ HOM \subset sl\text{-}DT^7 \circ nd\text{-}DT$. It should be clear that $sl\text{-}DT^3 \circ HOM \subseteq sl\text{-}DT^6 \circ HOM$. By statement (1), we have $sl\text{-}DT^6 \circ HOM \subset sl\text{-}DT^7 \circ nd\text{-}DT$. Hence $sl\text{-}DT^3 \circ HOM \subset sl\text{-}DT^7 \circ nd\text{-}DT$ should hold. □

With this, we finished the proof of the Lemma 4.3.

6. Conclusions

In this paper we have considered a monoid [M] generated by the tree transformation classes *HOM*, *sl-DT*, *l-DT*, *nd-DT* and *DT* with composition. This is the first work, where *sl-DT* is taken as a generator element of a tree transformation monoid. Using term rewriting techniques, we have developed an algorithm which, given any two elements $X_1 \circ X_2 \circ \cdots \circ X_m$ and $Y_1 \circ Y_2 \circ \cdots \circ Y_n$ of [M], can decide whether the inclusion $X_1 \circ X_2 \circ \cdots \circ X_m \subseteq Y_1 \circ Y_2 \circ \cdots \circ Y_n$ holds. Of course, it is also decidable whether $\supseteq$, $=$ or incomparability holds. We have represented elements of [M] by strings, and have given a terminating and confluent string rewriting system R as well as the inclusion diagram of the normal forms with respect to R. The inclusion between two elements of [M] can be decided in the following way. We reduce the strings representing the tree transformation classes $X_1 \circ X_2 \circ \cdots \circ X_m$ and $Y_1 \circ Y_2 \circ \cdots \circ Y_n$ to normal forms with respect to R. The string rewriting system R is constructed in such a way that $\subseteq$ ($\supseteq$, $=$, incomparability) holds between the two tree transformation classes if and only if the same relation holds between the tree transformation classes represented by the corresponding normal forms. This latter, however, can be read from the inclusion diagram depicted in Fig. 1.

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