International Journal of Approximate Reasoning 50 (2009) 1279–1286

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

On generalized Bonferroni mean operators for multi-criteria aggregation

Ronald R. Yager

Machine Intelligence Institute, Iona College, New Rochelle, NY 10801, United States

ARTICLE INFO

Article history:
Received 9 February 2009
Received in revised form 20 May 2009
Accepted 8 June 2009
Available online 14 June 2009

Keywords:
Aggregation operators
OWA operators
Multi-criteria modeling
Decision-making

ABSTRACT

We introduce the idea of multi-criteria aggregation functions and describe a number of properties desired in such functions. We emphasize the importance of having an aggregation function capture the expressed interrelationship between the criteria. A number of standard aggregation functions are introduced. We next introduce the Bonferroni mean operator. We provide an interpretation of this operator as involving a product of each argument with the average of the other arguments, a combined averaging and "anding" operator. This allows us to suggest generalizations of this operator by replacing the simple averaging by other mean type operators as well as associating differing importances with the arguments.

1. Introduction

Problems involving multi-criteria are pervasive in many areas of modern technology. Not only do they appear in decision-making but they also arise in such diverse areas as pattern recognition, information retrieval, case based reasoning and database querying among others. A central problem in multi-criteria problems is the aggregation of the satisfactions to the individual criteria to obtain a measure of satisfaction to the overall collection of criteria. This aggregation process must be guided by the interrelationship of the individual criteria, the criteria organization. As many different types of criteria relationships exist in the real world there is a need for many types of formal aggregation operations to enable the modeling of these numerous types of relationships. In response to this need a formal mathematical discipline called aggregation theory is emerging [1–4]. Here we contribute to this theory by looking at the Bonferroni mean operator [5,6] and suggesting some generalizations that enhance its modeling capability. We provide an interpretation of this operator as involving a product of each argument with the average of the other arguments, a combined averaging and "anding" operator. This allows us to suggest generalizations of this operator by replacing the simple averaging by other mean type operators such as the OWA operator and Choquet integral as well as associating differing importances with the arguments. We that various extensions of the Bonferroni mean can model different degrees of hard and soft partial conjunctions [7].

2. Multi-criteria aggregation functions

In multi-criteria decision-making have a collection $A_1,\ldots,A_n$ of criteria and a set $X = \{x_1,\ldots,x_m\}$ of alternatives. For each alternative x_i we have a value $A_j(x_i) \in [0, 1]$ indicating the degree to which alternative x_i satisfies criteria A_j. Our objective is to develop some procedure to select from these alternatives the one that best satisfies the collection of criteria. One property often required of such a procedure is what Arrow [8] called indifference to irrelevant alternatives. Essentially this

E-mail address: yager@panix.com

0888-613X/$ - see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.ijar.2009.06.004

property assures that the decision procedure is such that we can't affect the outcome by introducing alternatives whose sole purpose is to disturb the process. Formally this property requires that our procedure is such that if the application of procedure to $X = \{x_1, \ldots, x_m\}$ selects x^*, Procedure$(x_1, \ldots, x_m) \to x^*$, then application of Procedure to $\{x_1, \ldots, x_m, x_{m+1}\}$ must result in either x^* or x_{m+1}.

One way to guarantee this property of indifference to irrelevant alternatives is to obtain for each alternative x_j a valuation of $D(x_j)$ using a function $D(x_j) = F(A_1(x_j), \ldots, A_n(x_j))$ and then select the alternative with largest value of D. A function such as F is called a pointwise valuation function. The important feature here is that $D(x_j)$ just depends on the satisfaction of the criteria by x_j, it does not depend on the satisfactions by any of the other alternatives.

In addition to the above other properties are desired in the valuation procedure. One of these is monotonicity, if x_j and x_k are two alternatives such that $A_i(x_j) \geqslant A_i(x_k)$ for all A_i then we require $D(x_j) \geqslant D(x_k)$. Another property is what we shall call grounding, if $A_i(x_j) = 0$ for all i then $D(x_j) = 0$. If in addition $D(x_j) = 1$ if all $A_i\,(x_j) = 1$, a condition we shall refer to as being standard, then F is what is called an aggregation function [1,9]. Letting $I = [0, 1]$ then formally an aggregation function is a mapping Agg: $I^n \to I$ having the properties: $\mathrm{Agg}(0, \ldots, 0) = 0$, $\mathrm{Agg}(1, \ldots, 1) = 1$ and $\mathrm{Agg}(a_1, \ldots, a_n) \geqslant \mathrm{Agg}(b_1, \ldots, b_n)$ if $a_i \geqslant b_i$ for all i. We shall use the terms aggregation functions and aggregation operators synonymously.

Since the function F should be consistently chosen for all alternatives the pointwise nature of F allows us to simply focus on just one typical alternative, x, in discussing F. In the following we shall generally use a_j to indicate $A_j(x)$.

The actual choice of the aggregation function should be a reflection of our knowledge of the relational organization of the criteria. In the following we shall discuss some notable aggregation functions and indicate the type of criteria relationships they can model.

One formulation is $D(x) = T(a_1, \ldots, a_n)$ where T is a t-norm operator [10]. These aggregation functions are used to model situations when *all* the criteria are required to be satisfied by a solution. Notable among this class of functions are the following: $D(x) = \mathrm{Min}(a_1, \ldots, a_n)$, $D(x) = \prod_{j=1}^n a_j$ and $D(x) = \mathrm{Max}\left(0, \sum_{j=1}^n a_j - (n-1)\right)$.

Another class of functions is $D(x) = S(a_1, \ldots, a_n)$ where S is a t-conorm operator [10]. These are used to model situations where the satisfaction to *any* of the criteria is sufficient. Notable among this class of functions are the following: $D(x) = \mathrm{Max}(a_j), D(x) = 1 - \prod_{j=1}^n (1 - a_j)$ and $D(x) = \mathrm{Min}\left(1, \sum_{j=1}^n a_j\right)$.

A general class of functions that can be used to formulate the aggregation function F is the OWA operator [11]. Assume $w_j \in [0, 1]$ are a collection of parameters that sum to one. Letting $\pi(j)$ be the index of the jth largest of the a_i the OWA aggregation is calculated as

$$D(x) = F(a_1, \ldots, a_n) = \sum_{j=1}^n w_j a_{\pi(j)}$$

The w_j are referred to as the OWA weights and collectively they can viewed as a vector W whose jth component is w_j. The OWA operators are mean type aggregation functions [11].

By assigning different values to the OWA weights we can obtain a wide class of formulations for the aggregation function F. If $w_1 = 1$ and $w_j = 0$ for $j \neq 1$ then $D(x) = \mathrm{Max}_i(a_i)$ and if $w_n = 1$ and $w_j = 0$ for $j \neq n$ then $D(x) = \mathrm{Min}_i(a_i)$. If $w_j = 1/n$ for all j we get the usual average, $D(x) = \frac{1}{n}\sum_{i=1}^n a_i$. We can associate with an OWA operator a measure called its attitudinal character [11] defined as $A\text{–}C(W) = \sum_{j=1}^n w_j \frac{n-j}{n-1}$. It can be shown that $A\text{–}C(W) \in [0, 1]$. We note for the case of Max, $A\text{–}C(W) = 1$, for the case of Min, $A\text{–}C(W) = 0$ and for the average, $A\text{–}C(W) = 0.5$. Another measure associated with the OWA operator is the measure of dispersion [11,12], which is defined $\mathrm{Disp}(W) = -\sum_{j=1}^n w_j \ln(w_j)$.

In [13] Yager suggested a useful approach to obtain the OWA operator. Consider the class of functions f:$[0, 1] \to [0, 1]$ such that $f(0) = 0$, $f(1) = 1$ and $f(x) \geqslant f(y)$ if $x \geqslant y$. We refer to these as BUM functions. Using these functions we can generate valid weights for an OWA operator, $w_j = f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right)$. An important example is the case where $f(x) = x$, here we get $w_j = 1/n$. In [13] Yager related these BUM functions to Zadeh's concept [14] of linguistic quantifiers. This enabled the formulation of OWA operators based on linguistically expressed specifications.

The aggregation of criteria using the BUM function can be easily extended to the case where each of the criteria has an importance weight, $u_i \in [0, 1]$. If we let $u_{\pi(j)}$ indicate the importance weight of the criteria with the jth largest value for a_i then we generate the OWA weights as $w_j = f\left(\frac{T_j}{T}\right) - f\left(\frac{T_{j-1}}{T}\right)$ where $T_j = \sum_{k=1}^j u_{\pi(k)}$ and $T = \sum_{i=1}^n u_i$. In this special case where $f(x) = x$ we obtain that $w_j = u_j$ and hence we get the usual weighted average.

3. Bonferroni mean operators

The wide variety of possible relationships between the criteria in multi-criteria problems motivates great interest in seeking aggregation functions that can be used to model these various possibilities. Here we investigate the capabilities of a class of aggregation operators called Bonferroni means. The Bonferroni mean was originally introduced in [5] and discussed more recently in [1,6].

Let $(a_1, \ldots, a_n)$ be a collection of values so that $a_i \in [0, 1]$. Assume p and $q \geqslant 0$, then the general Bonferroni mean of these values is defined as

$$B^{p,q}(a_1,\ldots,a_n)=\left(\frac{1}{n}\frac{1}{n-1}\sum_{\substack{i,j=1\\ i\neq i}}^{n}a_i^p a_j^q\right)^{\frac{1}{p+q}}$$

It can easily be seen that

$$B^{p,q}(0,0,\ldots,0)=0$$
$$B^{p,q}(1,1,\ldots,1)=1$$

and $B^{p,q}$ is monotonic

$$B^{p,q}(a_1,\ldots,a_n)\geqslant B^{p,q}(d_1,\ldots,d_n)$$

if $a_i \geqslant d_i$ for all i. Thus $B^{p,q}$ is an aggregation operator.

Furthermore, if a^* = Max$_i$[b_i] then from the monotonicity

$$B^{p,q}(a_1,\ldots,a_n)\leqslant B^{p,q}(a^*,\ldots,a^*)=a^*$$

and if a^* = Min$_i$[a_i} then

$$B^{p,q}(a_1,\ldots,a_n)\geqslant B^{p,q}(a_*,\ldots,a_*)=a_*$$

thus

$$\mathrm{Min}_i[a]\leqslant B^{p,q}(a_1,\ldots,a_n)\leqslant \mathrm{Max}_i[a_i].$$

This boundedness implies that $B^{p,q}$ is a mean type aggregation operator [15].

We shall here consider for our purposes of aggregating multiple criteria the special case when $p = q = 1$. Here then we let a_j denote the satisfaction of an alternative x to the jth criteria A_j. Using this aggregation function and denoting $B^{1,1}$ simply as B we get as our aggregated value

$$D(x)=B(a_1,\ldots,a_n)=\left(\frac{1}{n}\frac{1}{n-1}\sum_{\substack{i,j=1\\ i\neq i}}^{n}a_i a_j\right)^{\frac{1}{2}}$$

One interpretation of this aggregation operator is as a kind of combined "anding" and "averaging" operator. In particular we earlier noted that the product operation could be used to implement an "anding" of criteria satisfaction. So then here we see that $a_i\,a_j$ indicates the degree to which both criteria A_i **and** A_j are satisfied. Here then we see that $B(a_1,\ldots,a_n)$ is calculating an average of the satisfaction of pairs of criteria.

There exists another interesting way to view this aggregation operator

$$D(x)=B(a_1,\ldots,a_n)=\left(\frac{1}{n}\sum_{i=1}^{n}a_i\left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n}a_j\right)\right)^{\frac{1}{2}}$$

We see that the term $\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n}a_j$ is the average satisfaction of all criteria except A_i. We shall denote this as u_i. Thus

$$D(x)=B(a_1,\ldots,a_n)=\left(\frac{1}{n}\sum_{i=1}^{n}u_i a_i\right)^{\frac{1}{2}}$$

Here then u_i is the average satisfaction to all criteria except A_i.

We can make the following observation. Assume a_i and a_k are two values then

$$u_i=\frac{1}{n-1}\left(a_k+\sum_{\substack{j=1\\ j\neq k,i}}^{n}a_j\right)$$
$$u_k=\frac{1}{n-1}\left(a_i+\sum_{\substack{j=1\\ j\neq k,i}}^{n}a_j\right)$$

We see that if $a_k > a_i$ then $u_i > u_k$. The u_i are inversely ordered with respect to the ordering of the a_i.

Thus we have that

$$D(x) = \left(\frac{1}{n}\sum_{i=1}^{n} u_i a_i\right)^{\frac{1}{2}}$$

where $u_i > u_k$ if $a_k > a_i$.

4. An OWA variation of Bonferroni means

As we indicated we can express the basic Bonferroni aggregation operator as

$$B(a_1, \ldots, a_n) = \left(\frac{1}{n}\sum_{i=1}^{n} u_i a_i\right)^{\frac{1}{2}}$$

where $u_i = \frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} a_j$

One can consider replacing the simple average used to obtain u_i by an OWA aggregation of the a_j, $j \neq i$. Let us denote V^i as the $n-1$ tuple $(a_1,\ldots,a_{i-1}\ a_{i+1}\ldots\ a_n)$. Let W be an OWA weighting vector of dimension $n-1$ with components $w_k \in [0,1]$ when $\sum_k w_k = 1$. We can use this to define the aggregation

$$\mathrm{OWA}_W(V^i) = \sum_{k=1}^{n-1} w_k a_{\pi_i(k)}$$

Here $a_{\pi i(k)}$ is the kth largest element in the tuple V^i. We note that for $a_i \in [0,1]$ we have $\mathrm{OWA}_W\,(V^i) \in [0,1]$.

Using this we can obtain

$$\mathrm{BON{-}OWA}(a_1, \ldots, a_n) = \left(\frac{1}{n}\sum_{i=1}^{n} a_i \mathrm{OWA}_W(V^i)\right)^{0.5}$$

We now observe that for the case where $w_k = \frac{1}{n-1}$ for all k then we get

$$\mathrm{OWA}_W(V^i) = u_i = \frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} a_j$$

which is the original case.

Let us now show that $\mathrm{BON{-}OWA}(a_1, \ldots, a_n) = \left(\frac{1}{n}\sum_i^n a_i \mathrm{OWA}_W(V^i)\right)^{0.5}$ is a mean type aggregation operator. First we see that if all $a_i = 0$, we get BON–OWA$(0,\ldots,0) = 0$. Consider now the case where all $a_i = 1$ here then $\mathrm{BON{-}OWA}(1, \ldots, 1) = \left(\frac{1}{n}\sum_i^n \mathrm{OWA}_W(V^i)\right)^{0.5}$. But we have that all V^i all have just ones, $V^i = (1,\ldots,1) = I$. Since the OWA operator is idempotent, the $\mathrm{OWA}_W(I) = 1$. Therefore, we get BON–OWA$(1, \ldots, 1) = 1$. The monotonicity of this aggregator is clear. Assume $b_i \geqslant a_i$ and now consider BON–OWA$(a_i,\ldots,a_n)$ and BON–OWA$(b_1,\ldots,b_n)$. Since the OWA operator is monotonic then $\mathrm{OWA}_W(a_1,\ldots,a_{i-1}, a_{i+1},\ldots,a_n) \leqslant \mathrm{OWA}_W(b_1,\ldots,b_{i-1}, b_{i+1},\ldots,b_n)$ and therefore BON–OWA$(a_1,\ldots,a_n) \leqslant$ BON–OWA$(b_1,\ldots,b_n)$. Finally we consider the boundedness. Let $a^* = \mathrm{Max}_i[a_i]$. We first observe that $\mathrm{OWA}_w(a^*,\ldots,a^*) = a^* \geqslant \mathrm{OWA}(V^i)$. Furthermore we see that

$$\mathrm{BON{-}OWA}_W(a_1, \ldots, a_n) = \left(\frac{1}{n}\sum_{i=1}^{n} a_i \mathrm{OWA}_W(V^i)\right)^{0.5} \leqslant \left(\left(\frac{1}{n}\sum_{i=1}^{n} a^* a^*\right)^{0.5} \leqslant ((a^*)^2\right)^{1/2} \leqslant a^*$$

Thus BON–OWA$(a_1,\ldots,a_n) \leqslant \mathrm{Max}_i[a_i]$. In an analogous way we can show that BON–OWA$(a_1,\ldots,a_n) \geqslant \mathrm{Min}_i[a_i]$. Thus we see that the BON–OWA operator provides a valid class of mean like aggregation operators.

Let us look at some special cases. First we consider the case where W is W^*, here $w_1 = 1$ and $w_k = 0$ for $k = 2$ to $n-1$. In this case

$$\mathrm{BON{-}OWA}_{W^*}(a_1, \ldots, a_n) = \left(\frac{1}{n}\sum_{i=1}^{n} a_i \mathrm{Max}[V^i]\right)^{0.5}$$

We further observe that $\mathrm{Max}[V^i]$ is the largest argument not equal to a_i. Let $a_{\mathrm{ind}(j)}$ be the jth largest of the a_i. We see that for $i \neq \mathrm{ind}(1)$ then $\mathrm{Max}[V^i] = a_{\mathrm{ind}(1)} = \mathrm{Max}_i[a_i]$ and for $i = \mathrm{ind}(1)$ then $\mathrm{Max}[V^i] = a_{\mathrm{ind}(2)}$. From this we see that

$$\mathrm{BON{-}OWA}_{W^*}(a_1, \ldots, a_n) = \left(\frac{1}{n}\left(\sum_{j=2}^{n} a_{\mathrm{ind}(j)} a_{\mathrm{ind}(1)} + a_{\mathrm{ind}(1)} a_{\mathrm{ind}(2)}\right)\right)^{0.5}$$

$$\mathrm{BON{-}OWA}_{W^*}(a_1, \ldots, a_n) = \left(\frac{1}{n}\left(a_{\mathrm{ind}(1)}\sum_{j=2}^{n} a_{\mathrm{ind}(j)} + a_{\mathrm{ind}(2)}\right)\right)^{0.5}$$

$$\mathrm{BON{-}OWA}_{W^*}(a_1,\dots,a_n) = \left(a_{\mathrm{ind}(1)}\frac{1}{n}\sum_{j=2}^{n}a_{\mathrm{ind}(j)} - (a_{\mathrm{ind}(1)} - a_{\mathrm{ind}(2)})\right)^{0.5}$$

$$\mathrm{BON{-}OWA}_{W^*}(a_1,\dots,a_n) \approx (\mathrm{Max}_i[a_i]\mathrm{Ave}(a_1,\dots,a_n))^{1/2}$$

Consider now the case where $W = W^*$, here $w_{n-1} = 1$ and $w_k = 0$ for $k = 1$ to $n-2$. In this case $\mathrm{BON{-}OWA}_{W^*}(a_1,\dots,a_n) = \left(\frac{1}{n}\sum_{i=1}^{n}a_i\mathrm{Min}[V^i]\right)^{0.5}$. We observe that $\mathrm{Min}[V^i]$ is the smallest argument not equal to a_i. We see that for $i \neq \mathrm{ind(n)}$ then $\mathrm{Min}[V^i] = a_{\mathrm{ind}(n)} = \mathrm{Min}_i[a_i]$ and for $i = \mathrm{ind}(n)$ the $\min[V^i] = a_{\mathrm{ind}(n-1)}$. From this we get after appropriate algebraic manipulation

$$\mathrm{BON{-}OWA}_{W_*}(a_1,\dots,a_n) \approx (\mathrm{Min}_i(a_i)\mathrm{Ave}(a_1,\dots,a_n))^{1/2}$$

As we have shown a valid form of the Bonferroni mean can be obtain using

$$\mathrm{BON{-}OWA}(a_1,\dots,a_n) = \left(\frac{1}{n}\sum_{i=1}^{n}a_i\mathrm{OWA}(V^i)\right)^{0.5}$$

where V^i is the $n-1$ arguments a_j for $j = 1$ to n excluding a_i. The performance of the OWA aggregation requires an $n-1$ vector W whose components w_j lie in the unit interval and sum to one. As discussed in [16] there are a number of ways that the OWA weighting vector W can be stipulated. One approach is to directly specify the vector W.

Another approach was suggested by O'Hagan [17]. In this approach we specify a value $\alpha \in [0,1]$ for the attitudinal character and then determine the weights by solving the following mathematical programming problem.

$$\begin{aligned}
\text{Max :}\quad & -\sum_{j=1}^{n-1}w_j\ln(w_j)\\
\text{Such that :}\quad & \sum_{j=1}^{n-1}w_j\frac{n-j}{n-1} = \alpha\\
& \sum_{j=1}^{n-1}w_j = 1\\
& 0 \leqslant w_j \leqslant 1
\end{aligned}$$

One nice benefit of this approach is that we need supply only one parameter α to obtain all the weights. The downside is the need to solve the mathematical programming problem.

Another general approach is via a BUM function f, a monotonic mapping $f{:}[0,1] \to [0,1]$, where $f(0) = 0$ and $f(1) = 1$. In this approach we get $w_j = f\left(\frac{j}{n-1}\right) - f\left(\frac{j-1}{n-1}\right)$. Some useful methods can be obtained based on this BUM function approach. One is to take advantage of the ability of fuzzy subsets to represent linguistic concepts, particularly linguistic quantifiers [13]. Here we start with some linguistic expression of the desired aggregation and then represent it as a fuzzy subset on the unit interval. This fuzzy subset can then be used to provide the desired BUM function.

Another method based on the BUM function approach is to start with a parameterized family of BUM functions and then define the desired aggregation by specifying the value of the associated parameter. A useful example of this is the *a* and *b* function shown in Fig. 1. Here we must specify the values a and b. Another parameterized function is $f(x) = x^r$ for $r > 0$ here by specifying r we get a particular function. In this case it is known that the attitudinal character is such that $\alpha = \frac{1}{r+1}$. Thus here if we specify α we can obtain $r = \frac{1-\alpha}{\alpha}$.

Since we have many ways to obtain the OWA vector W in addition to its direct specification we shall use a more generic term $\mathscr{W}$ to indicate the guiding principle of the OWA aggregation hence

$$\mathrm{BON{-}OWA}(a_1,\dots,a_n) = \left(\frac{1}{n}\sum_{i=1}^{n}a_i\mathrm{OWA}_{\mathscr{W}}(V^i)\right)^{0.5}$$

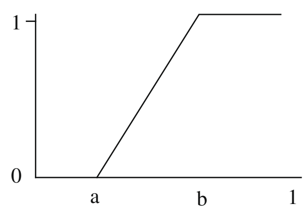

Fig. 1. (a and b) Type BUM function.

5. Weighted Bonferroni aggregation

In the preceding we showed that $\text{BON—OWA}(a_1, \ldots, a_n) = \left(\frac{1}{n}\sum_{i=1}^{n} a_i \text{OWA}_{\mathscr{W}}(V^i)\right)^{0.5}$ is an aggregation operator. It is monotonic with respect to the argument values and if $a_i = 0$ it has zero value and if $a_i = 1$ for all i then it takes the value one. In the framework of multi-criteria decision-making the arguments $a_i = A_i(x)$, the satisfaction of the alternative x to criteria A_i.

A further representational capability can be added by associating with each criterion A_i a value $p_i \in [0,1]$ called its personal importance. Using these personal importances we can generalize the BON–OWA aggregation operator to

$$\text{BON—OWA}(a_1, \ldots, a_n) = \left(\frac{1}{p}\sum_{i=1}^{n} p_i a_i \text{OWA}_{\mathscr{W}}(V^i)\right)^{0.5}$$

when $P = \sum_{i=1}^{n} p_i$. We easily can show that this is also an aggregation operator. It is monotonic with respect to the arguments and this satisfies the boundary conditions when all $a_i = 1$ or $a_i = 0$. In this case we can refer to the aggregation as BON–OWA$((a_1,p_1), (a_1,p_2), (a_3,p_3),\ldots,(a_n,p_n))$. The original case being where $p_i = 1$.

It is interesting and useful to observe that the properties making the BON–OWA an aggregation operator are retained even if a different type of OWA operator is used for each argument. Thus consider

$$\text{BON—OWA}(a_1, \ldots, a_n) = \left(\frac{1}{p}\sum_{i=1}^{n} p_i a_i \text{OWA}_{\mathscr{W}_i}(V^i)\right)^{0.5}$$

where $\mathscr{W}_i$ indicates the guiding type of aggregation associated with A_i. Let us look at this. Whatever method we use to specify $\mathscr{W}_i$ we end up with an $n-1$ vector W_i associated with A_i whose components w_{ij} are the OWA weights. In this case we easily see that when $a_i = 0$ that BON–OWA$(a_1,\ldots,a_n) = 0$. For $a_i = 1, \text{OWA}_{\mathscr{W}_i}(V^i) = 1$ and hence BON–OWA$(a_1,\ldots,a_n) = 1$. The monotonicity of BON–OWA$(a_1,\ldots,a_n)$ follows from the monotonic of each of the components. Thus we can now associate with each criterion A_i two parameters. One is p_i, its personal importance, and the other is $\mathscr{W}_i$. The term $\mathscr{W}_i$ is more complex then p_i, it is the guiding principle for the aggregation of the other criteria for their association with A_i. We shall refer to $\mathscr{W}_i$ as its social requirements. Thus here we can associate with each criterion A_i two parameters $(p_i, \mathscr{W}_i), p_i$ being its personal weight and $\mathscr{W}_i$ is the description of how the satisfaction of the other criteria are aggregated. We can express this

$$\text{BON—OWA}((a_i, p_i, \mathscr{W}_i)) = \left(\frac{1}{p}\sum_{i} p_i a_i u_i\right)^{1/2}$$

where $u_i = \text{OWA}_{\mathscr{W}_i}(V^i)$ with V^i being the vector of all a_j except a_i.

Some special simplifying cases are worth noting. If we calculate all W_i using the O'Hagan method all we need is a value α_i associated in A_i. If we use the $a - b$ function shown in Fig. 1, all we need is to specify is (a_i,b_i) for each A_i.

Example.

Assume we have three criteria A_1,A_2, and A_3 and these criteria we have $p_1 = 0.6, p_2 = 0.4$ and $p_3 = 1$. Furthermore we shall assume that W_i for each of these criteria is specified by a value α_i indicating the attitudinal character. Here we shall let $\alpha_1 = 0.5$, $\alpha_2 = 0.8$ and $\alpha_3 = 0$. Finally, let us assume that $A_1(x) = 0.7$, $A_2(x) = 0.6$ and $A_3(x) = 0.3$.

In this example $P = 2$, and hence $\frac{p_1}{P} = 0.3, \frac{p_2}{P} = 0.2, \frac{p_3}{P} = 0.5$. In addition

$$V^1 = [0.6, 0.3]$$
$$V^2 = [0.7, 0.3]$$
$$V^3 = [0.7, 0.6]$$

The calculation of the W_i is greatly simplified by the fact that $n - 1 = 2$. In particular

$$W_i = \begin{bmatrix} \alpha_i \\ 1 - \alpha_i \end{bmatrix}$$

and therefore

$$W_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Using this we get

$$u_1 = 0.5(0.6 + 0.3) = 0.45$$
$$u_2 = 0.8(0.7) + (0.2)(0.3) = 0.62$$
$$u_3 = (0)(0.7) + (1)(0.6) = 0.6$$
$$\text{BON—OWA}((a_i, p_i, \alpha_i)) = (0.3)(0.7)(0.45) + (0.2)(0.6)(0.62) + (0.5)(0.3)(0.6)$$
$$\text{BON—OWA}((a_i, p_i, \alpha_i)) = 0.095 + 0.074 + 0.09 = 0.259$$

6. Bonferroni choquet aggregation operator

In the preceding we have shown that the BON–OWA aggregation can be generalized to

$$\text{BON—OWA} = \left(\sum_{i=1}^{n} \frac{p_i}{P} a_i u_i\right)^{0.5}$$

Here a_i is the satisfaction of A_i, $\frac{p_i}{P}$ is the normalized importance of A_i and u_i is the external support.

We indicated that u_i can be obtained as an OWA aggregation of the argument vector V^i, where V^i is the collection of all a_j except a_i. We further noted each u_i could be calculated using a different type of OWA aggregation. Even more generally we can calculate u_i using a Choquet integral type aggregation [18]. The ability to perform this requires the availability of a monotonic measure over collections of criteria. We define this in the following

Let us denote $\Omega = \{A_1, \ldots, A_n\}$, it is the set of criteria. Let us denote $\Omega^i = \Omega - \{A_i\}$, the set of all criteria except A_i. It is a set of $n-1$ elements. We define a monotonic set measure m_i over Ω^i as follows. $m_i : 2^{\Omega^i} \to [0,1]$, it maps subsets of criteria into the unit interval and it has the following properties

$$m_i(0) = 0$$
$$m_i(\Omega^i) = 1$$
$$m_i(A) \geqslant m_i(B) \quad \text{if } B \subseteq A.$$

We shall use this to define the Choquet integral of the argument V^i with respect to the measure m_i.

First let us recall that each V^i is the collection of all $A_k(x)$, for $k \neq i$. For convenience let us denote the elements in V^i as v_{i1}, $v_{i2}, \ldots, v_{in-1}$, and assume the elements have been ordered so that $v_{ij1} \geqslant v_{ij2}$ if $j_1 < j_2$. In addition we shall let ind(j) be the index of the criteria with the jth largest value in V^i.

We further let H^i_j be the subset of Ω^i consisting the j criteria with the largest satisfactions. We define $H^i_0 = \emptyset$. We now can define the Choquet integral of V^i with respect to m_i as

$$C_{m_i}(V^i) = \sum_{j=1}^{n-1} v_{ij}(m_i(H^i_j) - m_i(H^i_{j-1}))$$

It can be shown that if we define $u_i = C_{m_i}(V^i)$ then the resulting Bonferroni operator still retains the properties of a mean aggregation operator.

Thus we define Bonferroni Choquet operator as

$$\text{BON—CHOQ}(a_1, \ldots, a_n) = \left(\sum_{i=1}^{n} \frac{p_i}{P} a_i C_{m_i}(V^i)\right)^{0.5}$$

An interesting special case of the preceding is the following. Again let p_i be the personal weight associated in A_i. Let $P = \sum_{i=1}^{n} p_i$ and let $p^i = P - p_i$. Let m_i be such that

$$m_i(D) = \frac{\sum_{A_j \in D} p_j}{P - p_i}$$

In this case

$$C_{m_i}(V^i) = \frac{\sum_{j \neq i} a_j p_j}{P - p_i}$$

In this case

$$\text{BON—CHOQ}(a_1, \ldots, a_n) = \left(\frac{1}{P} \sum_{i=1}^{n} \frac{p_i a_i}{P - p_i} \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} p_j a_j\right)\right)^{0.5}$$

7. Conclusion

We introduced the idea of multi-criteria aggregation functions and described a number of properties desired of such functions. We emphasized the importance of having an aggregation function capture the expressed interrelationship between the criteria. A number of standard aggregation functions were introduced. We then introduced the Bonferroni mean operator. We provided an interpretation of this operator as involving a product of each argument with the average of the other arguments, a combined averaging and "anding" operator. This allowed us to suggest generalizations of this operator by replacing the simple averaging by other mean type operators as well as associating differing importances with the arguments.

References

[1] G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions: A Guide for Practitioners, Springer, Heidelberg, 2007.
[2] V. Torra, Y. Narukawa, Modeling Decisions: Information Fusion and Aggregation Operators, Springer, Berlin, 2007.
[3] R. Mesiar, A. Kolesarova, T. Calvo, M. Komornikova, A review of aggregation functions, in: H. Bustince, F. Herrera, J. Montero (Eds.), Fuzzy Sets and Their Extensions: Representation, Aggregation and Models, Springer, Heidelberg, 2008, pp. 121–144.
[4] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, Aggregation Functions, Cambridge University Press, Cambridge, 2009.
[5] C. Bonferroni, Sulle medie multiple di potenze, Bolletino Matematica Italiana 5 (1950) 267–270.
[6] P.S. Bullen, Handbook of Means and Their Inequalties, Kluwer, Dordrecht, 2003.
[7] J.J. Dujmovic, H.L. Larsen, Generalized conjunction/disjunction, International Journal of Approximate Reasoning 46 (2007) 423–446.
[8] K.J. Arrow, Social Choice and Individual Values, John Wiley & Sons, New York, 1951.
[9] T. Calvo, A. Kolesarova, M. Komornikova, R. Mesiar, Aggregation operators: properties, classes and construction methods, in: T. Calvo, G. Mayor, R. Mesiar (Eds.), Aggregation Operators, Physica-Verlag, Heidelberg, 2002, pp. 3–104.
[10] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrecht, 2000.
[11] R.R. Yager, On ordered weighted averaging aggregation operators in multi-criteria decision making, IEEE Transactions on Systems, Man and Cybernetics 18 (1988) 183–190.
[12] M. Carbonell, M. Mas, G. Mayor, On a class of monotonic extended OWA operators, in: Proceedings of the 6th IEEE International Conference on Fuzzy Systems, Barcelona, Spain, vol. 3, 1997, pp. 1695–1700.
[13] R.R. Yager, Quantifier guided aggregation using OWA operators, International Journal of Intelligent Systems 11 (1996) 49–73.
[14] L.A. Zadeh, A computational approach to fuzzy quantifiers in natural languages, Computing and Mathematics with Applications 9 (1983) 149–184.
[15] D. Dubois, H. Prade, A review of fuzzy sets aggregation connectives, Information Sciences 36 (1985) 85–121.
[16] Z.S. Xu, An overview of methods for determining OWA weights, International Journal of Intelligent Systems 20 (2005) 843–865.
[17] M. O'Hagan, Using maximum entropy-ordered weighted averaging to construct a fuzzy neuron, in: Proceedings of the 24th Annual IEEE Asilomar Conference on Signals, Systems and Computers, Pacific Grove, CA, 1990, pp. 618–623.
[18] T. Murofushi, M. Sugeno, Fuzzy measures and fuzzy integrals, in: M. Grabisch, T. Murofushi, M. Sugeno (Eds.), Fuzzy Measures and Integrals, Physica-Verlag, Heidelberg, 2000, pp. 3–41.