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# On the links between triangular sets and dynamic constructible closure

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**Abstract**

Two kinds of triangular systems are studied: normalized triangular polynomial systems (a weaker form of Lazard's triangular sets (Discrete Appl. Math. 33 (1991) 33)) and constructible triangular systems (involved in the dynamic constructible closure programs of Gómez-Díaz (Quelques applications de l'évaluation dynamique, Ph.D. Thesis, Université de Limoges, 1994)). This paper shows that these notions are strongly related. In particular, combining the two points of view (constructible and polynomial) on the subject of square-free conditions, it allows us to effect dramatic improvements in the dynamic constructible closure programs. © 2001 Elsevier Science B.V. All rights reserved.

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## 0. Introduction

Dynamic evaluation is a general method for computing with parameters [9,13]. In 1994, Gómez-Díaz implemented the dynamic constructible closure in the scientific computation system Axiom [22]: by simulating dynamic evaluation, it offers the possibility to compute with parameters in a very large way [16]. Thus, a parameter $a$ can be subjected to algebraic constraints but also to inequalities:

$$Q_1(a) \neq 0, \ldots, Q_r(a) \neq 0,$$

where $Q_1, \ldots, Q_r$ are polynomials in one variable.

There are numerous applications of these programs [15]. We can mention polynomial system solving with parameters [14], automatic geometric theorem proving [17,18],

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computation of Jordan forms with parameters [19], computation of the gcd of polynomials with parameters [12], etc. In every case, the outputs are represented by a finite collection of constructible triangular systems [15, Definition, p. 106].

This notion of triangular system adds further to many concepts of triangular sets. We can mention the characteristic sets of Ritt–Wu [29,32], the regular chains of Kalkbrener [23], the triangular sets of Lazard [24], the regular sets of Moreno Maza [27] [1] and the simple systems of Wang [30] which have in common the fact that they are stated in a commutative algebra context.

On the opposite, the constructible triangular systems involved in Gómez-Díaz programs are defined within the constructible closure terminology. This may explain why nobody has been concerned with the analysis of the dynamical constructible triangular systems (see [2, Section 7.5, p. 50]). We think that [10] is a first step in this direction as we establish a relevant model of these systems within the framework of commutative algebra.

In this paper, we first present the basic ideas of this model inspired from the work of Aubry et al. [3], Aubry [2] and Moreno Maza [27] and justified in details in [10, Chapter 4]. This algebraic approach of Gómez-Díaz systems allows us to study a more practical problem in this paper. The square-free condition imposed in the dynamic constructible closure programs is too strong. Then, it is the source of many undesirable splits which grow up the number of systems in output. Our algebraic approach combined with a fundamental result (Theorem 4.1) solves this problem by relating this square-free condition with the one introduced by Lazard in [24, Definition 3.2, p. 150]. This provides us with a good strategy to improve the dynamic constructible closure programs.

The paper is structured as follows. We have collected in Section 1 some needed notations. In Section 2, we introduce the notions of weak and normalized polynomial triangular systems. We review then two main results of triangular sets theory. The reader is referred to [2,3,27]. We define in Section 3 the analogous properties in the *constructible* context. This leads to the notions of weak and normalized constructible triangular systems. We examine then in Section 4 the relationship between these two kind of triangular systems. Section 5 is devoted to study the square-free conditions of Lazard and Gómez-Díaz. This leads to the implementation of Lazard's condition in the dynamic constructible closure programs. We report in Section 6 some experimental results which justify our strategy.

## 1. Preliminaries

Given a commutative ring $A$ with identity, the zero divisors of $A$, the total ring of fractions of $A$ and the set of the units of $A$ are, respectively, denoted by $Div(A)$,

[1] The relationship between these notions have been studied by Aubry et al. in [3, Theorem 6.1, p. 19].

$Frac(A)$ and $A^\star$. Let $\mathcal{I}$ be an ideal of a ring $A$. We write $A/\mathcal{I}$ the residue class ring of $A$ by $\mathcal{I}$.

Throughout this paper, $K_0$ will denote a commutative field of characteristic zero. We set

$$P_0 = K_0.$$

Let $n$ be a positive integer. We denote by $\mathbb{Z}_n$ and $\mathbb{Z}_n^+$ the sets $\{0,\dots,n\}$ and $\{1,\dots,n\}$, respectively. Then for all $i \in \mathbb{Z}_n^+$, we define:

$$P_i = K_0[X_1,\dots,X_i].$$

Moreover, for all $i \in \mathbb{Z}_n^+$ and for all $f \in P_i - P_{i-1}$, we use the following terminology (nearly the one adopted by Lazard in [24] and by the authors of [3]):

- the *main variable* of $f$ is $X_i$;
- the *index* of $f$ ($ind(f)$) is $i$;
- the *degree* of $f$ ($deg(f)$) is its degree in $X_i$;
- the *leading coefficient* of $f$ ($lc(f)$) is the coefficient of $X_i^{deg(f)}$ in $P_{i-1}[X_i]$;
- we denote by $lc^j(f)$ with $j > 0$ the $j$th iteration of the function $lc$ applied to $f$ (for all $j$ such that $lc^j(f)$ is well defined) and $lc^0(f) = f$;
- the *discriminant* of $f$ ($Disc_{X_i}(f)$) is the resultant of the polynomials $f$ and $\partial f/\partial X_i$ with respect to $X_i$.

Working with triangular systems gives rise to consider the following kind of ideal of polynomials (see the beginning of [3, Section 2.3, p. 5] for more references).

**Definition 1.1** (*Cox et al.* [8, *Exercise* 8, *p.* 196]). Let $I \subseteq P_n$ be an ideal and fix $f \in P_n$. Then the saturation of $I$ with respect to $f$ is the ideal of $P_n$:

$$I\colon f^\infty = \{g \in P_n\colon f^m g \in I \text{ for some } m \geq 0\}.$$

Let $A, B$ be two commutative rings with identity and $\sigma : A \to B$ a ring-homomorphism. Then by $\sigma[T]$ we mean the ring homomorphism defined by

$$\begin{array}{ccc} A[T] & \xrightarrow{\sigma[T]} & B[T] \\ \\ \sum_{i=0}^{d} a_i T^i & \longmapsto & \sum_{i=0}^{d} \sigma(a_i) T^i. \end{array}$$

From now on, the words ring and homomorphism mean, respectively, commutative ring with identity and ring homomorphism.

Let $\{a_k\}_{k\in I}$ be a finite set of elements of $A$ and $\mathcal{I}$, $S$, respectively, the ideal and the multiplicative set of $A$ generated by the $a_k$ ($k \in I$). We write

$$\mathcal{I} = \langle a_k \rangle_{k\in I}$$

and

$$S = \prec a_k \succ_{k\in I}.$$

Moreover, given a multiplicative set $T$ of $A$, we write, respectively, $\mathscr{S}at(T)$ and $T^{-1}A$ the saturated multiplicative set generated by $T$ [1, Exercise 7, p. 44] and the ring of fractions of $A$ with respect to $T$. Finally, given $a \in A$, we note $A_a$ the ring $T^{-1}A$ with $T = \prec a \succ$.

## 2. Normalized polynomial triangular systems

This section is a brief overview of Lazard triangular sets theory. It was introduced in [24] to solve algebraic systems in the general case. We only focus on two of the six properties of the original definition. These are the notions of weak (Definition 2.1) and normalized (Definition 2.3) polynomial triangular systems. Then, we recall two fundamental results (Theorems 2.1 and 2.2) which appear in a more recent work (see [2,3,27]).

**Definition 2.1.** Let $n$ be a positive integer and $E$ be a subset of $\mathbb{Z}_n^+$. A *weak polynomial* triangular system in $P_n$ is a system of polynomials in $P_n$ $\{f_j = 0\}_{j\in E}$ such that for each $j \in E$:

$$ind(f_j) = j.$$

**Remark.** Let $\{f_j = 0\}_{j\in E}$ be a weak polynomial triangular system in $P_n$. If we only consider the subset of $\{f_j\}_{j\in E}$ of $P_n$, we obtain a so-called *triangular set* [3, Definition 2.2, p. 3].

**Definition 2.2.** Let $\{f_j = 0\}_{j\in E}$ be a weak polynomial triangular system in $P_n$. We denote by $\Phi_0$ the identity homomorphism of $K_0$. For all $i \in \mathbb{Z}_n^+$, we recursively define a ring $K_i$ and a homomorphism $\Phi_i : P_i \to K_i$ as follows:

- if $i \notin E$, we set

$$K_i = Frac(K_{i-1}[X_i]) \quad \text{and} \quad \Phi_i = inj_i \circ \Phi_{i-1}[X_i],$$

  where $inj_i$ is the canonical injection of $K_{i-1}[X_i]$ into $Frac(K_{i-1}[X_i])$;
- if $i \in E$, we set

$$K_i = Frac\left(\frac{K_{i-1}[X_i]}{\langle \Phi_{i-1}[X_i](f_i)\rangle}\right) \quad \text{and} \quad \Phi_i = inj'_i \circ \pi_i \circ \Phi_{i-1}[X_i],$$

  where $\pi_i$ is the projection of $K_{i-1}[X_i]$ over $K_{i-1}[X_i]/\langle \Phi_{i-1}[X_i](f_i)\rangle$ and $inj'_i$ is the canonical injection of $K_{i-1}[X_i]/\langle \Phi_{i-1}[X_i](f_i)\rangle$ into its total ring of fractions.

**Remark.** Note that it is the definition of the rings adopted by Lazard in [24] apart from the algebraic case: [2] he sets $K_i = K_{i-1}[X_i]/\langle \Phi_{i-1}[X_i](f_i)\rangle$. On the other hand,

[2] In fact, the two definitions coincide as soon as the polynomial triangular system is regular [10, p. 229]. In this case, for all $i \in E$, the homomorphism $inj'_i$ is the identity.

this corresponds with the construction operated in the definition of a *tower of simple extensions of* $K_0$ (see [2,3,27]).

**Example.** Let $K_0 = \mathbb{Q}$ be the field of rational numbers and $f_2$ be the polynomial $X_2^2 + X_1^2 - 1$ in $P_2$. We consider the system $\{f_2 = 0\}$. It is obviously a weak polynomial triangular system in $P_2$. We construct the rings $K_i$ $(i = 1, 2)$ associated with this system. By definition, since $E = \{2\}$, we have [3]

$$K_1 = Frac(K_0[X_1]) = \mathbb{Q}(X_1)$$

and

$$K_2 = Frac\left(\frac{K_1[X_2]}{\langle \Phi_1[X_2](f_2)\rangle}\right) = Frac\left(\frac{\mathbb{Q}(X_1)[X_2]}{\langle X_2^2 + X_1^2 - 1\rangle}\right),$$

where the homomorphism $\Phi_1$ is the canonical injection of $\mathbb{Q}[X_1]$ into $\mathbb{Q}(X_1)$.

**Definition 2.3.** A *normalized polynomial triangular system* in $P_n$ is a weak polynomial triangular system $\{f_j = 0\}_{j \in E}$ in $P_n$ such that for all $j \in E$:

$$\forall \alpha > 0, \quad ind(lc^{\alpha}(f_j)) \notin E.$$

We need another notation. Let $n$ be a positive integer and $E$ be a subset of $\mathbb{Z}_n^+$. We set $U_0(E) = K_0^{\star}$. For all $i \in \mathbb{Z}_n^+$, we define a subset $U_i(E)$ of $P_i$ by

$$U_i(E) = \{u \in P_i - \{0\},\ \forall \alpha \geq 0,\ ind(lc^{\alpha}(u)) \notin E\},$$

where we set $lc^0(u) = u$. We can now reformulate Definition 2.3 as follows.

**Lemma 2.1.** *A weak polynomial triangular system* $\{f_j = 0\}_{j \in E}$ *in* $P_n$ *is normalized if and only if the following condition holds for all* $j \in E$:

$$lc(f_j) \in U_{j-1}(E).$$

**Proof.** Given $j \in E$, we only need to remark the equivalence

$$\forall \alpha \geq 0,\ ind(lc^{\alpha}(lc(f_j))) \notin E \iff \forall \alpha > 0,\ ind(lc^{\alpha}(f_j)) \notin E. \qquad \square$$

**Proposition 2.1.** *Let* $\{f_j = 0\}_{j \in E}$ *be a normalized polynomial triangular system in* $P_n$. *Then for all* $i \in \mathbb{Z}_n$:

$$\forall u \in U_i(E), \quad \Phi_i(u) \in K_i^{\star}.$$

**Proof.** By induction on $i$ (see [10] for a more detailed proof). The key fact is that, for all $i \in \mathbb{Z}_n$, the homomorphisms in the definition of $\Phi_i$ preserve identity. $\square$

**Remark.** Let $\{f_j = 0\}_{j \in E}$ be a normalized polynomial triangular system in $P_n$. For all $j \in E$, we have $\phi_{j-1}(lc(f_j)) \in K_{j-1}^{\star}$ by Lemma 2.1. We find then the property of

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[3] One can see that the "*Frac*" is superfluous here. The ring $K_2$ is equal to $\mathbb{Q}(X_1)[X_2]/\langle X_2^2 + X_1^2 - 1\rangle$ (it is an illustration of the previous footnote).

*regularity* introduced by Moreno Maza in [27]. This concept appears in [3] under the name of *regular sets*. With our terminology, $\mathscr{T}=\{f_j\}_{j\in E}$ is a regular set if $\{f_j=0\}_{j\in E}$ is a weak triangular set with the property that the $lc(f_j)$ are units of $K_{j-1}$ ($j\in E$). Then we can restate the previous proposition as follows: the concept of normalization is *stronger* than the concept of regularity (this result appears in [27]). The converse is false. It suffices for example to consider the system

$$\begin{cases} (X_1-1)X_2-1=0,\\ X_1^2+X_1+1=0. \end{cases}$$

The following theorems [4] are the translation, in our context, of two results from [2,3,27]. They will be useful in Section 4.

**Notation.** Let $\{f_j=0\}_{j\in E}$ be a normalized polynomial triangular system in $P_n$. Then for all $i\in\mathbb{Z}_n$, we denote by $p_i$ the projection of $P_i$ over $P_i/ker\,\Phi_i$ and $can_i$ the canonical injection of $P_i/ker\,\Phi_i$ into its total ring of fractions.

The next result is very close to [27, Proposition III.18, p. 105; 3, Theorem 5.1, p. 18; 2, Proposition 4.5.7].

**Theorem 2.1.** *Let $\{f_j=0\}_{j\in E}$ be a normalized polynomial triangular system in $P_n$. Then for all $i\in\mathbb{Z}_n$, we have the following commutative diagram*:

$$\begin{tikzcd}
P_i \arrow[rr, "\Phi_i"'] \arrow[d, twoheadrightarrow, "p_i"'] \arrow[drr, phantom, "\circ"] & & K_i \\
\dfrac{P_i}{ker\,\Phi_i} \arrow[rr, hookrightarrow, "can_i"'] & & Frac\left(\dfrac{P_i}{ker\,\Phi_i}\right) \arrow[u, "\phi_i"', "\simeq"]
\end{tikzcd}$$

*where for any $f,g\in P_i$ such that $p_i(g)$ is not a zero divisor in $P_i/ker\,\Phi_i$, the isomorphism $\phi_i$ is defined by*

$$\phi_i\left(\frac{p_i(f)}{p_i(g)}\right)=\frac{\Phi_i(f)}{\Phi_i(g)}.$$

For the next theorem, we need further notations. Let $\{f_j=0\}_{j\in E}$ be a normalized polynomial triangular system in $P_n$. For all $i\in\mathbb{Z}_n^+$, we set $E_i=E\cap\mathbb{Z}_i^+$ and

$$h_i=\prod_{j\in E_i} lc(f_j).$$

[4] A proof of these theorems is also given in [10] under the weaker property of regularity.

Moreover, given $g \in P_i$, we write $prem(g, \{f_j\}_{j\in E_i})$ the pseudo-remainder of $g$ by the $\{f_j\}_{j\in E_i}$ (see for example [26, Theorem 5.2.2, p. 170] or [3, Notations 2.3, p. 5]). More precisely, if $E_i = \emptyset$ we define $prem(g, \{f_j\}_{j\in E_i}) = g$ otherwise if $E_i = \{i_1, \dots, i_{r-1}, i_r\}$ with $i_1 < \cdots < i_{r-1} < i_r$, we define

$$prem(g, \{f_j\}_{j\in E_i}) = prem(prem(g, f_{i_r}), \{f_j\}_{j\in E_{i_{r-1}}}).$$

**Theorem 2.2** (Aubry et al. [3, Proposition 5.1, p. 17]). *Let $\{f_j = 0\}_{j\in E}$ be a normalized polynomial triangular system in $P_n$. Then for $i \in \mathbb{Z}_n^+$:*

$$ker\, \Phi_i = \langle f_j\rangle_{j\in E_i} : h_i^\infty = \{g \in P_i;\, prem(g, \{f_j\}_{j\in E_i}) = 0\}.$$

## 3. Normalized constructible triangular systems

In [10], we show that a good algebraic model for the triangular systems involved in the dynamic constructible closure programs is what we called *square-free normalized constructible triangular systems*. In this section, we only focus on *normalized constructible triangular systems* (the square-free condition is studied in Section 5). For this purpose, we present a terminology similar to the polynomial case: for example, we work now with rings $L_i$ (introduced in Definition 3.2) instead of $K_i$ and homomorphisms $\Psi_i$ instead of $\Phi_i$. This will be very helpful in Section 4 to investigate the links between polynomial and constructible triangular systems. Using the kernel of the $\Psi_i$, we give in Theorem 3.1 another description of the rings $L_i$. Surprisingly, this ideal is only determined by the equations and admits the same characterization as in the polynomial case (Theorem 3.2).

**Definition 3.1.** Let $n$ be a positive integer and $E, F$ be subsets of $\mathbb{Z}_n^+$. A weak constructible triangular system in $P_n$ is a set $\{g_j\, \xi_j\, 0\}_{j\in E\cup F}$ verifying for all $j \in E \cup F$:
1. the polynomial $g_j$ belongs to $P_j$ with index $j$;
2. $\xi_j$ is the symbol "=" or the symbol "≠" accordingly as $j \in E$ or $j \in F$.

**Remark.** Let $\{g_j\, \xi_j\, 0\}_{j\in E\sqcup F}$ be a weak constructible triangular system in $P_n$. One can easily check that $E$ and $F$ are two disjoint subsets of $\mathbb{Z}_n^+$.

**Definition 3.2.** Let $\{g_j\, \xi_j\, 0\}_{j\in E\sqcup F}$ be a weak constructible triangular system in $P_n$. We set $L_0 = K_0$ and we denote by $\Psi_0$ the identity homomorphism of $L_0$. For all $i \in \mathbb{Z}_n^+$, we recursively define a ring $L_i$ and a homomorphism $\Psi_i : P_i \to L_i$ in the following way:
- if $i \notin E \sqcup F$, we set

$$L_i = L_{i-1}[X_i] \quad \text{and} \quad \Psi_i = \Psi_{i-1}[X_i];$$

- if $i \in F$, we set

$$L_i = (L_{i-1}[X_i])_{\Psi_{i-1}[X_i](g_i)} \quad \text{and} \quad \Psi_i = inj_{L_i} \circ \Psi_{i-1}[X_i],$$

where $inj_{L_i}$ is the canonical homomorphism:

$$L_{i-1}[X_i] \xrightarrow{inj_{L_i}} (L_{i-1}[X_i])_{\Psi_{i-1}[X_i](g_i)},$$

$$f \mapsto \frac{f}{1};$$

- if $i \in E$, we set

$$L_i = \frac{L_{i-1}[X_i]}{\langle \Psi_{i-1}[X_i](g_i)\rangle} \quad \text{and} \quad \Psi_i = \pi_{L_i} \circ \Psi_{i-1}[X_i],$$

where $\pi_{L_i}$ is the projection of $L_{i-1}[X_i]$ over $L_{i-1}[X_i]/\langle \Psi_{i-1}[X_i](g_i)\rangle$.

**Example.** Let us consider again the unit circle example. Let $K_0 = L_0 = \mathbb{Q}$ and $g_2$ be the polynomial $X_2^2 + X_1^2 - 1 \in P_2$. We consider the system $\{g_2 = 0\}$. It is obviously a weak constructible triangular system in $P_2$. Then, we can construct the rings $L_i$ $(i=1,2)$ associated with this system. By definition, since $E = \{2\}$ and $F = \emptyset$, we have

$$L_1 = L_0[X_1] = \mathbb{Q}[X_1] \quad \text{and} \quad L_2 = \frac{L_1[X_2]}{\langle \Psi_1[X_2](f_2)\rangle} = \frac{\mathbb{Q}[X_1, X_2]}{\langle X_2^2 + X_1^2 - 1\rangle},$$

where $\Psi_1$ is the identity homomorphism of $\mathbb{Q}[X_1]$.

**Notation.** Let $\{g_j\, \xi_j\, 0\}_{j \in E \sqcup F}$ be a weak constructible triangular system in $P_n$. For all $i \in \mathbb{Z}_n^+$, we write $E_i = E \cap \mathbb{Z}_i^+$ and $F_i = F \cap \mathbb{Z}_i^+$. Furthermore, we get $G_0 = \{1\}$ and for all $i \in \mathbb{Z}_n^+$, we define a multiplicative set $G_i$ of $P_i$ by

$$G_i = \prec g_k \succ_{k \in F_i} .$$

**Definition 3.3.** A *normalized constructible triangular system* in $P_n$ is a weak constructible triangular system $\{g_j\, \xi_j\, 0\}_{j \in E \sqcup F}$ in $P_n$ such that for all $j \in E \sqcup F$:

$$lc(g_j) \in \mathscr{S}at(G_{j-1}).$$

**Example.** Consider the weak constructible triangular system in $P_3$ with $K_0 = \mathbb{Q}$:

$$\mathscr{T} = \begin{cases} g_3 = (X_1X_2 - 1)X_3^2 + X_2 = 0, \\ g_2 = X_1X_2^2 - X_2 \neq 0, \\ g_1 = X_1(X_1 - 1) \neq 0. \end{cases}$$

By definition, the sets $G_1$ and $G_2$ are, respectively, equal to $\prec X_1(X_1 - 1)\succ$ and $\prec X_1(X_1 - 1),\, X_1X_2^2 - X_2 \succ$. Then, we have obviously that $lc(g_2) = X_1 \in \mathscr{S}at(G_1)$ and $lc(g_3) = X_1X_2 - 1 \in \mathscr{S}at(G_2)$. Thus $\mathscr{T}$ is a normalized constructible triangular system.

**Proposition 3.1.** *Let* $\{g_j\, \xi_j\, 0\}_{j \in E \sqcup F}$ *be a normalized constructible triangular system in* $P_n$. *Then for all* $i \in \mathbb{Z}_n$ *and* $g \in \mathscr{S}\mathrm{at}(G_i)$:

$$\Psi_i(g) \in L_i^{\star}.$$

**Proof** (*Sketch*)**.** By induction on $i$ (see [10] for a more detailed proof). The main ingredient of the proof is that, since the leading coefficient of the $g_j$ belongs to $\mathcal{S}at(G_{j-1})$ ($j \in E_i \sqcup F_i$), the homomorphism $\Psi_i$ preserves identity. □

**Notation.** Let $\{g_j \, \xi_j \, 0\}_{j \in E \sqcup F}$ be a normalized constructible triangular system in $P_n$. For all $i \in \mathbb{Z}_n$, we denote by $q_i$ the projection of $P_i$ over $P_i/ker\,\Psi_i$.

**Remark.** Let $\{g_j \, \xi_j \, 0\}_{j \in E \sqcup F}$ be a normalized constructible triangular system in $P_n$. It is easy to check that for all $i \in \mathbb{Z}_n$, the image $q_i(G_i)$ of $G_i$ is a multiplicative set of $P_i/ker\,\Psi_i$ which does not contain 0. It is an obvious corollary of previous lemma and [25, Proposition 5.5, p. 30].

**Notation.** Let $\{g_j \, \xi_j \, 0\}_{j \in E \sqcup F}$ be a normalized constructible triangular system in $P_n$. For all $i \in \mathbb{Z}_n$, we write $can_{\bar{G}_i}$ the canonical homomorphism:

$$\frac{P_i}{ker\,\Psi_i} \xrightarrow{can_{\bar{G}_i}} q_i(G_i)^{-1}\left(\frac{P_i}{ker\,\Psi_i}\right)$$

defined for all $f \in P_i$ by

$$q_i(f) \mapsto \frac{q_i(f)}{1}.$$

The next two results are very close to Theorems 2.1 and 2.2.

**Theorem 3.1.** *Let $\{g_j \, \xi_j \, 0\}_{j \in E \sqcup F}$ be a normalized constructible triangular system in $P_n$. Then for all $i \in \mathbb{Z}_n$, we have the following commutative diagram*:

$$\begin{tikzcd}
P_i \arrow[rr, "\Psi_i"'] \arrow[d, two heads, "q_i"'] \arrow[drr, phantom, "\circ"] & & L_i \\
\dfrac{P_i}{ker\,\Psi_i} \arrow[rr, hook, "can_{\bar{G}_i}"'] & & q_i(G_i)^{-1}\left(\dfrac{P_i}{ker\,\Psi_i}\right) \arrow[u, "\psi_i"', "\simeq"]
\end{tikzcd}$$

*where for all $f, g \in P_i$, with $q_i(g) \in q_i(G_i)$, the isomorphism $\psi_i$ is defined by*

$$\psi_i\left(\frac{q_i(f)}{q_i(g)}\right) = \frac{\Psi_i(f)}{\Psi_i(g)}.$$

**Proof** (*Sketch*)**.** By induction on $i$. Using Proposition 3.1 and [1, Proposition 3.1, p. 37], it is easy to show the existence of $\psi_i$ making the diagram commutative. The injectivity of $\psi_i$ is obvious. Proving that the homomorphism $\psi_i$ is surjective is more difficult (details are given in [10]): the case $i \notin E \sqcup F$ is trivial; if $i \in E \sqcup F$, the key fact is the isomorphism

$$L_{i-1}[X_i] \simeq q_{i-1}(G_{i-1})^{-1}\left(\frac{P_{i-1}}{ker\,\Psi_{i-1}}\right)[X_i].$$

Given $f \in L_{i-1}[X_i]$, one can check that there exists $h \in P_i$ and $g \in G_{i-1}$ such that $f = \Psi_{i-1}[X_i](h)/\Psi_{i-1}(g)$; the result follows then from the commutativity of the diagram. □

In the next result, we use one notation adopted in Theorem 2.2. More precisely, given a normalized constructible triangular system $\{g_j \,\xi_j\, 0\}_{j \in E \sqcup F}$ in $P_n$, we set for all $1 \le i \le n$:

$$h_i = \prod_{j \in E_i} lc(g_j).$$

**Theorem 3.2.** *Let $\{g_j \,\xi_j\, 0\}_{j \in E \sqcup F}$ be a normalized constructible triangular system in $P_n$. Then for all $i \in \mathbb{Z}_n^+$:*

$$ker\ \Psi_i = \langle g_j \rangle_{j \in E_i} : h_i^\infty = \{g \in P_i;\ prem(g, \{g_j\}_{j \in E_i}) = 0\}.$$

**Proof** (*Sketch*). There are two main ingredients in the proof (see [10]). Fix a positive integer $i \in \mathbb{Z}_n^+$. The first ingredient is that the image $\Psi_i(h_i)$ of $h_i$ is a unit of $L_i$ (since the constructible system is normalized). The second is that, given $g \in ker\ \Psi_i$, we have $prem(g, \{g_j\}_{j \in E_i}) = 0$. Using these two points, the proof is exactly the same as in the polynomial case. □

## 4. Links between polynomial and constructible systems

Given a normalized constructible triangular system $\{g_j \,\xi_j\, 0\}_{j \in E \sqcup F}$ in $P_n$, there is a natural way to construct a polynomial system: it suffices to keep the equations $\{g_j = 0\}_{j \in E}$. Several questions must be considered now. Is this a weak polynomial triangular system? In this case, we can construct the rings $K_i$, the homomorphisms $\Phi_i$ and then explore the links between the rings $K_i$ and $L_i$ ($i \in \mathbb{Z}_n$). Finally, one can wonder if the polynomial triangular system is normalized.

This section answers to these questions. In fact, the process

$$\{g_j \,\xi_j\, 0\}_{j \in E \sqcup F} \mapsto \{g_j = 0\}_{j \in E}$$

defines a map between normalized constructible triangular systems and normalized polynomial triangular systems. Proposition 4.2 and Theorem 4.1 present the two algebraic properties of this map. First, the kernels of the $\Psi_i$ and $\Phi_i$ ($i \in \mathbb{Z}_n$) are equal. Then, we show, in the guise of a commutative diagram, that $L_i$ can be viewed as a subring of $K_i$ ($i \in \mathbb{Z}_n$). Finally, Theorem 4.2 relates from a geometric point of view these two kinds of systems.

**Lemma 4.1.** *Let $\{g_j \,\xi_j\, 0\}_{j \in E \sqcup F}$ be a normalized constructible triangular system in $P_n$. Then for all $i \in \mathbb{Z}_n$:*

$$\mathscr{S}\mathrm{at}(G_i) \subseteq U_i(E).$$

**Proof** (*Sketch*)**.** By induction on $i$. The case $i \notin F$ is obvious by induction assumption. Conversely, let $f \in \mathscr{S}at(G_i)$ with $ind(f) = i$. Since the constructible system is normalized, it is easy to show that $lc(f)$ belongs to $\mathscr{S}at(G_{i-1})$. Moreover, given $g \in P_i$, one can verify that $g \in U_i(E)$ if and only if $ind(g) \notin E$ and $lc(g) \in U_{i-1}(E)$. Then the inclusion follows using the above argument with $g$ replaced by $f$. $\square$

**Proposition 4.1.** *Let* $\{g_j\, \xi_j\, 0\}_{j \in E \sqcup F}$ *be a normalized constructible triangular system in* $P_n$*. Then* $\{g_j = 0\}_{j \in E}$ *is a normalized polynomial triangular system.*

**Proof.** Since the constructible triangular system is normalized, the result easily follows from Lemmas 2.1 and 4.1. $\square$

Thus the map

$$\begin{array}{ccc} \text{Normalized constructible} & \xrightarrow{\mathscr{F}} & \text{Weak polynomial} \\ \text{triangular systems} & & \text{triangular systems} \end{array}$$

$$\{g_j \xi_j 0\}_{j \in E \sqcup F} \longmapsto \{g_j = 0\}_{j \in E}$$

preserves the normalization property.

**Remark.** In fact, we can construct another map $\mathscr{G}$:

$$\begin{array}{ccc} \text{Normalized polynomial} & \xrightarrow{\mathscr{G}} & \text{Weak constructible} \\ \text{triangular systems} & & \text{triangular systems} \end{array}$$

(see [10, p. 55] for more details). The map $\mathscr{G}$ does not preserve the normalization property but the results established in Proposition 4.2, Theorem 4.1 and Theorem 4.2 remain true in this case.

Our next task is to study the algebraic links between a normalized constructible triangular system and its image by $\mathscr{F}$.

**Example.** Consider again the following normalized constructible triangular system in $P_3$ with $K_0 = \mathbb{Q}$:

$$\mathscr{T} = \begin{cases} (X_1X_2 - 1)X_3^2 + X_2 = 0, \\ X_1X_2^2 - X_2 \neq 0, \\ X_1(X_1 - 1) \neq 0. \end{cases}$$

By definition, the image $\mathscr{F}(\mathscr{T})$ of $\mathscr{T}$ is the polynomial system in $P_3$:

$$f_3 = (X_1X_2 - 1)X_3^2 + X_2 = 0.$$

It is obviously a weak polynomial triangular system in $P_3$. Furthermore, we have $ind(lc(f_3)) = ind(X_1X_2 - 1) = 2$ and $ind(lc^2(f_3)) = ind(X_1) = 1$. Since there is no polynomial of index 1 or 2 in the system $\mathscr{F}(\mathscr{T})$, we conclude that $\mathscr{F}(\mathscr{T})$ is a normalized polynomial triangular system.

**Proposition 4.2.** *Let $\{g_j\ \xi_j\ 0\}_{j\in E\sqcup F}$ be a normalized constructible triangular system in $P_n$. Then for all $i\in\mathbb{Z}_n$*:

$$ker\ \Psi_i = ker\ \Phi_i.$$

**Proof.** The case $i{=}0$ is obvious. Now fix $i\in\mathbb{Z}_n^+$. Then, since $\{g_j{=}0\}_{j\in E_i}$ is normalized by previous proposition, the result follows directly from Theorems 2.2 and 3.2. □

**Theorem 4.1.** *Let $\{g_j\ \xi_j\ 0\}_{j\in E\sqcup F}$ be a normalized constructible triangular system in $P_n$. Then for all $i\in\mathbb{Z}_n$, there exists an injective homomorphism $\varepsilon_i$ such that the diagram is commutative*:

$$\begin{tikzcd}
p_i(G_i)^{-1}\left(\dfrac{P_i}{ker\ \Phi_i}\right) \arrow[d, "\psi_i"', "\simeq"] \arrow[r, phantom, "\circ" below, shift right=6ex] & \dfrac{P_i}{ker\ \Phi_i} \arrow[l, hook', "can_{\bar{G}_i}"'] \arrow[r, hook, "can_i"] & Frac\left(\dfrac{P_i}{ker\ \Phi_i}\right) \arrow[d, "\phi_i", "\simeq"'] \arrow[l, phantom, "\circ" below, shift left=6ex] \\
L_i \arrow[rr, bend right, "\varepsilon_i"'] \arrow[rr, phantom, "\circ", shift right=3ex] & P_i \arrow[l, "\Psi_i"] \arrow[r, "\Phi_i"'] \arrow[u, two heads, "p_i"'] & K_i
\end{tikzcd}$$

*Furthermore, for all $i\in\mathbb{Z}_n$, there exists a isomorphism $\Delta_i$ such that the diagram is commutative*:

$$\begin{tikzcd}
L_i \arrow[r, hook, "\varepsilon_i"] \arrow[d, hook] \arrow[dr, phantom, "\circ"] & K_i \\
Frac\ (L_i) \arrow[ur, "\Delta_i", "\simeq"'] &
\end{tikzcd}$$

**Proof** (*Sketch*)**.** The case $i{=}0$ is obvious. Now let $i\in\mathbb{Z}_n^+$. We first prove the existence of $\varepsilon_i$. By Proposition 4.1, we know that the polynomial triangular system $\{g_j=0\}_{j\in E}$ is normalized. Then, using Theorems 2.1, 3.1 and Proposition 4.2, it remains to show that there exists an injective homomorphism $\varepsilon_i$ such that $\Phi_i=\varepsilon_i\circ\Psi_i$. Let $g\in G_i$. By Proposition 2.1 and Lemma 4.1, the image $\Phi_i(g)$ of $g$ is an unit of $K_i$. The result follows then from [1, Proposition 3.1, p. 37].

The main ingredient in the proof of the second diagram is that using Lemmas 2.1, 4.1 and Proposition 2.1, one can check that for all $g\in G_i$, we have $p_i(g)\notin Div(P_i/ker\ \Phi_i)$. From [2, Lemma 4.5.6, p. 61], we deduce the isomorphism

$$Frac\left(\frac{P_i}{ker\ \Phi_i}\right)\simeq Frac\left(p_i(G_i)^{-1}\left(\frac{P_i}{ker\ \Phi_i}\right)\right).$$

The result follows then easily from the previous diagram. □

**Example.** Let us return to our previous example. By definition, the rings $L_i$ $(1 \le i \le 3)$ are equal to

$$L_1 = G_1^{-1}\mathbb{Q}[X_1], \quad L_2 = G_2^{-1}\mathbb{Q}[X_1,X_2], \quad L_3 = \frac{(G_2^{-1}\mathbb{Q}[X_1,X_2])\,[X_3]}{\langle (X_1X_2-1)X_3^2+X_2\rangle}$$

with $G_1 = \prec X_1(X_1-1)\succ$ and $G_2 = \prec X_1(X_1-1), X_1X_2^2-X_2 \succ$ whereas the rings $K_i$ $(1 \le i \le 3)$ defined from $\mathscr{F}(\mathscr{T})$ are [5]

$$K_1 = \mathbb{Q}(X_1), \quad K_2 = \mathbb{Q}(X_1,X_2), \quad K_3 = \frac{\mathbb{Q}(X_1,X_2)\,[X_3]}{\langle (X_1X_2-1)X_3^2+X_2\rangle}.$$

So, it appears clearly that $\varepsilon_1$ and $\varepsilon_2$ are, respectively, the canonical injections of $G_1^{-1}\mathbb{Q}[X_1]$ into $\mathbb{Q}(X_1)$ and $G_2^{-1}\mathbb{Q}[X_1,X_2]$ into $\mathbb{Q}(X_1,X_2)$. Finally, it is easy to show the existence of an injective homomorphism $\alpha$ such that $\pi_3 \circ \varepsilon_2[X_3] = \alpha \circ \pi_{L_3}$. Then $\varepsilon_3$ is equal to $\alpha$.

In fact, one can also study *geometric* connections between a normalized constructible triangular system and its image by $\mathscr{F}$ [10, Chapter 2]. To investigate this geometric point of view, we recall two notions of zeros.

**Notation.** We set $\tilde{K}_0$ to be an algebraic closure of $K_0$. Given an ideal $\mathscr{J}$ of $P_n$, we write $V(\mathscr{J})$ the affine variety of $\tilde{K}_0^n$ defined by $\mathscr{J}$. By extension, given a polynomial $g \in P_n$, we denote by $V(g)$ the affine variety defined by the ideal $\langle g\rangle$ of $P_n$. For all subset $W$ of $\tilde{K}_0^n$, we write $\overline{W}$ the Zariski closure of $W$ [8, Definition 2, p. 192]. Finally, let $\{g_j\ \xi_j\ 0\}_{j\in E\sqcup F}$ be a weak constructible triangular system in $P_n$. For all $i \in \mathbb{Z}_n^+$ we set

$$h_i = \prod_{j\in E_i} lc(g_j), \quad H_i = \prod_{j \in F_i} g_j.$$

**Definition 4.1.** Let $T = \{g_j\ \xi_j\ 0\}_{j\in E\sqcup F}$ be a weak constructible triangular system in $P_n$. For all $i \in \mathbb{Z}_n^+$ we define a subset $Z_i$ of $\tilde{K}_0^i$ by

$$Z_i = V(\langle g_j\rangle_{j\in E_i}) - V(H_i).$$

This is the set of zeros of $T$. Furthermore for all $i \in \mathbb{Z}_n^+$ we set

$$W_i = V(\langle g_j\rangle_{j\in E_i}) - V(h_i).$$

This is the *set of regular zeros* of $\mathscr{F}(T)$ [27, Definition III.19, p. 102].

One can note that the definition of the zeros of $T$ is very natural. In fact, it can be characterized by a less trivial property under the normalization property. For all $i \in \mathbb{Z}_n^+$ the set $Z_i$ is the standard open set $D_i$ [28, Definition 4.13, p. 21] of $V(ker\ \Psi_i)$ defined by the class of $H_i$ in the ring $P_i/\sqrt{ker\ \Psi_i}$ (see [10, Proposition 2.3.2, p. 84] for more details). This allows us to relate the sets $Z_n$ and $W_n$.

[5] The "*Frac*" is obvious here again (see footnote 1).

**Theorem 4.2.** *Let $\{g_j \, \xi_j \, 0\}_{j \in E \sqcup F}$ be a normalized constructible triangular system in $P_n$. Then*

$$Z_n \subseteq W_n \subseteq \overline{Z_n} = \overline{W_n}.$$

**Proof.** There are three steps in the proof. First one can show that the standard open set $D_n$ (see the notation above) and so $Z_n$ is equal to

$$D_n = Z_n = V(\langle g_j \rangle_{j \in E}) - V(H_n h_n)$$

[10, Proposition 2.3.1, p. 82]. This leads immediately to the inclusion $Z_n \subseteq W_n$. Furthermore, we deduce from [2, Proposition A.1.16, p. 142] that

$$\overline{Z_n} = V(\langle g_j \rangle_{j \in E} \, : \, (H_n h_n)^{\infty}).$$

The second point is that the kernel of $\Psi_n$ is equal to $\langle g_j \rangle_{j \in E} \, : \, (H_n h_n)^{\infty}$ [10, Lemma 2.3.2, p. 82]. Therefore we have $\overline{Z_n} = V(\mathit{ker}\, \Psi_n)$. Finally, it suffices to apply Proposition 4.2, Theorem 2.2 and again [2, Proposition A.1.16, p. 142] to conclude that $\overline{Z_n} = \overline{W_n}$. □

## 5. Application: about square-freeness

Applying the dynamic constructible closure programs, we get a finite collection of triangular constructible systems. Each polynomial of these systems verifies a square-free condition. Unfortunately, this condition is, in general, too strong. Consider (again) the unit circle example:

$$\{X_2^2 + X_1^2 - 1\} = 0.$$

It can be introduced into the dynamic constructible closure as follows:

$x$: $CL$:=`parameter('`$x$`)`

$y$: $CL$:=`parameter('`$y$`)`

`mustBeEqual(`$y$`**2+`$x$`**2-1,0)`

with the result

`[value is true in case` $y = 0$ `and` $x^2$`-1` $= 0$,
`value is true in case` $y^2$`+`$x^2$`-1` $= 0$ `and` $x^2$`-1/` $= 0$`]`

Thus, the programs of Gómez-Díaz describe the unit circle by isolating the points $(-1, 0)$ and $(1, 0)$. We say that it *splits* the system $\{X_2^2 + X_1^2 - 1 = 0\}$: the set of the solutions of this system is the disjoint union of the solutions of the two systems $\{X_2 = 0, X_1^2 - 1 = 0\}$ and $\{X_2^2 + X_1^2 - 1 = 0, X_1^2 - 1 \neq 0\}$. This can be desirable for very specific problems (for example, finding the possible vertical tangents of a plane curve) but it is quite uninteresting in general. *Our goal is to avoid these undesirable splits.*

For this purpose, we first use the concept of normalized constructible triangular system to present, in a algebraic way, the square-free condition of Gómez-Díaz. We also introduce another square-free condition, close to Lazard's one [24] (Definition 5.1). Using Theorem 4.1, it appears that the new condition is weaker than the original one (Lemma 5.1). This result gives us a practical way to solve our problem.

Let $\{g_j \; \xi_j \; 0\}_{j \in E \sqcup F}$ be a normalized constructible triangular system in $P_n$. According to Theorem 4.1, for all $i \in \mathbb{Z}_n$, there exists an injective homomorphism $\varepsilon_i$ which makes commutative the following diagram:

$$\begin{tikzcd} P_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & K_i \\ L_i \arrow[ur, hook, "\varepsilon_i"'] & \end{tikzcd}$$

As a result, for all $i \in \mathbb{Z}_n$, one can view $L_i$ as a subring of $K_i$ (by the injective homomorphism $\varepsilon_i$). Therefore, throughout this section, for all $i \in \mathbb{Z}_n$, we will not distinguish an element $f$ of $L_i$ and its image by $\varepsilon_i$.

**Definition 5.1.** Let $\{g_j \; \xi_j \; 0\}_{j \in E \sqcup F}$ be a normalized constructible triangular system in $P_n$. For all $i \in \mathbb{Z}_n^+$ and $p \in L_{i-1}[X_i]$ with $ind(p) = i$, the polynomial $p$ is said to be *Gómez-Díaz square-free* if

$$Disc_{X_i}(p) \in L_{i-1}^{\bigstar},$$

and *Lazard square-free* if

$$Disc_{X_i}(p) \in K_{i-1}^{\bigstar}.$$

**Remark.** Note that the polynomial $p$ in the previous definition belongs to $L_{i-1}[X_i]$ and not to $P_i$. It is very important because, in practice, the dynamic constructible closure programs use these kinds of polynomials (and not elements of $P_i$) [10, Chapter 4]. Next, one can observe that the second point of previous definition is very close to the square-free condition of Lazard triangular sets (see [24]).

**Example.** Assume $n = 2$ and let $K_0 = \mathbb{Q}$. We still consider the system:

$$\mathscr{T} = \left\{ X_2^2 + X_1^2 - 1 = 0 \right. .$$

It is obviously a normalized constructible triangular system in $\mathbb{Q}[X_1, X_2]$. It is clear that $L_1$ is the ring $\mathbb{Q}[X_1]$ and $\Psi_1$ is the identity homomorphism of $\mathbb{Q}[X_1]$. Moreover, this is also a normalized polynomial triangular system and $K_1$ is equal to $\mathbb{Q}(X_1)$ by definition. Now, we look at $g_2$ viewed as a polynomial of $L_1[X_2]$. Its discriminant $d$ is equal to $-4(X_1^2 - 1)$. Then $d$ is a unit in $K_1$ but not in $L_1$. Therefore, the polynomial $X_2^2 + X_1^2 - 1$ is Lazard square-free but not Gómez-Díaz square-free. *That is why, in practice, the dynamic constructible closure programs split the system* $\mathscr{T}$. After the computation, we obtain the two systems

$$\begin{cases} X_2 = 0, \\ X_1^2 - 1 = 0, \end{cases} \qquad \begin{cases} X_2^2 + X_1^2 - 1 = 0, \\ X_1^2 - 1 \neq 0, \end{cases}$$

in output.

Note that in the second system, the polynomial $X_2^2 + X_1^2 - 1$ is now Gómez-Díaz square-free. Indeed, its discriminant is a unit of the ring $L_1 = \mathbb{Q}[X_1]_{X_1^2-1}$.

The following lemma is trivial (it follows easily from the previous commutative diagram) but states the main result of this section.

**Lemma 5.1.** *Lazard square-free condition is weaker than Gómez-Díaz's one.*

Note that this appears clearly in the unit circle example. Hence the idea of substituting Gómez-Díaz square-free condition by Lazard's one in the dynamic constructible closure programs. This work has been done [10, Chapter 5] and has led to an implementation in the scientific computation systems Axiom [22] and Axiom-XL [31]. The key fact is the second commutative diagram in Theorem 4.1. Indeed, at each step of a computation with our programs, we deal with polynomials of $L_{i-1}[X_i]$. So, the diagram allows us to *consider these polynomials as elements of the rings* $K_{i-1}[X_i]$. In practice, it is not so difficult: it suffices to "forget" the inequalities ($\neq$) of the normalized constructible triangular system $\mathscr{T}$ (called the *current case* with Gómez-Díaz terminology) which defines the ring $L_{i-1}[X_i]$ or, in other words, to consider at each step, the image $\mathscr{F}(\mathscr{T})$ of $\mathscr{T}$ [10, Chapter 5].

The main part of this work is the implementation of Lazard square-free condition in the dynamic constructible closure programs. It is inspired by an algorithm called *invertible*? given by Lazard in [24], which tests whether an element of $K_i$ is a unit or not (see [10, Section 5.1] for more details).

**Remark.** In fact, there was one problem with this strategy. In case of splits, the square-free condition was not always verified by the next system to be treated by the programs (also called *next case*) [10, Section 5.2]. This has led us to rewrite a function called *parameter*. It is the function *newElement* of [15]. Indeed the goal of *parameter* (except introducing parameters) is to transform a next case to the current case (see [10] or [15] for more details). Originally, this mainly consisted of reduction operations. But it was not adapted to our new strategy and was at the source of our problem. Therefore in the step

next case $\rightarrow$ current case

the function *parameter* verifies now and if necessary imposes the square-free condition of each polynomial of the future current case. This solves our problem [10, Section 5.3].

Table 1
Number of constructible triangular systems in output

| | Gómez-Díaz square-free condition | Lazard square-free condition |
|---|---|---|
| Bronstein1 | 10 | 3 |
| Bronstein2 | 13 | 6 |
| Kinematic problem | 74 | 4 |
| Robot plano dificil | 27 | 11 |
| Bifurcation problem | 3 | 3 |
| Cyclohexane | 3 | 3 |
| Matrice de passage | 8 | 5 |
| Robot ROMIN | 48 | 4 |

## 6. Experimental results

We present now six examples of triangular decompositions for *polynomial* systems and two examples of triangular decompositions for *constructible* systems. The descriptions and the sources of our examples are specified below:

*Bronstein 1* [6]: $\{x^2+y^2+z^2-R^2=0,\ x+y-z=0,\ xy+z^2-1=0\}$ with $R<x<y<z$;
*Bronstein 2* [6]: $\{x^2+y^2+z^2-R^2=0,\ xy+z^2-1=0,\ xyz-x^2-y^2-z+1=0\}$ with $R<z<x<y$;
*Kinematic problem* [7]: $\{x_1+a_1c_1=0,\ y_1-a_1s_1=0,\ x_2+a_4+a_3c_2=0,\ y_2-a_3s_2=0,\ (x_2-x_1)^2+(y_2-y_1)^2-a_2^2=0,\ s_1^2+c_1^2-1=0,\ s_2^2+c_2^2-1=0\}$ with $x_1<y_1<x_2<y_2<a_1<a_2<a_3<a_4<s_1<c_1<s_2<c_2$;
*Robot plano dificil* [27]: $\{-l_3s_2s_1+(l_3c_2+l_2)c_1-a=0,\ (l_3c_2+l_2)s_1+l_3s_2c_1-b=0,\ s_1^2+c_1^2-1=0,\ s_2^2+c_2^2-1=0\}$ with $b<a<l_3<l_2<c_2<s_2<c_1<s_1$;
*Bifurcation problem* [5]: $\{y^2+x^2-\frac{17}{64}=0,\ 2yz+2x^2y^3-2x^6y=0,\ 2xz+xy^4-6x^5y+5x^9=0\}$ with $x<y<z$;
*Cyclohexane* [5]: $\{-(1+x^2)y^2+24xy-x^2-13=0,\ -(1+x^2)z^2+24xz-x^2-13=0,\ -(1+y^2)z^2+24yz-y^2-13=0\}$ with $x<y<z$;
*Matrice de Passage* [15]: $\{xz-y^2\neq 0,\ ay+bz-cx-dy=0\}$ with $a<b<c<d<x<y<z$;
*Robot ROMIN* ([20], see also [21]): $\{l_2\neq 0,\ l_3\neq 0,\ -ds_1-a=0,\ dc_1-b=0,\ l_2c_2+l_3c_3-d=0,\ l_2s_2+l_3s_3-c=0,\ s_1^2+c_1^2-1=0,\ s_2^2+c_2^2-1=0,\ s_3^2+c_3^2-1=0\}$ with $d<c<b<a<l_3<l_2<s_1<c_1<s_2<c_2<s_3<c_3$.

Tables 1 and 2 contain two kinds of informations: the number of constructible triangular systems in output and the computation time (evaluation).[6] In each table, we

[6] A detailed analysis of these results can be found in [10, Chapter 7].

Table 2
Timings (in s)

| | Gómez-Díaz square-free condition | Lazard square-free condition |
|---|---|---|
| Bronstein1 | 14.27 | 1.97 |
| Bronstein2 | 109.13 | 5.95 |
| Kinematic problem | 60.25 | 3.68 |
| Robot plano dificil | 739.20 | 649.26 |
| Bifurcation problem | 1.45 | 1.72 |
| Cyclohexane | 30.38 | 29.75 |
| Matrice de passage | 19.67 | 4.78 |
| Robot ROMIN | 3164.97 | 272.51 |

have put, respectively in the columns *Gómez-Díaz square-free condition* and *Lazard square-free condition* the results obtained with the original version (in Axiom) of the dynamic constructible closure programs and with our version (in Axiom) of these programs. All these examples have been tested with a machine which has 500 MHz chip and 128 Meg of RAM memory and which runs under OSF1(V4). Furthermore, they have been tested with two kinds of subresultant algorithms. We report here the best timing for each example.

One can note that if our strategy is sometimes not very interesting (as in the *Cyclohexane* example), it can however lead to dramatic improvements in the number of constructible triangular systems in output and in the timings. Thus there are approximately a factor 18 in the *kinematic problem* example and a factor 12 in the *Robot ROMIN* example (see [10, Section 7.4] for a complete study of this system).

These examples (and others treated in [10, Chapter 7]) confirm the good behaviour of Lazard square-free condition from the programming point of view and then confirm the interest of our strategy.

**Remark.** One may wonder what happens if we remove all square-free conditions in our programs. In fact, with the *Robot ROMIN* example, we obtain 19 systems in output (instead of four).

Furthermore, our implementation in Axiom-XL of our programs gives more better timings. Thus with the *Robot ROMIN* example, we obtain an union of four systems in 66.533 s (with the same machine). But it is a *first* implementation and we think that this factor 4 obtained in this example (272.51 s with the Axiom version) should be improved with a more efficient implementation in Axiom-XL.

## 7. Conclusion

First one must keep in mind that our programs are not specifically designed to solve polynomial or constructible systems. Thus, our goal is not to obtain a program for solving polynomial systems as powerful as the methods developed by Aubry and Moreno Maza [4] and Lazard [24] for example. Our programs are more general (see the introduction of this paper for a brief list of others applications), note for example that we can solve *constructible* systems. Nevertheless we have shown in this paper that there exists strong connections between Lazard triangular sets and the triangular sets involved in the dynamic constructible closure programs. Furthermore, this work has allowed us to improve the efficiency of these programs. Now we show in [11] that there are stronger connections between the triangular sets involved in these programs and Wang simple systems [30]. This theoretical work done in [11] may lead to another improvement of the dynamic constructible closure programs.

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