A Recurring Theorem About Pairs of Quadratic Forms and Extensions: A Survey

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ABSTRACT

This is a historical and mathematical survey of work on necessary and sufficient conditions for a pair of quadratic forms to admit a positive definite linear combination and various extensions thereof.

INTRODUCTION

Work on classifying definite matrix pencils has been going on for over 40 years. Following a chronological survey, we will indicate how some of the numerous proofs now known for the Main Theorem are interconnected mathematically. As extensions of this work, we will deal with recent results: a classification of positive semidefinite and indefinite pencils, as well as a network of interrelated theorems about the simultaneous diagonalization of two quadratic forms and very recent work on related questions on general fields and on the stability analysis of a generalized eigenvalue problem.

NOTATION. S and T will denote two real symmetric matrices of the same dimension n throughout, while H,K will stand for two complex (or quaternion) hermitian matrices. Moreover, to avoid dealing in trivialities, let us assume that S and T (or H and K) are not scalar multiples of each other.

We define the quadratic hypersurface associated with S as $Q_S := \{x \in R^n | x'Sx = 0\}$, while $Q_H := \{x \in C^n | x^*Hx = 0\}$. Moreover, we call a *pencil* $P(S,T) := \{aS + bT | a,b \in R\}$ (or $P(H,K) := \{aH + bK | a,b \in R\}$, respectively) a *d-pencil* if there exists a definite matrix in $P(S,T)$ (or $P(H,K)$).

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Such a pencil is called a *s.d. pencil* if there exists a nonzero semidefinite but no definite matrix in it, while we call it an *i-pencil* if every nonzero matrix in it is indefinite.

We are concerned with the history and mathematical background of the

MAIN THEOREM.

(I) *If* $n \geq 3$, *then* $P(S,T)$ *is a d-pencil iff* $Q_S \cap Q_T = \{0\}$.

(II) *For arbitrary* n, $P(S,T)$ *is a d-pencil iff the quadratic form of* S (*or* T) *does not change sign on* Q_T (*or* Q_S, *respectively*).

(III) *For arbitrary* n, $P(H,K)$ *is a d-pencil iff* $Q_H \cap Q_K = \{0\}$.

HISTORY

Theorems (I) and (II) were originally proved in 1936 by P. Finsler of Zürich [21], who is also known for a branch of differential geometry and for having discovered the Finsler comet 1937v. At the time he was studying a special class of algebraic surfaces called *Freigebilde* (see [22]). The *Zentralblatt* review of [21] unfortunately does not mention (I).

Almost concurrently with Finsler, in the summer of 1937, G. A. Bliss proposed Theorem (II) in his seminar on multidimensional calculus of variations at the University of Chicago. Following this seminar, several proofs of (II) were found by W. T. Reid [47], A. A. Albert [2], E. J. McShane [43] and M. Hestenes [26]. Later Hestenes and McShane [27] jointly proved the following generalization to more than two quadratic forms:

(II for r forms) *Assume that* $x'Sx > 0$ *for all* $x \in Q_{T_1} \cap \cdots \cap Q_{T_r}$, $x \neq 0$, *and let* T_i *be such that* $\Sigma_i a_i T_i$ *is indefinite for any nontrivial choice of* $a_i \in R$. *Moreover assume that for any subspace* $L \subsetneq \mathrm{C}(Q_{T_1} \cap \cdots \cap Q_{T_r})$ *there are constants* $b_i \in R$ *such that* $x'(\Sigma_i b_i T_i)x > 0$ *for all* $0 \neq x \in L$. ($\mathrm{C}(\cdot)$ *denotes the complement.*) *Then* $P(S, T_1, \ldots, T_r)$ *is a d-pencil.*

For $r = 1$ *only the first assumption needs to be made.*

This sufficient condition for d-pencils is certainly not necessary, but it is well suited for minimizing certain types of integrals arising in the multidimensional calculus of variations, an area that was being developed at that time.

In 1941, L. Dines [17] used convexity type arguments for the first time. In his paper, (II) is called the "Bliss-Albert theorem" in Corollary 2. It is intriguing to read the following footnote there [17, p. 494]: "While the present paper was in press, Professor N. H. McCoy kindly called the author's attention to the fact that this theorem was first proven by Paul Finsler: 'Über das Vorkommen...' ([21] in our bibliography). Apparently this work had been overlooked by the authors referred to above." The authors mentioned by Dines [17] were Albert [2], Reid [47] and Hestenes and McShane [27].

Dines [18] also proved this result about r forms:

(D1) (a) *$P(T_1,\dots,T_r)$ is a d-pencil iff there exist no $0\neq z_j\in R^n$ with $\sum_{j=1}^k m_j T_i(z_j)=0$ for $m_j>0$, for all $k\geqslant 1$ and all $i=1,\dots,r$.*

(b) *If $T_1,\dots,T_r$ are linearly independent, then $P(T_1,\dots,T_r)$ is a s.d. pencil iff*

(i) *there exist $z_j\neq 0$ and $m_j>0$ with $\sum_{j=1}^k m_j T_i(z_j)=0$ for all $i=1,\dots,r$ and some $k\geqslant 1$, and*

(ii) $\{(a_1,\dots,a_r)|a_i=\sum_{j=1}^k m_j T_i(z_j),\ z_j\neq 0,\ m_j>0\}\neq R^r$.

These results were combined with work by F. John [30] to yield still other classifications of d-, $s.d.$ and i-pencils of r real quadratic forms in Dines [19]:

(D2) (a) *$P(T_1,\dots,T_r)$ is a d-pencil iff* trace$(ST_i)=0$ *for $S=S'$ and all i implies S is indefinite,*

(b) *$P(T_1,\dots,T_r)$ is a s.d. pencil iff there exists a semidefinite $S=S'$ with* trace$(ST_i)=0$ *for all i, but no such definite S,*

(c) *$P(T_1,\dots,T_r)$ is an i-pencil iff there exists a definite $S=S'$ with* trace$(ST_i)=0$ *for all i.*

Surprisingly, none of Dines's work on the classification of symmetric pencils was quoted for nearly 20 years: in 1960, R. Bellman [10, p. 88] quoted both [17] and [19].

In between, in 1958, (I) was given as a problem in the first edition of W. Greub's book *Linear Algebra* [25, p. 263, Problem 1]—erroneously, though, without any restriction on n. The problem was posed after J. Milnor's proof [44] of the following theorem on the simultaneous diagonalization of two symmetric matrices, a theorem which will be studied more closely in the section on "Extensions," subsection (b):

(PM) *If $Q_S\cap Q_T=\{0\}$, then S and T can be diagonalized simultaneously by a real congruence transformation, provided $n\geqslant 3$.*

This theorem was originally discovered by E. Pesonen [45, Satz 1.2] for pairs of hermitian matrices. Further proofs of (PM) are due to K. N. Majindar [39, 40], H. Kraljevic [32], M. Wonenburger [72] and W. C. Waterhouse [69, Theorem 5.2]. Pesonen [45, Satz 1.1] also showed that $Q_H\cap Q_K=\{0\}$ implies that x^*Hx does not change sign for $x\in Q_K$, as did R. Kühne [35, Lemma 1.1], i.e., (III) implies (II).

In Hilbert space, pairs of symmetric operators A, B with $Q_A \cap Q_B = \{0\}$ are called Pesonen operator pairs by J. Bognar [12, Chapter II.9], since Pesonen had originated work on (PM) and Finsler type problems in Hilbert space. But it was E. Calabi [14, Theorem 2] who first showed that Pesonen operator pairs admit a semidefinite linear combination. This result was later given a quantitative form by R. Kühne [34, Satz 1.1; 35, Lemma 1.2] and by M. G. Krein and J. L. Smuljan [33, Theorem 1.1]:

(CKKS) *If A, B are symmetric operators in a Hilbert space $\mathcal{H}$ such that $x^*Ax \geqslant 0$ for all $0 \neq x \in Q_B$, where B is indefinite, then*

(a) $\inf_{x^*Bx=1} x^*Ax =: m_A > -\infty$ *and*
(b) $A - m_A B$ *is positive semidefinite on* $\mathcal{H}$.

An even earlier—but rather complicated—extension of Finsler's theorem to Hilbert space is due to M. R. Hestenes [29, Sec. 13, pp. 559–562]. He considers Legendre pairs of quadratic forms P, Q, that satisfy several weak-strong continuity conditions; see [29, p. 559] or [28, p. 404, 405]. For these pairs, Hestenes [29, Theorem 13.1] shows that there always is a positive definite linear combination $P + bQ$ provided that $P(x) \leqslant 0$, $Q(x) \leqslant 0$ implies $x = 0$. Further sufficient conditions for the existence of positive definite linear combinations $P + bQ$ are given in [29, Lemma 13.1, Theorems 13.2, 13.3].

For further results on Finsler's problem in Hilbert space, and specifically on Pesonen operators, see J. Bognar [12, Chapter II.9] and the "Notes to Chapter II" in [12, pp. 56, 57], as well as [12, Chapter II, Theorem 6.2]. Note, though, that none of [12], [33], [34], [35], [45] contain any references to previous work on Finsler's problem. Another extension of (PM)—this time for finite dimensional spaces only—is due to M. Marcus [42]: Assuming $(\det X'SX)^2 + (\det X'TX)^2 > 0$ for all $n \times 2$ real matrices X of rank 2, Marcus uses Thompson's real pair form theorem in [59] to determine the finest simultaneous block diagonalizations possible for such S and T.

It is not known whether Greub's problem of deducing (I) from (PM) in [25] had been solved before 1964; in fact, the problem was omitted in the latest edition (*Heidelberger Taschenbücher*, Bd. 179, 1976) of Greub's book. In 1964, in a remark added to E. Calabi's paper [14], O. Taussky indicated how this could be done via Stiemke's theorem (see [54] or Theorem (1.6.4) in [55]).

Calabi [14] was led to consider (I) from studying differential geometry and matrix differential equations. Unaware of any previous work on the subject, he gave a new and short topological proof of (I), together with the just-mentioned extension of (I) to infinite dimensional spaces.

Taussky later detailed her proof of (I) via Stiemke's transposition theorem in [56, pp. 313, 314]. In [56, pp. 314, 315] she also deduced (I) from Brickman's theorem [13], which proved the convexity of the "real field of values." By the same argument, (III) can actually be proved for arbitrary n. In [56], there is a quotation of unpublished work by H. F. Bohnenblust on more than two forms, as well as a hint about work by H. Wielandt—also unpublished, as I have recently learned.

Bohnenblust's result was given a different proof by S. Friedland and R. Loewy in [23]:

(BFL) *Let V be a k-dimensional subspace of R_{nn} consisting only of symmetric matrices, and let $1 \leqslant r \leqslant n-1$. If $\sum_{i=1}^{r} x_i' S x_i = 0$ for every $S \in V$ implies $x_i = 0$ for $i = 1, \cdots, r$, where $k < (r+1)(r+2)/2 - \delta_{n,r+1}$, then V contains a definite matrix.*

Through Dines's work, Bellman [10, Chapter 5] also learned about Finsler's results, but unfortunately he attributes only a very special case of (I) and (II) to Finsler, namely:

(Be) *Let $T = T' \in R_{nn}$ be positive semidefinite. If $x'Sx > 0$ for all $x \in$ Ker T, then $P(S,T)$ is a d-pencil.*

In a later edition [10, 1970, Chapter 5, Exercise 21, p. 88], Calabi [14] is given credit for Theorem (I).

Theorem (Be) has nevertheless found various applications in equilibrium studies by economic analysts: see e.g. A. Afriat [1], G. Debreu [16, Theorem 3], R. Farebrother [20], K. Lancaster [37, Part IV, R6.3, p. 301] or P. Samuelson [48, Math. Appendix A, Sec. V, pp. 376–379].

Later M. Hestenes [28] studied the set of "multipliers" a, b such that $aS + bT > 0$ (or $\geqslant 0$) and reestablished (I) with methods similar to Dines's [17]. Hestenes's bibliography includes the papers that came forth from Bliss's seminar, as well as Calabi [14] and Taussky [56], but surprisingly neither Finsler nor Dines is quoted, though Hestenes had written the MR review of [19] himself.

More recently, Y.-H. Au-Yeung [4]—only mentioning Greub [25] and Calabi [14] in the bibliography—re-proved (I) and (PM) for real symmetric and complex or quaternion hermitian matrix pairs. (I) was also proved by the author via (PM) in [64, p. 566]. And A. Berman and A. Ben-Israel [11] re-proved Dines's result (D2)(a) via the theory of convex cones, unaware of Dines's earlier proof. Most recently, D. Saunders [49] studied convexity properties of the norm-numerical range $V_\nu(A) = \{ y^*Ax | (x,y) \in \Pi \}$ where $\Pi := \{(x,y) \in C^n \times C^n | \nu(x) = \nu^D(y) = y^*x = 1\}$ for an arbitrary norm ν of C^n

and the dual norm $\nu^D(y)=\sup\{|y^*x||\nu(x)=1\}$ as introduced by F. Bauer [9]. If ν is the euclidean norm, then $V_\nu(A)=W_A$, the field of values of A. And the following analog to the Hausdorff-Toeplitz theorem for W_A holds for $V_\nu(A)$ [49, Theorem 7]:

(S) *If A and B are ν-hermitian for a vector norm ν of C^n, then $V_\nu(A+iB)$ is convex.*

Here A is called *ν-hermitian* if $V_\nu(A)\subseteq R$.

Hence by Taussky's proof in [56] one obtains:

(SLS) *If $A,B\in C_{nn}$ are ν-hermitian matrices, then the following are equivalent:*

(i) *For all $(x,y)\in\Pi$ either $y^*Ax\neq 0$ or $y^*Bx\neq 0$, and*
(ii) *$P(A,B)$ is a ν-positive definite pencil.*

"SLS" here stands for D. Saunders, R. Loewy and H. Schneider, who communicated this result to me. For the euclidean norm, (i) just says that $Q_A\cap Q_B=\{0\}$ as in (I).

MATHEMATICS

In this section we outline various proofs for the Main Theorem and show their relationships to each other. It has proved quite useful to use a "field of values" approach: For a square matrix A, the field of values $W_A=\{x^*Ax|x\in C^n, \|x\|=1\}$ is convex and contains all the eigenvalues of A (see Marcus and Minc [41, pp. 168, 169], for example). Specifically, for S symmetric, $W_S=[\lambda_{\min}(S),\lambda_{\max}(S)]\subseteq R$, since symmetric matrices are normal and have only real eigenvalues.

With this notation, S is said to be *definite* iff $0\notin W_S$.

For pairs of symmetric matrices S and T, these concepts and results can be generalized thus: If $(0,0)\notin W_S\times W_T\subseteq R^2$, then clearly one of S or T is definite and hence $P(S,T)$ is a d-pencil. The converse is false, though: take $S=\text{diag}(1,-1)$, $T=\text{diag}(-2,1)$ for example. Then $4S+3T=\text{diag}(-2,-1)$ is definite, while $0\in W_S\cap W_T$.

Hence more useful results have come from looking at

$$N(S,T):=\{(x'Sx,x'Tx)|x\in R^n\}\subseteq R^2$$

instead. Namely, (I) states (for $n\geqslant 3$) that $P(S,T)$ is a d-pencil iff $0=(x'Sx,x'Tx)\in N(S,T)$ implies $x=0$. Dines [17, Theorem 1] showed that

$N(S,T)$ is convex and if $Q_S \cap Q_T = \{0\}$, then $N(S,T)$ is closed and either $N(S,T) = R^2$ or $N(S,T)$ is an angular sector with angle less than 180° [17, Theorem 2]. Whence he concludes (I) for $n \geqslant 3$ in Corollary 2. (D1) is proved in [18] by considering the convex hull of $N(T_1,\dots,T_r)$, for $N(T_1,\dots,T_r)$ need not be convex if $r>2$. (D2) is proved directly in [19] by a very short manipulation with trace(ST).

In [28, Theorem 2], Hestenes also showed that $N(S,T)$ is a convex cone —even for infinite dimensional vector spaces. Moreover if $\infty \geqslant n \geqslant 3$ and $Q_S \cap Q_T = \{0\}$ then S and T allow a nonzero semidefinite linear combination [28, Theorem 3], re-proving the result of Calabi [14, Theorem 2]. M. R. Hestenes then proved that the two assumptions $Q_S \cap Q_T = \{0\}$, $N(S,T)$ closed (and for $n=2$ additionally $N(S,T) \neq R^2$) together imply that there is a closed set $M \subseteq R^2$ such that $\lambda S + \mu T \geqslant 0$ for all $(\lambda,\mu) \in M$ (Theorem 5). Such sets of "multipliers" M are also studied for Legendre pairs of quadratic forms in Hilbert space (Theorem 8).

Instead of considering the cone $N(S,T)$ itself, several authors have worked with its cross-sections in order to prove Theorem (I). In [56], Taussky makes use of Brickman's theorem [13]:

(Br) *If* $n \geqslant 3$, *then* $R(S,T) := \{(x'Sx, x'Tx) | x \in R^n,\ \|x\| = 1\} \subseteq R^2$ *is convex.*

To prove (I), Taussky notices that if $Q_S \cap Q_T = \{0\}$, then $R(S,T)$ is separated from $(0,0) \in R^2$ by a line G through $(0,0)$; hence the angle $\beta(x)$ between $G^{\perp}$ and any point $(x'Sx, x'Tx) \in R(S,T)$ is less than 90°, i.e.,

$$\cos\beta(x) = \frac{ax'Sx + bx'Tx}{\|\cdots\| \cdot \|\cdots\|} > 0,$$

where $a,b \in R$ are the constants defining $G^{\perp}$, i.e. $aS + bT$ is positive definite. In fact, (Br) holds in a more general context too, as has been shown by Y.-H. Au-Yeung [7]:

(AY) *If* $n \geqslant 3$ *and* $P = P' \in R_{nn}$ *is positive definite, then* $R_P(S,T) := \{(x'Sx, x'Tx) | x \in R^n,\ x'Px = 1\}$ *is convex.*

This result also holds over C and the quaternions for hermitian pairs.

R. Loewy—in private communication—remarked that the set $C(H,K) := \{(x^*Hx, x^*Kx) | x \in C^n,\ \|x\| = 1\} \subseteq R^2$ is closed and convex as well for every n, since W_{H+iK} is closed and convex. Hence Taussky's argument above can be applied to prove (III) for hermitian matrix pairs and arbitrary n. The proof of (SLS) hinges on the same principle again.

It should be noted here that an effective criterion for checking whether $0 \in R(S,T)$ [or $C(H,K)$] has recently been developed by C. S. Ballantine in [8, Fact 2.3]; see specifically Remark 1.10 on p. 127 there.

In [3] and [4], Au-Yeung takes still another cross-section of $N(S,T)$: He studies the map

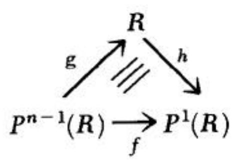
$$f(x) = \left(\frac{x'Sx}{\|(x'Sx, x'Tx)\|}, \frac{x'Tx}{\|(x'Sx, x'Tx)\|} \right) \qquad \text{for} \quad x \notin Q_S \cap Q_T.$$

For finite $n \geqslant 3$ and real symmetric S and T, or for $\infty \geqslant n \geqslant 2$ and H,K complex (or quaternion) hermitian, Au-Yeung [3, Theorem 1] shows by algebraic manipulations with two simultaneous quadratic equations that the image under f is either the whole real unit circle, an arc (relatively open, half-open or closed) of length $\leqslant \pi$ or two opposite points. In [3, Theorem 2], the result about semidefinite pencils of E. Çalabi [14, Theorem 2] is reestablished, while in [4] it is shown for n as above but finite that $\operatorname{Im} f$ is a closed arc for an angle less than 180° if $Q_S \cap Q_T = \{0\}$ (or $Q_H \cap Q_K = \{0\}$). From this, (I) and (III) are deduced in [4, p. 547], as well as (PM) in [4, Lemma 2].

Assuming that $Q_S \cap Q_T = \{0\}$, Calabi [14] considered the same mapping f for $x \in R^n$ with $\|x\| = 1$ as a map of real projective spaces P^k, namely, $f: P^{n-1}(R) \to P^1(R)$. His proof uses concepts from algebraic topology. Since the fundamental group satisfies

$$\Pi_1(P^{r-1}(R)) = \begin{cases} Z_2 & \text{for} \quad r > 2, \\ Z & \text{otherwise} \end{cases}$$

(see e.g. Schubert [50, p. 282]), the induced map f_* between the fundamental groups must be trivial. Thus $f_*(\Pi_1(P^{n-1}(R))) = \{0\}$ is clearly contained in $\Pi_1(P^1(R))$, so that one can factor f via the universal covering map h of R onto $P^1(R)$:



Calabi then shows that $L := h(g(P^{n-1}(R))) \subsetneq P^1(R)$ and that L is closed, two results that are reminiscent of Au-Yeung's results in [4, Theorem 1] and [3, Theorem 1]. Continuing with the proof of (I), the relative complement of L is nonempty and open, and a definite linear combination of S and T can be exhibited. This is then applied to give a slightly weaker result than (CKKS) in [14, Theorem 2] for infinite dimensional spaces.

Unfortunately, though, Calabi's proof does not carry over to hermitian matrix pairs, since there $\Pi_1(P^k(C))=\{0\}$ for all k (see e.g. Schubert [50, p. 282]). But Calabi's method seems to come closest mathematically to Finsler's original proof [21], done in homogeneous coordinates too. Finsler [21] uses the fact that a quadratic hypersurface Q_S is arcwise connected for $n \geqslant 3$—this same fact helped Calabi show that $L \not\cong P^1(R)$ [14, p. 845, first paragraph]. Since the sets $\{x \in R^n | x'Sx > 0\}$ and $\{x \in R^n | x'Sx < 0\}$ are also arcwise connected, then if $Q_S \cap Q_T = \{0\}$, Q_S and Q_T cannot "cross" each other, or—in Bliss's formulation—"$x'Sx$ does not change sign for $x \in Q_T$." Finsler then completes his proof by continuity arguments on $Q_{S+\lambda T}$ as λ ranges over $R \cup \{\infty\}$. In [21, Satz 4] he also gives results for more than two forms as long as $2 \leqslant n \leqslant 4$.

The proof of (CKKS) is based on a powerful lemma about pairs of "inner products," which we state for finite dimensions here only :

(KKS) *Let* $H = H^*$ *be indefinite and* $K = K^*$ *be arbitrary. If for all* $x \in Q_H$ $x^*Kx \geqslant 0$, *then* $y^*Ky/y^*Hy \leqslant z^*Kz/z^*Hz$ *for every pair* $y,z \in C^n$ *with* $y^*Hy < 0, z^*Hz > 0$.

This follows easily from Hestenes's observation in [28] that $N(H,K)$ is always convex. R. Kühne [34, Lemma 1.2] and M. G. Krein and J. L. Smuljan [33, Theorem 1.1] give proofs that hold in Hilbert spaces too (see J. Bognar [12, Lemma 6.1, p. 42]). Their proof, in fact, also underlies Au-Yeung's methods in [4, p. 546].

The proofs of (II) stemming from Bliss's seminar (i.e., [2], [27] and [47]) generally seem to be less involved with modern concepts like algebraic topology, convexity theory etc. Albert [2] determines explicitly all real λ such that $S-\lambda T$ is definite by simultaneously diagonalizing the generators of a d-pencil. His proof of (II) is by induction on n. Hestenes and McShane, in their proof of (II) for r forms in [27], use minimax type arguments on the quadratic form $x'(S+\Sigma_i\lambda_i T_i)x$ by considering $\max_{\lambda_i} \min_{\|x\|=1} x'(S+\Sigma_i\lambda_i T_i)x$ in [27, Lemmas A, B, Theorem 1].

In order to prove (II), Reid [47] shows analytically that the roots of $g(\lambda) := \det(S-\lambda T)$ are all simple and real, given that $x'Sx$ does not change sign for $x \in Q_T$. Reid then specifies an interval $A \subseteq R$ from the roots of g such that $S-\lambda T$ is positive definite for all $\lambda \in A$ [47, p. 440].

Very recently, K. N. Majindar [40] re-proved (I), (III) and (PM) all by elementary means. For his proof the pencil $P(S,T)$ [or $P(H,K)$] is reduced to a suitable equivalent form from which it is easy to read off the properties of $g(\lambda) = \det(S-\lambda T)$ if $Q_S \cap Q_T = \{0\}$, i.e., Reid's results [47, Lemmas 1,2,3]. Finsler's theorem is then proved by induction.

RECENT EXTENSIONS

Extensions of Finsler's original results (I) and (II) to the hermitian, quaternion hermitian, ν-hermitian and Hilbert space cases and to semidefinite and indefinite pencils as well as to more than two forms have already been treated in the previous sections. In this section we will deal with rather new extensions in the following directions:

(a) How else can semidefinite or indefinite pencils be characterized?

(b) How does Theorem (PM) of Pesonen and Milnor relate to Finsler's Theorems (I) and (II)?

(c) Do Finsler type results hold for arbitrary base fields F?

(d) The stability analysis of the generalized eigenvalue problem $Hx=\lambda Kx$.

Several aspects of these questions cannot be answered fully as of today. We will try to give as complete a picture as possible of the answers that do exist already, but many open questions will remain.

(a)

Two symmetric matrices S,T form a *nonsingular pair* if S is nonsingular. $P(S,T)$ is called a *nonsingular pencil* if there is a nonsingular matrix in $P(S,T)$.

With these notions, the following answer to question (a) has been obtained by the author in [65]—an answer that has no apparent relationship to Dines's classification of s.d. and i-pencils (D2) in [19], but seems much closer in spirit to (I) than (D2):

(U1) *Let* $l:=\max\{k|\text{there exist } k \text{ lin. indep. vectors in } Q_S\cap Q_T\}$. *If* $n\geqslant 3$, *then*

(a) $P(S,T)$ *is a d-pencil iff* $l=0$,

and for a nonsingular pair S,T:

(b) $P(S,T)$ *is a s.d. pencil iff* $1\leqslant l\leqslant n-1$ *and in case* $l=n-1$, S *and* T *are simultaneously congruent to*

$$\pm\operatorname{diag}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},\varepsilon_3,\ldots,\varepsilon_n\right)\quad and\quad \pm\operatorname{diag}\left(\begin{pmatrix}0&\lambda\\\lambda&1\end{pmatrix},\varepsilon_3\lambda,\ldots,\varepsilon_n\lambda\right)\qquad \text{(D)}$$

for $n\geqslant 4$, $\lambda\in R$, $\varepsilon_i=\pm 1$ *such that* $\varepsilon_m\varepsilon_k=-1$ *for at least one pair of indices*

$3 \leqslant m, k \leqslant n$, *or to*

$$\operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n) \quad \textit{and} \quad \operatorname{diag}(\varepsilon_1\lambda, \ldots, \varepsilon_{n-1}\lambda, \varepsilon_n\mu), \tag{E}$$

with $\lambda, \mu \in \mathrm{R}$, $\lambda \neq \mu$, $\varepsilon_j = \pm 1$ and $\varepsilon_m \varepsilon_k = -1$ for some $1 \leqslant m, k \leqslant n-1$.

(c) *$P(S,T)$ is an i-pencil iff $n-1 \leqslant l \leqslant n$ and in case $l = n-1$, S and T are simultaneously congruent to*

$$\pm \operatorname{diag}\left[\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, 1, \ldots, 1\right]$$

$$\textit{and} \quad \pm \operatorname{diag}\left[\begin{bmatrix} 0 & 0 & \lambda \\ 0 & \lambda & 1 \\ \lambda & 1 & 0 \end{bmatrix}, \lambda, \ldots, \lambda\right] \tag{A}$$

with $\lambda \in R$, or to

$$\pm \operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1, \ldots, 1\right)$$

$$\textit{and} \quad \pm \operatorname{diag}\left(\begin{pmatrix} b & a \\ a & -b \end{pmatrix}, \lambda, \ldots, \lambda\right) \tag{B}$$

with $a, b, \lambda \in R$ and $b \neq 0$.

The actual proof in [64], [65] and [66] is based on the author's thesis [63] and specifically on the real canonical pair form as developed there and in [61] for nonsingular real symmetric pairs S, T. Such pair forms go back more than 100 years to both Kronecker and Weierstrass. Recently R. C. Thompson [59, 60] has worked on a canonical pair form for arbitrary symmetric or hermitian pairs S and T or H and K. The resulting canonical form is similar to Uhlig's [61] for the nonsingular part of S and T, while the singular parts of S and T can be made congruent to a more complicated but still very sparse block configuration. Thus one may hope that a result similar to Theorem (U1) can be proved by means analogous to those in [64], [65] and [66], for both arbitrary symmetric pairs S and T and hermitian pairs H and K.

(b)

It is classical knowledge that S and T can be diagonalized simultaneously by real congruence if one of S or T is definite. Thus Theorem (PM) becomes obvious, once Finsler's theorem (I) is presupposed.

Clearly neither S nor T need be definite in order that S and T can be diagonalized simultaneously. In fact, the following theorem—most likely due to Greub [25, p. 255]—holds regarding simultaneous diagonability (see also Gantmacher [24, Vol. 2, Theorem 7, p. 43], Uhlig [62, 68], M. Wonenburger [72], Rao and Mitra [73, Chapter 6], Au-Yeung [74] and Becker [75]).

(Gr) *A nonsingular pair of symmetric matrices S and T over an arbitrary field F can be diagonalized simultaneously by F-congruence iff $S^{-1}T$ is similar to a diagonal matrix over F.*

Specifically, for $F = R, n \geq 3$ and a nonsingular pair S and T, we thus obtain two levels of "if and only if" theorems with the following implications in between:

(I):	$Q_S \cap Q_T = \{0\}$	iff	$P(S,T)$ is a d-pencil
	Personen-Milnor $\Downarrow$ (PM)	Weierstrass $\swarrow$ (W)	Taussky $\Downarrow$ ((T))
(Gr):	S and T are simultaneously diagonable	iff	$S^{-1}T$ is similar to a real diagonal matrix

The implication ((T)) attributed to Taussky in the diagram is part of Taussky's theorem from [57], [58]:

(T) *Let A be a real square matrix. Then there exists P positive definite and T symmetric such that $A = P^{-1}T$ iff A is similar to a real diagonal matrix.*

And the theorem associated with Weierstrass goes back to his 1868 paper [71]. This paper—historically speaking—is the earliest work on a real canonical pair form for nonsingular symmetric pairs. The following appears as a corollary there [71, p. 337, 338]:

(W) *If $P(S,T)$ is a d-pencil, then S and T are simultaneously diagonable over R by congruence.*

The problem of simultaneous diagonability for nonsingular pairs S and T has been further extended via the real and rational pair form theorems in [61], [67] and [68] to treat the concept of the finest simultaneous block diagonal structure that can be obtained by congruence for a given pair S and T; see Uhlig [62, 67, 68]. Au-Yeung [5, 6] has obtained results on simultaneous two-block diagonalizations for arbitrary real symmetric, hermitian and quaternion hermitian matrix pairs. His results—considered only for the nonsingular real symmetric case—are weaker than and contained in the corresponding results in [66]. The methods used are different, though.

(c)

Here we will mention some current results for arbitrary fields F with $\operatorname{char} F \neq 2$, namely by W. Givens on anisotropic matrix pencils, by the author on a generalization of Taussky's theorem (T) and by W. C. Waterhouse on classifying fields via (T).

W. Givens (unpublished) considers the Lyapunov map

$$L_A(G) := \tfrac{1}{2}(GA + A'G)$$

for nonsingular symmetric $G \in F_{nn}$ and gives the

Definition. Let $G, H \in F_{nn}$ be symmetric and $\operatorname{char} F \neq 2$.

(i) Then we set $\mathfrak{A}(G,H) := \{A \mid L_A(G) = H\}$. Moreover,

(ii) G is called *anic* if $x'Gx = 0$ for $x \in F^n$ implies $x = 0$.

(iii) G is called *positive definite* if $x'Gx > 0$ for all $x \neq 0$ and all orderings of F, provided F is formally real.

(iv) $\operatorname{index} G := \max\{\dim V \mid V \subseteq F^n \text{ subspace with } v'Gv = 0 \text{ for all } v \in V\}$.

Givens is able to classify both anic matrices and anic pencils via the set $\mathfrak{A}(G,H)$:

(Gi1) *H is anic iff every nonzero matrix in $\mathfrak{A}(G,H)$ is nonsingular.*

(Gi2) *$P(G,H)$ is an anic pencil with $H - aG$ anic iff no nonzero matrix in $\mathfrak{A}(G,H)$ has $a \in F$ as an eigenvalue.*

Via the rational canonical pair form theorem for nonsingular symmetric matrix pairs S, T in [67], [68], the author started to investigate generalizations of (T) for arbitrary fields F with $\operatorname{char} F \neq 2$ in [67]. The current results—to be published elsewhere in detail—are these:

(U2) (a) *If $A \in F_{nn}$ has elementary divisors $p_i^{t_i}$ over F, then every nonsingular symmetric $S \in F_{nn}$ with $SA = A'S$ satisfies*

$$\operatorname{index} S \geqslant \sum_i \deg p_i \left[\frac{t_i}{2} \right].$$

(b) *If* $S=S'\in F_{nn}$ *is nonsingular, then the elementary divisors* $p_i^{t_i}$ *of every* $A\in F_{nn}$ *with* $SA=A'S$ *satisfy*

$$\sum_i \deg p_i\left[\frac{t_i}{2}\right]\leqslant \operatorname{index} S.$$

Here $[\cdot]$ denotes the greatest integer function.

Hence this partial analog to Taussky's theorem (T) holds for arbitrary fields:

COROLLARY. *If* $S=S'$ *is anic, then every* A *with* $SA=A'S$ *is* F*-similar to a direct sum of companion matrices for* F*-irreducible polynomials.*

This result was also found by C. S. Ballantine (unpublished) and by W. C. Waterhouse [70, Proposition 1], where in addition it is shown via Springer's theorem (see e.g. T. Y. Lam [36, Theorem 2.3, p. 198]) that the characteristic polynomial of any such A must be separable.

An intriguing phenomenon occurs when the "greatest integer" brackets $[\cdot]$ are applied differently in the formulas of (U2). W. C. Waterhouse [70, Theorem 8, Proposition 9], using his version of the rational pair form in [69], has shown:

(Wa1) (a) *If* F *is hereditarily euclidean and* $S=S'\in F_{nn}$, *then the elementary divisors* $p_i^{t_i}$ *of all* $A\in F_{nn}$ *with* $SA=A'S$ *satisfy*

$$\operatorname{index} S\geqslant \sum_i\left[\deg p_i\frac{t_i}{2}\right].$$

(b) *If* $\sum_i[\deg p_i t_i/2]\leqslant \operatorname{index} S$ *for all* $A\in F_{nn}$ *such that* $SA=A'S$ *for a fixed* $S=S'$, *then* F *is hereditarily euclidean or every element of* F *is a square.*

For an introduction to euclidean and hereditarily euclidean fields see A. Prestel and M. Ziegler [46]. For $F=R$, such relations between the index of a symmetrizer S and the elementary divisors of a symmetrized matrix A were first obtained about 80 years ago by F. Klein [31, Section 16] and A. Loewy [38].

Waterhouse [70, Theorem 2, Proposition 6] was also able to settle part of the open question from [67, Sec. 7]: classify the fields for which (T) holds.

(Wa2) (a) *If*

(T anic) *For* $A\in F_{nn}$ *there exists an anic* P *with* $PA=A'P$ *iff* A *is* F*-diagonable* *holds over F for every n, then F is euclidean or every element in F is a square.*

(b) *If F is formally real, then*

(T pos. def.) *For $A \in F_{nn}$ there exists a positive definite P with $PA = A'P$ iff A is F-diagonable*

holds for F and every n iff F is the intersection of its real closures.

(d)

The stability analysis of the definite generalized eigenvalue problem, i.e. of $Hx = \lambda Kx$ with $P(H,K)$ a d-pencil, has recently been linked to our Main Theorem by G. W. Stewart in [53]. One sets

$$c(H,K) := \min_{(a,b) \in C(H,K)} \|(a,b)\|,$$

where $C(H, K) := \{(x^*Hx, x^*Kx) | x \in C^n, \|x\| = 1\}$ as before. But instead of the relation $c(H,K) > 0$ alone as in (I), the actual value of $c(H,K)$ plays the key role here. This was first noticed by C. R. Crawford in [15].

Stewart [53, Theorem 2.2] proves that for a d-pencil there is a matrix $H_0 \in P(H,K)$ with $\lambda_{\min}(H_0) = c(H,K)$. Since $c(H+E, K+F) \geqslant c(H,K) - (\|E\|^2 + \|F\|^2)^{1/2}$ (see [53, Theorem 2.4]), the perturbed problem $(H+E)x = \lambda(K+F)x$ remains definite as long as E and F are small relative to $c(H,K)$. Thus Stewart [53, Theorem 3.2] can prove:

(St) *If the definite problem $Hx = \lambda Kx$ has eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$, and E and F are such that the perturbed problem $(H+E)x = \lambda(K+F)x$ is definite, then*

$$|\lambda_i - \tilde{\lambda}_i| \leqslant \arcsin \frac{(\|E\|^2 + \|F\|^2)^{1/2}}{c(H,K)},$$

where $\tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \cdots \leqslant \tilde{\lambda}_n$ are the eigenvalues for the perturbed problem.

If $\lambda = \mu/\nu$, $\tilde{\lambda} = \tilde{\mu}/\tilde{\nu}$, then this result can also be expressed in terms of the chordal metric

$$\mathrm{X}(\lambda, \tilde{\lambda}) := \frac{|\mu\tilde{\nu} - \tilde{\mu}\nu|}{\|(\mu,\nu)\| \, \|(\tilde{\mu},\tilde{\nu})\|},$$

namely,

$$\mathrm{X}(\lambda_i, \tilde{\lambda}_i) \leqslant \frac{(\|E\|^2 + \|F\|^2)^{1/2}}{c(H,K)}$$

(see [53, p. 13]). Stewart [53, Chapter 4] also studies perturbation bounds for the associated eigenspaces via the chordal metric. For earlier uses of the chordal metric with respect to perturbation analysis see Stewart [51, 52].

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