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On a class of hemivariational inequalities at resonance

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Abstract

We consider a class of noncoercive hemivariational inequalities involving the p-Laplacian at resonance. We use the unilateral growth condition so the energy functional is nonsmooth, nonconvex and its effective domain does not coincide with the whole space $W_0^{1,p}(\Omega)$. To avoid this difficulty we study the problem in finite-dimensional spaces using the mountain-pass theorem for locally Lipschitz functionals and then we pass to the limit to obtain the existence of solutions.

Keywords: Noncoercive hemivariational inequality; Resonance; Critical point theory; Locally Lipschitz functionals; Unilateral growth condition

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a compact connected C^2-boundary $\partial\Omega$. The problem under consideration is as follows: Find $u \in W^{1,p}(\Omega)$ such that

$$\begin{cases} \operatorname{div}(|Du(x)|^{p-2}Du(x)) + \lambda_1|u(x)|^{p-2}u(x) \in \partial j(x, u(x)) \quad \text{a.e. on } \Omega, \\ u|_{\partial\Omega} = 0, \quad 2 \leqslant p < \infty. \end{cases} \tag{1.1}$$

By $\partial j(x, u)$ we denote the generalized gradient for locally Lipschitz functionals due to Clarke [4]. For the right hand side of (1.1) we suppose only that it satisfies the unilateral

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growth condition due to Naniewicz [14]. Thus the functions $j^0(\cdot, u; v)$ and $j(\cdot, u)$ is not in general summable for every $u, v \in W_0^{1,p}(\Omega)$. Therefore the energy functional has no longer as its effective domain the whole space $W_0^{1,p}(\Omega)$, so we cannot use directly the mountain-pass theorem but we have to study the problem in finite-dimensional spaces (subspaces of $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$), in which we can use the mountain-pass theorem and then pass to the limit using the Dunford–Pettis criterion.

In order to prove that our energy functional satisfies the (PS) condition we use an extended Poincaré inequality which appears very recently in the paper of Fleckinger-Pellé and Takáč [7]. So for this purpose we assume that our boundary is a compact connected C^2-manifold.

Hemivariational inequalities have been introduced by Panagiotopoulos (cf. [17,18], see also [13,16]) in order to describe mechanical problems with nonmonotone and multivalued conditions. For hemivariational inequalities with resonance involving the classical growth conditions we refer to [8,11,13].

Let us recall some facts and definitions from the critical point theory for locally Lipschitz functionals and the subdifferential of Clarke [4].

Let Y be a subset of a Banach space X. A function $f : Y \to \mathbb{R}$ is said to satisfy a Lipschitz condition (on Y) provided that, for some nonnegative scalar K, one has

$$\bigl|f(y) - f(x)\bigr| \leqslant K\|y - x\|_X$$

for all points $x, y \in Y$. Let f be Lipschitz near a given point x, and let v be any other vector in X. The generalized directional derivative of f at x in the direction v, denoted by $f^0(x; v)$, is defined as follows:

$$f^0(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t},$$

where y is a vector in X and t a positive scalar. If f is Lipschitz of rank K near x then the function $v \to f^0(x; v)$ is finite, positively homogeneous, subadditive and satisfies the conditions $|f^0(x; v)| \leqslant K\|v\|_X$ and $f^0(x; -v) = (-f)^0(x; v)$. Now we are ready to introduce the generalized gradient $\partial f(x)$ defined by [4]

$$\partial f(x) = \bigl\{w \in X^*:\ f^0(x; v) \geqslant \langle w, v\rangle_X \text{ for all } v \in X\bigr\}.$$

Some basic properties of the generalized gradient of locally Lipschitz functionals are the following:

(a) $\partial f(x)$ is a nonempty, convex, weakly compact subset of $X^\star$ and $\|w\|_{X^\star} \leqslant K$ for every w in $\partial f(x)$.
(b) For every v in X, one has

$$f^0(x; v) = \max\bigl\{\langle w, v\rangle:\ w \in \partial f(x)\bigr\}.$$

If f_1, f_2 are locally Lipschitz functions then

$$\partial(f_1 + f_2) \subseteq \partial f_1 + \partial f_2.$$

Let us recall the (PS) condition introduced by Chang [3].

Definition 1.1. We say that a Lipschitz function f satisfies the Palais–Smale condition if any sequence $\{x_n\}$ along which $|f(x_n)|$ is bounded and

$$\lambda(x_n) = \min_{w \in \partial f(x_n)} \|w\|_{X^\star} \to 0$$

possesses a convergent subsequence.

The (PS) condition can also be formulated as follows (see Costa and Goncalves [5]).

$(\mathrm{PS})^*_{c,+}$ Whenever $(x_n) \subseteq X$, $(\varepsilon_n), (\delta_n) \subseteq R_+$ are sequences with $\varepsilon_n \to 0$, $\delta_n \to 0$, and such that

$$f(x_n) \to c,$$
$$f(x_n) \leqslant f(x) + \varepsilon_n \|x - x_n\| \quad \text{if } \|x - x_n\| \leqslant \delta_n,$$

then (x_n) possesses a convergent subsequence: $x_{n'} \to \hat{x}$.

Similarly, we define the $(\mathrm{PS})^*_c$ condition from below, $(\mathrm{PS})^*_{c,-}$, by interchanging x and x_n in the above inequality. Finally we say that f satisfies $(\mathrm{PS})^*_c$ provided it satisfies $(\mathrm{PS})^*_{c,+}$ and $(\mathrm{PS})^*_{c,-}$.

Note that these two definitions are equivalent when f is locally Lipschitz functional.

Let us mention some facts about the first eigenvalue of the p-Laplacian. Consider the first eigenvalue λ_1 of $(-\Delta_p, W_0^{1,p}(\Omega))$. From Lindqvist [10] we know that $\lambda_1 > 0$ is isolated and simple, that any two solutions u, v of

$$\begin{cases} -\Delta_p u := -\operatorname{div}(|Du|^{p-2} Du) = \lambda_1 |u|^{p-2} u & \text{a.e. on } \Omega, \\ u\,|_{\partial\Omega} = 0, \quad 2 \leqslant p < \infty, \end{cases} \tag{1.2}$$

satisfy $u = cv$ for some $c \in \mathbb{R}$. In addition, the λ_1-eigenfunctions do not change sign in Ω. Finally we have the following variational characterization of λ_1 (Rayleigh quotient):

$$\lambda_1 = \inf\left[\frac{\|Du\|_p^p}{\|u\|_p^p}\colon\ u \in W_0^{1,p}(\Omega),\ u \neq 0\right].$$

We are going to use the mountain-pass theorem of Chang [3] and the generalization of the Poincaré inequality of Fleckinger-Pellé and Takáč [7]: There exists a positive constant $c > 0$ such that

$$\int_\Omega |Du|^p\,dx - \lambda_1 \int_\Omega |u|^p\,dx \geqslant c\left(|e|^{p-2} \int_\Omega |D\theta|^{p-2} |D\hat{u}|^2\,dx + \int_\Omega |D\hat{u}|^p\,dx\right),$$
$$\forall u \in W_0^{1,p}(\Omega), \tag{1.3}$$

where λ_1 is the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$, θ is the λ_1-eigenfunction and $u = e\theta + \hat{u}$ is an orthogonal decomposition of u in $L^2(\Omega)$, $e = \|\theta\|^{-2}_{L^2(\Omega)} \langle u, \theta\rangle_{L^2(\Omega)}$, $\langle \hat{u}, \theta\rangle_{L^2(\Omega)} = 0$.

Theorem 1.1. *If a locally Lipschitz functional $f : X \to \mathbb{R}$ on the reflexive Banach space X satisfies the* (PS) *condition and the hypotheses*

(i) *there exist positive constants* ρ *and* a *such that*

$$f(u) \geqslant a \quad \text{for all } u \in X \text{ with } \|u\| = \rho;$$

(ii) $f(0) = 0$ *and there is a point* $e \in X$ *such that*

$$\|e\| > \rho \quad \text{and} \quad f(e) \leqslant 0,$$

then there exists a critical value $c \geqslant a$ *of* f *determined by*

$$c = \inf_{g \in G} \max_{t \in [0,1]} f\big(g(t)\big),$$

where

$$G = \big\{g \in C\big([0,1], X\big)\colon\ g(0) = 0,\ g(1) = e\big\}.$$

2. Preliminary results

Let us denote by $V_0 = \{s\theta\}_{s \in \mathbb{R}}$ the one-dimensional eigenspace spanned by the eigenfunction θ corresponding to the first eigenvalue λ_1 of $(-\Delta_p, W_0^{1,p}(\Omega))$, normalized by $\theta > 0$ in Ω and $\|\theta\|_{W_0^{1,p}(\Omega)} = 1$. Due to Anane [1] we have $\theta \in L^\infty(\Omega)$. By $V^\perp$ we denote the orthogonal complement in $L^2(\Omega)$ of V_0. Thus for any $u \in W_0^{1,p}(\Omega)$ the decomposition follows

$$u = e\bar{\theta} + \hat{u} \quad \text{with } e \geqslant 0,\ \bar{\theta} \in \{\pm\theta\} \subset V_0,\ \hat{u} \in \hat{V}, \tag{2.1}$$

where $\hat{V} := V^\perp \cap W_0^{1,p}(\Omega)$.

Lemma 2.1. *Assume that*

(H0) $j(\cdot, 0) \in L^1(\Omega)$ *and* $j(x, \cdot)$ *is Lipschitz continuous on the bounded subsets of* $\mathbb{R}$ *uniformly with respect to* $x \in \Omega$, *i.e.,* $\forall r > 0\ \exists K_r > 0$ *such that* $\forall |y_1|, |y_2| \leqslant r$,

$$\big|j(x, y_1) - j(x, y_2)\big| \leqslant K_r |y_1 - y_2|, \quad \text{for a.e. } x \in \Omega;$$

(H1) *There exist* $\mu > p$, $1 \leqslant \sigma < p$, $a \in L^1(\Omega)$ *and a constant* $k \geqslant 0$ *such that*

$$\mu j(x, \xi) - j^0(x, \xi; \xi) \geqslant -a(x) - k|\xi|^\sigma, \quad \forall \xi \in \mathbb{R} \text{ and for a.e. } x \in \Omega;$$

(H2) *Assume that*

$$\liminf_{\substack{t \to +\infty \\ \eta \to \bar{\theta}}} \frac{1}{t^{p-1}} \int_\Omega -j^0\big(x, t\eta(x); -\bar{\theta}(x)\big)\, dx > 0, \quad \forall \bar{\theta} \in V_0 \text{ with } \|\bar{\theta}\|_{W_0^{1,p}(\Omega)} = 1.$$

Moreover, suppose that for a sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ *there exists* $\varepsilon_n \searrow 0$ *such that the conditions below are fulfilled*:

$$\int_\Omega \left|Du_n(x)\right|^{p-2}\left\langle Du_n(x), Dv(x) - Du_n(x)\right\rangle_{\mathbb{R}^N} dx - \lambda_1 \int_\Omega \left|u_n(x)\right|^{p-2} u_n(x)\left(v(x) - u_n(x)\right) dx + \int_\Omega j^0\left(x, u_n(x); v(x) - u_n(x)\right) dx \geqslant -\varepsilon_n \|v - u_n\|_{W_0^{1,p}(\Omega)}, \quad \forall v \in \operatorname{Lin}\left(\{u_n, \theta\}\right), \tag{2.2}$$

and

$$\frac{1}{p}\int_\Omega \left|Du_n(x)\right|^p dx - \frac{\lambda_1}{p}\int_\Omega \left|u_n(x)\right|^p dx + \int_\Omega j\left(x, u_n(x)\right) dx \leqslant C, \quad C > 0, \tag{2.3}$$

where $\operatorname{Lin}(\{u_n, \theta\})$ *is the linear subspace of* $W_0^{1,p}(\Omega)$ *spanned by* $\{\theta, u_n\}$. *Then the sequence* $\{u_n\}$ *is bounded in* $W_0^{1,p}(\Omega)$, *i.e., there exists* $M > 0$ *such that*

$$\|u_n\|_{W_0^{1,p}(\Omega)} \leqslant M. \tag{2.4}$$

Proof. Suppose on the contrary that the claim is not true, i.e., there exists a sequence $\{u_n\}_{n=1}^\infty \subset W_0^{1,p}(\Omega)$ with $\|u_n\|_{W_0^{1,p}(\Omega)} \to \infty$ for which (2.2) and (2.3) hold. Combining (2.3) and (2.2) with $v = 2u_n$ yields

$$C + \varepsilon_n \|u_n\|_{W_0^{1,p}(\Omega)} \geqslant \frac{\mu - p}{p}\left(\|Du_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1 \|u_n\|^p_{L^p(\Omega)}\right) + \int_\Omega \left(\mu j(u_n) - j^0(u_n; u_n)\right) dx. \tag{2.5}$$

By the generalization of the Poincaré inequality (1.3) the decomposition results in $u_n = e_n\theta_n + \hat{u}_n$, where $\hat{u}_n \in \hat{V}$, $e_n \geqslant 0$, $\theta_n \in \{\pm\theta\}$, $\|\theta\|_{W_0^{1,p}(\Omega)} = 1$, such that

$$\|Du_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1 \|u_n\|^p_{L^p(\Omega)} \geqslant c\left(e_n^{p-2}\int_\Omega |D\theta_n|^{p-2}|D\hat{u}_n|^2 dx + \|D\hat{u}_n\|^p_{L^p(\Omega;\mathbb{R}^N)}\right). \tag{2.6}$$

Thus by (H1) we have

$$C + \varepsilon_n \|u_n\|_{W_0^{1,p}(\Omega)} \geqslant c\frac{\mu - p}{p} e_n^{p-2}\int_\Omega |D\theta|^{p-2}|D\hat{u}_n|^2 dx + c\frac{\mu - p}{p}\left\|D(\hat{u}_n)\right\|^p_{L^p(\Omega;\mathbb{R}^N)} - c_1\|u_n\|^\sigma_{L^p(\Omega)}. \tag{2.7}$$

Hence

$$C+\varepsilon_n\big(\|\hat u_n\|_{W_0^{1,p}(\Omega)}+e_n\big)\geqslant c\frac{\mu-p}{p}\|D\hat u_n\|^p_{L^p(\Omega;\mathbb{R}^N)}-c_1\|\hat u_n\|^\sigma_{L^p(\Omega)}-c_2e_n^\sigma. \tag{2.8}$$

Thus it follows that $e_n\to\infty$ because, otherwise, we would have the boundedness of $\{u_n\}$ in $W_0^{1,p}(\Omega)$. Consequently we arrive at the estimate

$$\begin{aligned}\frac{C}{e_n}+\varepsilon_n\left(\left\|\frac{\hat u_n}{e_n}\right\|_{W_0^{1,p}(\Omega)}+1\right)\geqslant e_n^{p-1}c\frac{\mu-p}{p}\left\|D\left(\frac{\hat u_n}{e_n}\right)\right\|^p_{L^p(\Omega;\mathbb{R}^N)}\\ -e_n^{\sigma-1}c_1\left\|\frac{\hat u_n}{e_n}\right\|^\sigma_{L^p(\Omega)}-e_n^{\sigma-1}c_2,\end{aligned} \tag{2.9}$$

which in view of $e_n\to\infty$ leads to the conclusion that

$$\left\|\frac{\hat u_n}{e_n}\right\|_{W_0^{1,p}(\Omega)}\to 0. \tag{2.10}$$

Now let us turn back to (2.2). By passing to a subsequence one can suppose also that $\theta_n=\theta$ (or $\theta_n=-\theta$). Thus, substituting $v=\hat u_n$ into (2.2) yields

$$\begin{aligned}&e_n^p\int_\Omega\left|D\left(\frac{\hat u_n}{e_n}\right)+D\theta\right|^{p-2}\left\langle D\left(\frac{\hat u_n}{e_n}\right)+D\theta,-D\theta\right\rangle_{\mathbb{R}^N}dx\\&\quad-e_n^p\lambda_1\int_\Omega\left|\frac{\hat u_n}{e_n}+\theta\right|^{p-2}\left(\frac{\hat u_n}{e_n}+\theta\right)(-\theta)\,dx+e_n\int_\Omega j^0\left(e_n\left(\frac{\hat u_n}{e_n}+\theta\right);-\theta\right)dx\\&\quad\geqslant-\varepsilon_ne_n.\end{aligned}$$

Hence

$$\begin{aligned}\varepsilon_n\geqslant\frac{1}{e_n^{p-1}}&\int_\Omega -j^0\left(e_n\left(\frac{\hat u_n}{e_n}+\theta\right);-\theta\right)dx\\&+\int_\Omega\left|D\left(\frac{\hat u_n}{e_n}\right)+D\theta\right|^{p-2}\left\langle D\left(\frac{\hat u_n}{e_n}\right)+D\theta,D\theta\right\rangle_{\mathbb{R}^N}dx\\&-\lambda_1\int_\Omega\left|\frac{\hat u_n}{e_n}+\theta\right|^{p-2}\left(\frac{\hat u_n}{e_n}+\theta\right)\theta\,dx.\end{aligned} \tag{2.11}$$

Now we are ready to pass to the limit with $n\to\infty$. For this purpose notice that in view of (2.10) it results

$$\begin{aligned}\lim_{n\to\infty}\Bigg\{&\int_\Omega\left|D\left(\frac{\hat u_n}{e_n}\right)+D\theta\right|^{p-2}\left\langle D\left(\frac{\hat u_n}{e_n}\right)+D\theta,D\theta\right\rangle_{\mathbb{R}^N}dx\\&-\lambda_1\int_\Omega\left|\frac{\hat u_n}{e_n}+\theta\right|^{p-2}\left(\frac{\hat u_n}{e_n}+\theta\right)\theta\,dx\Bigg\}=\|D\theta\|^p_{L^p(\Omega;\mathbb{R}^N)}-\lambda_1\|\theta\|^p_{L^p(\Omega)}=0,\end{aligned}$$

and by (iii) we have

$$\liminf_{n\to\infty} \frac{1}{e_n^{p-1}} \int_\Omega -j^0\left(e_n\left(\frac{\hat{u}_n}{e_n}+\theta\right); -\theta\right) dx > 0.$$

Thus from (2.11) we arrive at the inequality $0 > 0$ which is a contradiction. Thus the proof of Lemma 2.1 is complete. □

Lemma 2.2. *Assume that* (H0) *and the hypotheses below hold*:

(H3) *The unilateral growth condition* [14]: *there exist* $p < q < p^* = Np/(N-p)$, *and a constant* $\kappa \geqslant 0$ *such that*

$$j^0(x,\xi;-\xi) \leqslant \kappa\big(1+|\xi|^q\big), \quad \forall \xi \in \mathbb{R} \text{ and for a.e. } x \in \Omega;$$

(H4) *Uniformly for a.e.* $x \in \Omega$,

$$\liminf_{\xi\to 0} \frac{pj(x,\xi)}{|\xi|^p} \geqslant \phi(x) \geqslant 0,$$

with $\phi(x) \in L^\infty(\Omega)$ *and* $\phi(x) > 0$ *on a set of positive measure.*

Then there exists $\rho > 0$ *such that*

$$\mathcal{R}(u) := \frac{1}{p}\|Du\|^p_{L^p(\Omega;\mathbb{R}^N)} - \frac{\lambda_1}{p}\|u\|^p_{L^p(\Omega)} + \int_\Omega j(u)\,dx \geqslant \eta, \quad \eta = \text{const} > 0, \tag{2.12}$$

is valid for any $u \in W_0^{1,p}(\Omega)$ *with* $\|u\|_{W_0^{1,p}(\Omega)} = \rho$.

Proof. Suppose the assertion is not true. Thus there exist sequences $\{u_n\} \subset W_0^{1,p}(\Omega)$ and $\rho_n \searrow 0$ such that $\|u_n\|_{W_0^{1,p}(\Omega)} = \rho_n$ and $\mathcal{R}(u_n) \leqslant \rho_n^{p+1}$. So we have

$$\|Du_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|u_n\|^p_{L^p(\Omega)} + \int_\Omega pj(u_n)\,dx \leqslant p\rho_n^{p+1}. \tag{2.13}$$

Further, from (H4) it follows that for any $\varepsilon > 0$, uniformly for all $x \in \Omega$ one can find $\delta > 0$ such that

$$pj(x,\xi) \geqslant \phi(x)|\xi|^p - \varepsilon|\xi|^p, \quad |\xi| \leqslant \delta.$$

Moreover, (H3) allows to conclude that (see Lemma 2.1 in [15, pp. 119–120])

$$j(x,\xi) \geqslant -\kappa_0\big(1+|\xi|^q\big), \quad \forall \xi \in \mathbb{R},\ \kappa_0 = \text{const} > 0. \tag{2.14}$$

Thus it is easy to see that

$$pj(x,\xi) \geqslant \big(\phi(x)-\varepsilon\big)|\xi|^p - \gamma|\xi|^q, \quad \forall \xi \in \mathbb{R}, \tag{2.15}$$

for some positive $\gamma = \gamma(\delta) > 0$. Then by (2.13) it follows

$$\|Du_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|u_n\|^p_{L^p(\Omega)} + \int\limits_\Omega \big(\phi(x)-\varepsilon\big)\big|u_n(x)\big|^p\,dx$$
$$\leqslant p\rho_n^{p+1} + \gamma\int\limits_\Omega \big|u_n(x)\big|^q\,dx. \tag{2.16}$$

Let us set $y_n = (1/\rho_n)u_n$. Dividing inequality (2.16) by ρ_n^p yields

$$\|Dy_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|y_n\|^p_{L^p(\Omega)} + \int\limits_\Omega \big(\phi(x)-\varepsilon\big)\big|y_n(x)\big|^p\,dx$$
$$\leqslant p\rho_n + \gamma\rho_n^{q-p}\int\limits_\Omega \big|y_n(x)\big|^q\,dx. \tag{2.17}$$

Since $W_0^{1,p}(\Omega)$ is continuously embedded into $L^q(\Omega)$ we have

$$\|Dy_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|y_n\|^p_{L^p(\Omega)} + \int\limits_\Omega \big(\phi(x)-\varepsilon\big)\big|y_n(x)\big|^p\,dx$$
$$\leqslant p\rho_n + \gamma_1\rho_n^{q-p}, \quad \gamma_1 = \text{const} > 0. \tag{2.18}$$

Taking into account that $\|y_n\|_{W_0^{1,p}(\Omega)} = 1$ we can suppose that for a subsequence (again denoted by the same symbol) $y_n \to y$ weakly in $W_0^{1,p}(\Omega)$ and $y_n \to y$ strongly in $L^p(\Omega)$ (the Rellich theorem) for some $y \in W_0^{1,p}(\Omega)$. Passing to the limit and the weak lower semicontinuity of the norm allows the conclusion

$$\|Dy\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|y\|^p_{L^p(\Omega)} + \int\limits_\Omega \big(\phi(x)-\varepsilon\big)\big|y(x)\big|^p\,dx \leqslant 0, \tag{2.19}$$

which is valid for an arbitrary $\varepsilon > 0$. Therefore we get

$$\|Dy\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|y\|^p_{L^p(\Omega)} + \int\limits_\Omega \phi(x)\big|y(x)\big|^p\,dx \leqslant 0. \tag{2.20}$$

Using the Rayleigh quotient characterization of λ_1 and (H4) leads to the equalities

$$\|Dy\|^p_{L^p(\Omega;\mathbb{R}^N)} = \lambda_1\|y\|^p_{L^p(\Omega)}, \tag{2.21}$$
$$\int\limits_\Omega \phi(x)\big|y(x)\big|^p\,dx = 0. \tag{2.22}$$

Now we show that $y \neq 0$. Indeed, from the results obtained it follows that

$$\|Dy_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|y_n\|^p_{L^p(\Omega)} \to 0,$$

and by the compactness of the embedding $W_0^{1,p}(\Omega) \subset L^p(\Omega)$ we get

$$\|y_n\|_{L^p(\Omega)} \to \|y\|_{L^p(\Omega)}.$$

Since $\|Dy_n\|_{L^p(\Omega;\mathbb{R}^N)} \geqslant c\|y_n\|_{W_0^{1,p}(\Omega)} = c$, $c > 0$ (the equivalence of the norms), we arrive at $\lambda_1\|y\|^p_{L^p(\Omega)} \geqslant c^p$ which establishes the assertion. Therefore, taking into account (2.21) we conclude that $y \neq 0$ is an λ_1-eigenfunction. Since $\phi(x) > 0$ on a set of positive measure (by (H4)), and, as it is well known (cf. [10]), $|y(x)| > 0$ for a.e. $x \in \Omega$, we are led to the contradiction with (2.22). The proof of Lemma 2.2 is complete. □

Lemma 2.3. *Assume that* (H0)–(H1) *hold and that*

(H5) $\int_\Omega j(x,0)\,d\Omega \leqslant 0$ *and either for some* $\bar\theta \in V_0$, $\bar\theta \neq 0$,

$$\liminf_{s\to+\infty} \int_\Omega j\big(x, s\bar\theta(x)\big)\,dx < 0, \tag{2.23}$$

or there exists $v_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *such that*

$$\liminf_{s\to+\infty} s^{-\sigma} \int_\Omega j\big(x, s v_0(x)\big)\,dx < \frac{k}{\sigma-\mu}\|v_0\|^\sigma_{L^\sigma(\Omega)}, \tag{2.24}$$

with the positive constants k, μ, σ *entering* (H1).

Then there exists $e \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $e \neq 0$, *such that*

$$\mathcal{R}(se) \leqslant 0, \quad \forall s \geqslant 1.$$

Proof. If (2.23) is fulfilled then the assertion holds for $e = s_0\bar\theta$ with sufficiently large $s_0 > 0$.

For the case (2.24) we follow the lines of [13]. For all $\tau \neq 0$, $x \in \Omega$ and $\xi \in \mathbb{R}$, the formula below of generalized gradient (with respect to τ) holds:

$$\partial_\tau\big(\tau^{-\mu} j(x,\tau\xi)\big) = \tau^{-\mu-1}\big[-\mu j(x,\tau\xi) + \partial_\xi j(x,\tau\xi)(\tau\xi)\big],$$

for the constant $\mu > p$ fulfilling (H1). Since the function $\tau \mapsto \tau^{-\mu} j(x,\tau\xi)$ is differentiable a.e. on $\mathbb{R}$, the equality above and a classical property of Clarke's generalized directional derivative imply that

$$\begin{aligned} t^{-\mu} j(x,t\xi) - j(x,\xi) &= \int_1^t \frac{d}{d\tau}\big(\tau^{-\mu} j(x,\tau\xi)\big)\,d\tau \\ &\leqslant \int_1^t \tau^{-\mu-1}\big[-\mu j(x,\tau\xi) + j^0(x,\tau\xi;\tau\xi)\big]\,d\tau, \quad \forall t > 1,\ \text{a.e. } x \in \Omega,\ \xi \in \mathbb{R}. \end{aligned}$$

In view of assumption (H1) we infer that

$$t^{-\mu} j(x,t\xi) - j(x,\xi) \leqslant \int_1^t \tau^{-\mu-1}\big[a(x) + k\tau^\sigma|\xi|^\sigma\big]\,d\tau$$

$$= \left[a(x)\left(-\frac{1}{\mu}t^{-\mu} + \frac{1}{\mu}\right) + k|\xi|^{\sigma}\left(\frac{1}{\sigma-\mu}t^{\sigma-\mu} - \frac{1}{\sigma-\mu}\right)\right]$$
$$\leqslant \mu^{-1}a(x) + (\mu-\sigma)^{-1}k|\xi|^{\sigma}, \quad \forall t > 1,\ \text{a.e. } x \in \Omega,\ \xi \in \mathbb{R}. \tag{2.25}$$

Set $\xi = sv_0(x)$ with $x \in \Omega$ and $s > 0$. We find from (2.25) the estimate

$$j\big(x, tsv_0(x)\big) \leqslant t^{\mu}\big[j\big(x, sv_0(x)\big) + \mu^{-1}a(x) + (\mu-\sigma)^{-1}ks^{\sigma}\big|v_0(x)\big|^{\sigma}\big],$$
$$\forall t > 1,\ s > 0,\ \text{a.e. } x \in \Omega. \tag{2.26}$$

Combining (2.26) with (2.24) yields

$$\mathcal{R}(tsv_0) \leqslant \frac{1}{p}t^p s^p\big(\|Dv_0\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|v_0\|^p_{L^p(\Omega)}\big)$$
$$+ t^{\mu}s^{\sigma}\Bigg[s^{-\sigma}\int_{\Omega} j\big(x, sv_0(x)\big)\,dx + k(\mu-\sigma)^{-1}\|v_0\|^{\sigma}_{L^{\sigma}(\Omega)}$$
$$+ s^{-\sigma}\mu^{-1}\|a\|_{L^1(\Omega)}\Bigg], \quad \forall t > 1,\ s > 0. \tag{2.27}$$

Assumption (2.24) allows to fix some number $s_0 > 0$ such that

$$s_0^{-\sigma}\int_{\Omega} j\big(x, s_0v_0(x)\big)\,dx + k(\mu-\sigma)^{-1}\|v_0\|^{\sigma}_{L^{\sigma}(\Omega)} + s_0^{-\sigma}\mu^{-1}\|a\|_{L^1(\Omega)} < 0. \tag{2.28}$$

With such an $s_0 > 0$ we can pass to the limit as $t \to +\infty$ in (2.27) and obtain (in view of $\mu > p$) that $\mathcal{R}(ts_0v_0) \to -\infty$ as $t \to +\infty$. Consequently, setting $e = t_0s_0v_0$ with sufficiently large $t_0 > 0$ we establish the assertion. This completes the proof of Lemma 2.3. □

3. Finite-dimensional approximation

Let us denote by Λ the family of all finite-dimensional subspaces F of $W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ satisfying the conditions:

$$F \in \Lambda \quad \Leftrightarrow \quad F = V_0 + \hat{F} \text{ for some finite-dimensional subspace } \hat{F} \subset \hat{V} \cap L^{\infty}(\Omega)$$
$$\text{and } e \in F, \tag{3.1}$$

with $e \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ as explained in Lemma 2.3.

For every subspace $F \in \Lambda$ we introduce the functional $\mathcal{R}_F : F \to \mathbb{R}$ which is the restriction of $\mathcal{R}$ to F, i.e.,

$$\mathcal{R}_F(v) = \frac{1}{p}\|v\|^p_{L^p(\Omega;\mathbb{R}^N)} - \frac{\lambda_1}{p}\|v\|^p_{L^p(\Omega)} + \int_{\Omega} j\big(x, v(x)\big)\,dx, \quad \forall v \in F. \tag{3.2}$$

It is obvious that the functional $\mathcal{R}_F$ is locally Lipschitz and its generalized gradient is expressed by

$$\partial\mathcal{R}_F(v) \subset i_F^{\star}Ai_Fv + \bar{i}_F^{\star}\partial J(v), \quad \forall v \in F, \tag{3.3}$$

where $i_F : F \to W_0^{1,p}(\Omega)$ and $\bar{i}_F : F \to L^\infty(\Omega)$ are the inclusion maps with their dual projections $i_F^\star : W^{-1,p'}(\Omega) \to F^\star$ and $\bar{i}_F^\star : L^1(\Omega) \to F^\star$, respectively, while $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is defined by

$$\langle Au, v\rangle_{W_0^{1,p}(\Omega)} = \int_\Omega |Du|^{p-2}\langle Du, Dv\rangle_{\mathbb{R}^N}\, d\Omega - \lambda_1 \int_\Omega |u|^{p-2}uv\, d\Omega, \quad u, v \in W_0^{1,p}(\Omega). \tag{3.4}$$

By $\partial J(\cdot)$ the generalized Clarke gradient of $J : L^\infty(\Omega) \to \mathbb{R}$ given by

$$J(v) = \int_\Omega j\big(x, v(x)\big)\, dx, \quad \forall v \in L^\infty(\Omega),$$

has been denoted. Notice that in view of (H0), J is locally Lipschitz on $L^\infty(\Omega)$, so the generalized gradient $\partial J(\cdot)$ is well defined. The pairing over $F^\star \times F$ will be denoted by $\langle\cdot,\cdot\rangle_F$.

Proposition 3.1. *Assume the hypotheses* (H0)–(H5)*. Then for each $F \in \Lambda$ problem (P_F): Find $u_F \in F$ such as to satisfy the hemivariational inequality*

$$\int_\Omega |Du_F|^{p-2}\langle Du_F, Dv - Du_F\rangle_{\mathbb{R}^N}\, d\Omega - \lambda_1 \int_\Omega |u_F|^{p-2}u_F(v - u_F)\, d\Omega + \int_\Omega j^0(u_F; v - u_F)\, d\Omega \geqslant 0, \quad \forall v \in F, \tag{3.5}$$

has at least one solution $u_F \neq 0$. Moreover, there exist constants $M > 0$, $\gamma_1 > 0$ and $\gamma_2 > 0$ not depending on $F \in \Lambda$ such that

$$\|u_F\|_{W_0^{1,p}(\Omega)} \leqslant M, \quad \forall F \in \Lambda, \tag{3.6}$$

$$\gamma_1 \leqslant \mathcal{R}(u_F) \leqslant \gamma_2, \quad \forall F \in \Lambda. \tag{3.7}$$

Proof. First we show that the functional $\mathcal{R}_F : F \to \mathbb{R}$ satisfies the Palais–Smale condition in the sense of Chang [3]. Let $\{u_n\} \subset F$ and $\{w_n\} \subset F^\star$ be sequences such that $|\mathcal{R}_F(u_n)| \leqslant c$, for all $n \geqslant 1$, with a constant $c > 0$, and $w_n \in \partial\mathcal{R}_F(u_n)$, $\|w_n\|_{F^\star} = \varepsilon_n \to 0$ as $n \to \infty$. Since F is finite-dimensional, it remains to show that $\{u_n\}$ is bounded in F. According to (3.3) we see that w_n can be expressed as follows:

$$w_n = i_F^\star Au_n + \bar{i}_F^\star \chi_n, \quad \text{with } \chi_n \in \partial J(u_n). \tag{3.8}$$

Let us notice that the hypothesis of Theorem 2.7.3 in [4, p. 80] is verified. Therefore we obtain

$$\partial J(v) \subset \int_\Omega \partial j\big(x, v(x)\big)\, dx, \quad \forall v \in L^\infty(\Omega). \tag{3.9}$$

Thus

$$\langle Au_n, v-u_n\rangle_{W_0^{1,p}(\Omega)} + \int_\Omega j^0(u_n; v-u_n)\,d\Omega \geqslant \langle w_n, v-u_n\rangle_F \geqslant -\varepsilon_n\|v-u_n\|_F$$
$$\geqslant -c\varepsilon_n\|v-u_n\|_{W_0^{1,p}(\Omega)}, \quad \forall v\in F,\ c=\text{const}>0,$$

because the norms $\|\cdot\|_F$ and $\|\cdot\|_{W_0^{1,p}(\Omega)}$ are equivalent in F (F is finite-dimensional). Since $\mathrm{Lin}(\theta, u_n)\subset F$, the hypotheses of Lemma 2.1 are verified. Consequently $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ which means that

$$\|u_F\|_{W_0^{1,p}(\Omega)} \leqslant M_F \tag{3.10}$$

for some $M_F>0$.

Following the lines of the proof of Lemma 2.2 (with $W_0^{1,p}(\Omega)$ replaced by F) we conclude the existence of positive constants $\rho_F>0$ and $\eta_F>0$ such that

$$\mathcal{R}_F(v)\geqslant \eta_F, \quad \forall v\in\bigl\{w\in F\colon \|w\|_F=\rho_F\bigr\}. \tag{3.11}$$

By Lemma 2.3 we know that $\mathcal{R}(te)\leqslant 0$ for any $t\geqslant 1$, therefore $\rho_F<\|e\|_F$. Thus taking into account that $\mathcal{R}_F(0)\leqslant 0$ and $\mathcal{R}_F(e)\leqslant 0$ we are allowed to apply the mountain-pass theorem and deduce the existence of a critical point $u_F\in F$ of $\mathcal{R}_F$. This leads to the finite-dimensional hemivariational inequality (3.5) (cf. [13]).

Let us recall that the critical value $\mathcal{R}_F(u_F)$ is characterized by (cf. [13])

$$\mathcal{R}_F(u_F)=\inf_{\gamma\in C_F}\max_{t\in[0,1]}\mathcal{R}_F\bigl(\gamma(t)\bigr), \tag{3.12}$$

where

$$C_F=\bigl\{\gamma\in C\bigl([0,1],F\bigr)\colon \gamma(0)=0,\ \gamma(1)=e\bigr\}$$

is the family of all continuous curves in F joining points 0 and e in F, i.e., $\gamma(0)=0$ and $\gamma(1)=e$, $\gamma(t)\subset F$. Further, from Lemma 2.2 it follows that for a certain positive $\rho>0$ one can find $\eta>0$ with

$$\mathcal{R}(v)\geqslant\eta, \quad \forall v\in S_\rho := \bigl\{v\in W_0^{1,p}(\Omega)\colon \|v\|_{W_0^{1,p}(\Omega)}=\rho\bigr\}, \tag{3.13}$$

while Lemma 2.3 ensures the existence of $e\in W_0^{1,p}(\Omega)$, $e\neq 0$, such that

$$\mathcal{R}(te)\leqslant 0, \quad \forall t\geqslant 1. \tag{3.14}$$

Therefore, for any $F\in\Lambda$, if $\gamma\in C_F([0,1];F)$ then γ meets points of S_ρ which means that

$$\max_{t\in[0,1]}\mathcal{R}_F\bigl(\gamma(t)\bigr)\geqslant\eta. \tag{3.15}$$

Hence

$$\eta\leqslant\mathcal{R}(u_F)=\inf_{\gamma\in C_F}\max_{t\in[0,1]}\mathcal{R}_F\bigl(\gamma(t)\bigr)\leqslant\max_{t\in[0,1]}\mathcal{R}(te), \quad \forall F\in\Lambda, \tag{3.16}$$

and (3.7) results.

Now we are ready to show that $M_F>0$ in (3.10) is independent of $F\in\Lambda$. For this purpose suppose that a sequence $\{u_{F_n}\}_{F_n\in\Lambda}$ of solutions of (P_{F_n}) has the property that

$\|u_{F_n}\|_{W_0^{1,p}(\Omega)} \to \infty$. Taking into account (3.5) and (3.16) it is easy to check that the hypotheses (2.3) and (2.2) of Lemma 2.1 hold (with F replaced by F_n and $\varepsilon_n = 0$). Following the lines of the proof of Lemma 2.1 we arrive at the contradiction which establishes the assertion. The proof of Proposition 3.1 is complete. □

For the restriction of J to F, $J_F := J|_F : F \to \mathbb{R}$, we have $\partial J_F(u_F) \subset \bar{i}_F^{\star} \partial J(u_F)$. Therefore Proposition 3.1 can be reformulated as follows.

Corollary 3.1. *Assume the hypotheses* (H0)–(H5). *Then for each $F \in \Lambda$ there exist $u_F \in F$ and $\chi_F \in L^1(\Omega)$ such that*

$$\int_\Omega |Du_F|^{p-2} \langle Du_F, Dv - Du_F\rangle_{\mathbb{R}^N}\, d\Omega - \lambda_1 \int_\Omega |u_F|^{p-2} u_F (v - u_F)\, d\Omega + \int_\Omega \chi_F (v - u_F)\, d\Omega = 0, \quad \forall v \in F, \text{ and } \chi_F \in \partial j(u_F) \text{ a.e. in } \Omega. \tag{3.17}$$

According to the results obtained we know that to any $F \in \Lambda$ a pair $(u_F, \chi_F) \in F \times L^1(\Omega)$ can be assigned for which (3.17) holds. Moreover, the family $\{u_F\}_{F\in\Lambda}$ is uniformly bounded in $W_0^{1,p}(\Omega)$ ((3.6) holds). The question arises concerning the behavior of $\{\chi_F\}_{F\in\Lambda}$.

Proposition 3.2. *Assume that $(u_F, \chi_F) \in F \times L^1(\Omega)$ satisfies* (3.17). *Then the set $\{\chi_F\}_{F\in\Lambda}$ is weakly precompact in $L^1(\Omega)$.*

Proof. Since Ω is bounded, according to the Dunford–Pettis theorem (see, e.g., [6, p. 239]) it suffices to show that for each $\varepsilon > 0$ a number $\delta > 0$ can be determined such that for any $\omega \subset \Omega$ with $|\omega| < \delta$,

$$\int_\omega |\chi_F|\, dx < \varepsilon, \quad \forall F \in \Lambda. \tag{3.18}$$

Choose $\bar{q} \in (q, p^\star)$. Then the injection $W_0^{1,p}(\Omega) \subset L^{\bar{q}}(\Omega)$ is compact. Further, from (H3) it follows that there exists a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that (cf. Remark 5.6 [16, p. 156] and Lemma 1 [12, p. 95])

$$j^0(x, \xi; \eta - \xi) \leqslant \alpha(r)\big(1 + |\xi|^q\big), \quad \forall \xi, \eta \in \mathbb{R},\ |\eta| \leqslant r,\ r \geqslant 0. \tag{3.19}$$

Fix $r > 0$ and let $\eta \in \mathbb{R}$ be such that $|\eta| \leqslant r$. Then, by (3.17), $\chi_F(\eta - u_F) \leqslant j^0(x, u_F; \eta - u_F)$, from which we get

$$\chi_F\, \eta \leqslant \chi_F u_F + \alpha(r)\big(1 + |u_F|^q\big) \quad \text{for a.e. } x \in \Omega. \tag{3.20}$$

Let us set $\eta \equiv r \operatorname{sgn} \chi_F(x)$ where $\operatorname{sgn} y = 1$ if $y > 0$, $\operatorname{sgn} y = 0$ if $y = 0$, $\operatorname{sgn} y = -1$ if $y < 0$. One obtains that $|\eta| \leqslant r$ and $\chi_F(x)\eta = r|\chi_F(x)|$ for almost all $x \in \Omega$. Therefore from (3.20) it results

$$r|\chi_F| \leqslant \chi_F u_F + \alpha(r)\big(1 + |u_F|^q\big).$$

Integrating this inequality over $\omega \subset \Omega$ yields

$$\int_\omega |\chi_F|\,dx \leqslant \frac{1}{r}\int_\omega \chi_F u_F\,dx + \frac{1}{r}\alpha(r)|\omega| + \frac{1}{r}\alpha(r)|\omega|^{(\bar{q}-q)/\bar{q}}\|u_F\|^q_{L^{\bar{q}}(\Omega)}. \tag{3.21}$$

Consequently, from (3.6) and (3.21) it follows that

$$\int_\omega |\chi_F|\,dx \leqslant \frac{1}{r}\int_\omega \chi_F u_F\,dx + \frac{1}{r}\alpha(r)|\omega| + \frac{1}{r}\alpha(r)|\omega|^{(\bar{q}-q)/\bar{q}}\gamma^q M^q, \tag{3.22}$$

where $\gamma > 0$ is a constant satisfying $\|\cdot\|_{L^{\bar{q}}(\Omega)} \leqslant \gamma\|\cdot\|_{H^1_0(\Omega)}$ (which holds since $\hat{q} < p^\star$).

We claim

$$\int_\omega \chi_F u_F\,dx \leqslant C \tag{3.23}$$

for some positive constant C not depending on $\omega \subset \Omega$ and $F \in \Lambda$. Indeed, from (3.19) we derive that

$$\chi_F u_F + \alpha(0)\big(|u_F|^q + 1\big) \geqslant 0 \quad \text{for a.e. in } \Omega.$$

Thus it follows

$$\begin{aligned}\int_\omega \chi_F u_F\,dx &\leqslant \int_\omega \big(\chi_F u_F + \alpha(0)\big(|u_F|^q + 1\big)\big)\,dx \\ &\leqslant \int_\Omega \big(\chi_F u_F + \alpha(0)\big(|u_F|^q + 1\big)\big)\,dx \\ &\leqslant \int_\Omega \chi_F u_F\,dx + \bar{k}_1\big(\|u_F\|^q_{H^1_0(\Omega)} + |\Omega|\big),\end{aligned}$$

where $\bar{k}_1 > 0$ is a constant. By (3.6) and (3.17) (with $v = 0$) it turns out that

$$\int_\Omega \chi_F u_F\;dx = -\int_\Omega |Du_F|^p\,dx + \lambda_1\int_\Omega |u_F|^p\,dx \leqslant 0.$$

The estimates above imply (3.23).

Further, (3.22) and (3.23) entail

$$\int_\omega |\chi_F|\,dx \leqslant \frac{1}{r}C + \frac{1}{r}\alpha(r)|\omega| + \frac{1}{r}\alpha(r)|\omega|^{(\bar{q}-q)/\bar{q}}\gamma^q M^q, \quad \forall r > 0. \tag{3.24}$$

Corresponding to $\varepsilon > 0$, fix $r > 0$ with

$$\frac{1}{r}C < \frac{\varepsilon}{2} \tag{3.25}$$

and then take $\delta > 0$ small enough to have

$$\frac{1}{r}\alpha(r)|\omega| + \frac{1}{r}\alpha(r)|\omega|^{(\bar{q}-q)/\bar{q}}\gamma^q M^q < \frac{\varepsilon}{2} \tag{3.26}$$

provided that $|\omega| < \delta$. Using this together with (3.24) and (3.25) it follows that (3.18) is justified whenever $|\omega| < \delta$. This completes the proof. □

4. Main result

To formulate the main result we shall need the following hypothesis:

(H6) For any sequence $\{v_k\} \subset L^\infty(\Omega)$, $v_k \to 0$ strongly in $L^p(\Omega)$, if

$$\int_\Omega \min\{\psi(x)v_k(x)\colon\ \psi(x) \in \partial j(x, v_k(x))\}\, d\Omega \leqslant 0,$$

then

$$\limsup_{k\to\infty} \int_\Omega j(x, v_k(x))\, d\Omega \leqslant 0.$$

Theorem 4.1. *Assume the hypotheses* (H0)–(H6). *Then there exists* $u \in W_0^{1,p}(\Omega)$ *with* $u \neq 0$ *and* $j(u) \in L^1(\Omega)$, *such as to satisfy the hemivariational inequality*

$$\int_\Omega |Du|^{p-2}\langle Du, Dv - Du\rangle_{\mathbb{R}^N}\, d\Omega - \lambda_1 \int_\Omega |u|^{p-2}u(v-u)\, d\Omega + \int_\Omega j^0(u; v-u)\, d\Omega \geqslant 0, \quad \forall v \in W_0^{1,p}(\Omega). \tag{4.1}$$

Moreover, there exists $\chi \in L^1(\Omega)$ *with the property that*

$$\int_\Omega |Du|^{p-2}\langle Du, Dv - Du\rangle_{\mathbb{R}^N}\, d\Omega - \lambda_1 \int_\Omega |u|^{p-2}u(v-u)\, d\Omega + \int_\Omega \chi(v-u)\, d\Omega = 0, \quad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{4.2}$$

$$\chi u \in L^1(\Omega) \quad \textit{and} \quad \chi \in \partial j(u) \quad \textit{a.e. in } \Omega. \tag{4.3}$$

Proof. The proof is carried out in a sequence of steps.

Step 1. For every $F \in \Lambda$ we introduce

$$U_F = \{u_F \in W_0^{1,p}(\Omega)\colon \text{ for some } \chi_F \in L^1(\Omega; \mathbb{R}^N),\ (u_F, \chi_F) \text{ is a solution of } (P_F)\}$$

and

$$W_F = \bigcup_{\substack{F' \in \Lambda \\ F' \supset F}} U_{F'}.$$

By Proposition 3.1, W_F is nonempty (even U_F is nonempty) and contained in the ball $B_M = \{v \in W_0^{1,p}(\Omega)\colon \|v\|_{W_0^{1,p}(\Omega)} \leqslant M\}$. We denote by $\mathrm{weakcl}(W_F)$ the closure of W_F in the weak topology of $W_0^{1,p}(\Omega)$. Proposition 3.1 ensures that $\mathrm{weakcl}(W_F)$ is weakly compact in $W_0^{1,p}(\Omega)$. We claim that the family $\{\mathrm{weakcl}(W_F)\}_{F\in\Lambda}$ has the finite intersection property. Indeed, if $F_1,\dots,F_k \in \Lambda$ then $W_{F_1} \cap \cdots \cap W_{F_k} \supset W_F$, with $F = F_1 + \cdots + F_k$ and the assertion follows. Thus we are allowed to conclude that there exists an element $u \in W_0^{1,p}(\Omega)$ with

$$u \in \bigcap_{F\in\Lambda} \mathrm{weakcl}(W_F).$$

Let us choose $G \in \Lambda$ arbitrarily. Since $W_0^{1,p}(\Omega)$ is reflexive, one can extract an increasing sequence of subspaces $\{G_n\}$, each containing G, and for each n an element $u_n \in U_{G_n}$ such that $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$ as $n \to \infty$ (Proposition 11 [2, p. 274]). Let us denote by $\{\chi_n\} \subset L^1(\Omega)$ the corresponding sequence with the property that for each n a pair (u_n,χ_n) is a solution of (P_{G_n}). By Proposition 3.2 we can suppose without loss of generality that $\chi_n \to \chi^G$ weakly in $L^1(\Omega)$ for some $\chi^G \in L^1(\Omega)$. Thus we have asserted that

$$u_n \to u \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{4.4}$$

$$\chi_n \to \chi^G \quad \text{weakly in } L^1(\Omega), \tag{4.5}$$

and that (3.17) with F replaced by G_n reads

$$\langle Au_n, v-u_n\rangle_{W_0^{1,p}(\Omega)} + \int_\Omega \chi_n(v-u_n)\,d\Omega = 0, \quad \forall v \in G_n, \tag{4.6}$$

where $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is defined by (3.4).

Step 2. Now we prove that $\chi^G \in \partial j(u)$ a.e. in Ω. Since $W_0^{1,p}(\Omega)$ is compactly embedded into $L^p(\Omega)$, due to (3.6) one may suppose that

$$u_n \to u \quad \text{strongly in } L^p(\Omega). \tag{4.7}$$

This implies that for a subsequence of $\{u_n\}$ (again denoted by the same symbol) one gets $u_n \to u$ a.e. in Ω. Thus Egoroff's theorem can be applied from which it follows that for any $\varepsilon > 0$ a subset $\omega \subset \Omega$ with $|\omega| < \varepsilon$ can be determined such that $u_n \to u$ uniformly in $\Omega\setminus\omega$ with $u \in L^\infty(\Omega\setminus\omega)$. Let $v \in L^\infty(\Omega\setminus\omega)$ be an arbitrary function. From the estimate

$$\int_{\Omega\setminus\omega} \chi_n v\,d\Omega \leqslant \int_{\Omega\setminus\omega} j^0(u_n;v)\,d\Omega$$

combined with the weak convergence in $L^1(\Omega)$ of χ_n to χ^G, (4.7) and with the upper semicontinuity of

$$L^\infty(\Omega\setminus\omega) \ni u_n \mapsto \int_{\Omega\setminus\omega} j^0(u_n;v)\,d\Omega$$

it follows

$$\int_{\Omega\setminus\omega} \chi^G v\,d\Omega \leqslant \int_{\Omega\setminus\omega} j^0(u;v)\,d\Omega, \quad \forall v \in L^\infty(\Omega\setminus\omega).$$

But the last inequality amounts to saying that $\chi^G \in \partial j(u)$ a.e. in $\Omega\setminus\omega$. Since $|\omega| < \varepsilon$ and ε was chosen arbitrarily,

$$\chi^G \in \partial j(u) \quad \text{a.e. in } \Omega, \tag{4.8}$$

as claimed.

Step 3. Now it will be shown that

$$\limsup_{n\to\infty} \int_\Omega j^0(u_n; v-u_n)\,d\Omega \leqslant \int_\Omega j^0(u; v-u)\,d\Omega \tag{4.9}$$

holds for any $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. It can be supposed that $u_n \to u$ a.e. in Ω, since $u_n \to u$ in $L^q(\Omega)$. Fix $v \in L^\infty(\Omega)$ arbitrarily. In view of $\chi_n \in \partial j(u_n)$ and (3.19) we get

$$j^0(u_n; v-u_n) \leqslant \alpha\big(\|v\|_{L^\infty(\Omega)}\big)\big(1+|u_n|^q\big). \tag{4.10}$$

From Egoroff's theorem it follows that for any $\varepsilon > 0$ a subset $\omega \subset \Omega$ with $|\omega| < \varepsilon$ can be determined such that $u_n \to u$ uniformly in $\Omega\setminus\omega$. One can also suppose that ω is small enough to fulfill $\int_\omega \alpha(\|v\|_{L^\infty(\Omega)})(1+|u_n|^q)\,d\Omega \leqslant \varepsilon$, $n = 1, 2, \ldots,$ and $\int_\omega \alpha(\|v\|_{L^\infty(\Omega)})(1+|u|^q)\,d\Omega \leqslant \varepsilon$. Hence

$$\int_\Omega j^0(u_n; v-u_n)\,d\Omega \leqslant \int_{\Omega\setminus\omega} j^0(u_n; v-u_n)\,d\Omega + \varepsilon$$

which by Fatou's lemma and upper semicontinuity of $j^0(\cdot\,;\cdot)$ yields

$$\limsup_{n\to\infty} \int_\Omega j^0(u_n; v-u_n)\,d\Omega \leqslant \int_\Omega j^0(u; v-u)\,d\Omega + 2\varepsilon.$$

By arbitrariness of $\varepsilon > 0$ one obtains (4.9), as required.

Step 4. Now we show that

$$\chi^G u \in L^1(\Omega), \tag{4.11}$$

$$\liminf_{n\to\infty} \int_\Omega \chi_n u_n\,d\Omega \geqslant \int_\Omega \chi^G u\,d\Omega. \tag{4.12}$$

For this purpose let $\{\epsilon_k\} \subset L^\infty(\Omega)$ be such that [9]

$$\big\{(1-\epsilon_k)u\big\} \subset W_0^{1,p}(\Omega)\cap L^\infty(\Omega), \quad 0 \leqslant \epsilon_k \leqslant 1,$$
$$\tilde{u}_k := (1-\epsilon_k)u \to u \quad \text{strongly in } W_0^{1,p}(\Omega) \text{ as } k\to\infty. \tag{4.13}$$

Without loss of generality it can be assumed that $\tilde{u}_k \to u$ a.e. in Ω. Since it is already known that $\chi^G \in \partial j(u)$, one can apply (H3) to obtain $\chi^G(-u) \leqslant j^0(u; -u) \leqslant \kappa(1+|u|^q)$. Hence

$$\chi^G \tilde{u}_k = (1-\epsilon_k)\chi^G u \geqslant -\kappa\bigl(1+|u|^q\bigr). \tag{4.14}$$

This implies that the sequence $\{\chi^G \tilde{u}_k\}$ is bounded from below by integrable function and $\chi^G \tilde{u}_k \to \chi^G u$ a.e. in Ω. On the other hand, one gets

$$\int_\Omega \chi_n(\tilde{u}_k - u_n)\,d\Omega \leqslant \int_\Omega j^0(u_n; \tilde{u}_k - u_n)\,d\Omega.$$

Thus passing to the limit with $n \to \infty$ yields

$$\int_\Omega \chi^G \tilde{u}_k\,d\Omega - \liminf_{n\to\infty} \int_\Omega \chi_n u_n\,d\Omega \leqslant \limsup_{n\to\infty} \int_\Omega j^0(u_n; \tilde{u}_k - u_n)\,d\Omega,$$

and due to (4.9) we are led to the estimate

$$\begin{aligned}\int_\Omega \chi^G \tilde{u}_k\,d\Omega &\leqslant \liminf_{n\to\infty} \int_\Omega \chi_n u_n\,d\Omega + \int_\Omega j^0(u; \tilde{u}_k - u)\,d\Omega \\ &\leqslant \liminf_{n\to\infty} \int_\Omega \chi_n u_n\,d\Omega + \int_\Omega j^0(u; -\epsilon_k u)\,d\Omega \\ &\leqslant \liminf_{n\to\infty} \int_\Omega \chi_n u_n\,d\Omega + \int_\Omega \epsilon_k \kappa\bigl(1+|u|^q\bigr)\,d\Omega \leqslant C, \quad C = \text{const}.\end{aligned}$$

Thus by Fatou's lemma we are allowed to conclude that $\chi^G u \in L^1(\Omega)$, i.e., (4.11) holds. Taking into account that $\epsilon_k \to 0$ a.e. in Ω as $k \to \infty$ (passing to a subsequence if necessary) we establish (4.12), as required.

Step 5. It will be shown that

$$\langle Au, v-u\rangle_{W_0^{1,p}(\Omega)} + \int_\Omega \chi^G(v-u)\,d\Omega = 0, \quad \forall v \in \bigcup_{n=1}^\infty G_n \supset G,\ \chi^G \in \partial j(u). \tag{Q^G}$$

Since A is bounded and $\{u_F\}_{F\in\Lambda} \subset \{v \in W_0^{1,p}(\Omega)\colon \|v\|_{W_0^{1,p}(\Omega)} \leqslant M\}$, there exists $K > 0$ such that $\{Au_F\}_{F\in\Lambda} \subset \{l \in W^{-1,p'}(\Omega)\colon \|l\|_{W^{-1,p'}(\Omega)} \leqslant K\}$. From (4.6) it follows that for any fixed $G \in \Lambda$ we get

$$\left|\int_\Omega \chi^G v\,d\Omega\right| \leqslant K\|v\|_V, \quad \forall v \in \bigcup_{n=1}^\infty G_n,\ \chi^G \in \partial j(u), \tag{4.15}$$

because $\{G_n\}$ is an increasing sequence. Further, by making use of (4.11) and (4.12) we have $\chi^G u \in L^1(\Omega)$ and

$$\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle_{W_0^{1,p}(\Omega)} \leqslant \int_\Omega \chi^G(v-u)\,d\Omega, \quad \forall v \in \bigcup_{n=1}^\infty G_n. \tag{4.16}$$

Since $u_n \in G_n$ and $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$, the closure of $\bigcup_{n=1}^\infty G_n$ in the strong topology of $W_0^{1,p}(\Omega)$, $\overline{\bigcup_{n=1}^\infty G_n}$, must contain u. Thus there exists a sequence $\{w_i\} \subset \bigcup_{n=1}^\infty G_n$ converging strongly to u in $W_0^{1,p}(\Omega)$ as $i \to \infty$. We claim that for such a sequence,

$$\int_\Omega \chi^G w_i\,d\Omega \to \int_\Omega \chi^G u\,d\Omega \quad \text{as } i \to \infty. \tag{4.17}$$

Indeed, let $\{\tilde{u}_k\}_{k=1}^\infty$ be given by (4.13). From (4.14) it follows

$$-\kappa\big(1+|u|^q\big) \leqslant \chi^G\tilde{u}_k \leqslant |\chi^G u|, \quad k = 1, 2, \ldots, \tag{4.18}$$

with the bounds $-\kappa(1+|u|^q)$ and $|\chi^G u|$ being integrable in Ω. Thus there exists a constant $C > 0$ such that

$$\left|\int_\Omega \chi^G\tilde{u}_k\,d\Omega\right| \leqslant C\|\tilde{u}_k\|_{W_0^{1,p}(\Omega)}, \quad k = 1, 2, \ldots. \tag{4.19}$$

Denote by $\mathcal{A}$ a linear subspace spanned by $\{\tilde{u}_k\}_{k=1}^\infty$ and define a linear functional $\hat{l}_{\chi^G} : \bigcup_{n=1}^\infty G_n + \mathcal{A} \to \mathbb{R}$ by the formula

$$\hat{l}_{\chi^G}(v) := \int_\Omega \chi^G v\,d\Omega, \quad v \in \bigcup_{n=1}^\infty G_n + \mathcal{A}.$$

Taking into account (4.15) and (4.19), from the Hahn–Banach theorem it follows that $\hat{l}_{\chi^G}$ admits its linear continuous extension onto $W_0^{1,p}(\Omega)$, $l_{\chi^G} \in W^{-1,p'}(\Omega)$. By the dominated convergence,

$$\int_\Omega \chi^G\tilde{u}_k\,d\Omega \to \int_\Omega \chi^G u\,d\Omega, \quad \text{as } k \to \infty,$$

so we get $l_{\chi^G}(u) = \int_\Omega \chi^G u\,d\Omega$ which, in particular, implies (4.17), as claimed.

Taking into account (4.16) and (4.17) we conclude

$$\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle_{W_0^{1,p}(\Omega)} \leqslant 0, \tag{4.20}$$

which by the pseudomonotonicity of A implies

$$Au_n \to Au \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{4.21}$$

$$\langle Au_n, u_n\rangle_{W_0^{1,p}(\Omega)} \to \langle Au, u\rangle_{W_0^{1,p}(\Omega)}. \tag{4.22}$$

Hence from (4.6) we are led to (Q^G), as desired. Notice that (4.21) and (4.22) imply the strong convergence $u_n \to u$ in $W_0^{1,p}(\Omega)$.

Step 6. It remains to show that there exists $\chi \in \partial j(u)$ with the associated linear functional defined by

$$\hat{l}_\chi(v) := \int_\Omega \chi v \, d\Omega, \quad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

admitting a continuous extension $l_\chi \in W^{-1,p'}(\Omega)$ such that

$$Au + l_\chi = 0, \qquad \langle l_\chi, u \rangle_{W_0^{1,p}(\Omega)} = \int_\Omega \chi u \, d\Omega. \tag{4.23}$$

For every $G \in \Lambda$ let us introduce

$$V^{(G)} = \{\chi^G \in L^\infty(\Omega): \ (Q^G) \text{ holds}\}$$

and

$$Z^{(G)} = \bigcup_{\substack{G' \in \Lambda \\ G' \supset G}} V^{(G')}.$$

As in the proof of Proposition 3.2 we show that the family $\{\chi^G\}_{G \in \Lambda}$ is weakly precompact in $L^1(\Omega)$. Denoting by weakcl$(Z^{(G)})$ the closure of $Z^{(G)}$ in the weak topology of $L^1(\Omega)$ we prove analogously that the family $\{\text{weakcl}(Z^{(G)})\}_{G \in \Lambda}$ has the finite intersection property. Thus there exists an element $\chi \in \partial j(u)$ such that for any $G \in \Lambda$ it holds

$$\langle Au, v \rangle_{W_0^{1,p}(\Omega)} + \int_\Omega \chi v \, d\Omega = 0, \quad \forall v \in G.$$

Since $G \in \Lambda$ has been chosen arbitrarily and Λ is dense in $W_0^{1,p}(\Omega)$, (4.23) results, as desired.

Step 7. It remains to show (4.1). From (4.2) we obtain easily its validity for any $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Let us consider the case $j^0(u; v-u) \in L^1(\Omega)$ with $v \in W_0^{1,p}(\Omega)$. There exists a sequence $\tilde{v}_k = (1-\epsilon_k)v$ such that $\{\tilde{v}_k\} \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\tilde{v}_k \to v$ strongly in $W_0^{1,p}(\Omega)$. Since, as already has been established,

$$\langle Au, \tilde{v}_k - u \rangle_{W_0^{1,p}(\Omega)} + \int_\Omega j^0(u; \tilde{v}_k - u) \, d\Omega \geqslant 0,$$

so in order to show (4.1) it remains to deduce that

$$\limsup_{k \to \infty} \int_\Omega j^0(u; \tilde{v}_k - u) \, d\Omega \leqslant \int_\Omega j^0(u; v - u) \, d\Omega.$$

For this purpose let us observe that $\tilde{v}_k - u = (1-\epsilon_k)(v-u) + \epsilon_k(-u)$ which combined with the convexity of $j^0(u; \cdot)$ yields the estimate

$$\begin{aligned} j^0(u; \tilde{v}_k - u) &\leqslant (1-\epsilon_k) j^0(u; v-u) + \epsilon_k j^0(u; -u) \\ &\leqslant \left| j^0(u; v-u) \right| + \kappa\left(1 + |u|^q\right). \end{aligned}$$

Thus Fatou's lemma implies the assertion.

Consider the case $j^0(u; v-u) \notin L^1(\Omega)$. Recall that if $j^0(u; v-u) \notin L^1(\Omega)$ then according to the convention that $+\infty - \infty = +\infty$ we have

$$\int_\Omega j^0(u; v-u)\, d\Omega = \begin{cases} +\infty & \text{if } \int_\Omega [j^0(u; v-u)]^+ \, d\Omega = +\infty, \\ -\infty & \text{if } \int_\Omega [j^0(u; v-u)]^+ \, d\Omega < +\infty \text{ and } \int_\Omega [j^0(u; v-u)]^- \, d\Omega = +\infty, \end{cases}$$

where the following notation has been used: $r^+ := \max\{r, 0\}$ and $r^- := \max\{-r, 0\}$ for any $r \in \mathbb{R}$.

Thus, if $\int_\Omega j^0(u; v-u)\, d\Omega = +\infty$ then (4.1) holds immediately.

Now we show that the case $\int_\Omega j^0(u; v-u)\, d\Omega = -\infty$ is not allowed for any $v \in W_0^{1,p}(\Omega)$. Indeed, if we suppose that for some $v \in W_0^{1,p}(\Omega)$, $\int_\Omega j^0(u; v-u)\, d\Omega = -\infty$; then one can find a sequence $\tilde{v}_k = (1-\epsilon_k)v$ such that $\{\tilde{v}_k\} \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\tilde{v}_k \to v$ strongly in $W_0^{1,p}(\Omega)$. Since, as already has been established,

$$\langle Au, \tilde{v}_k - u\rangle_{W_0^{1,p}(\Omega)} + \int_\Omega j^0(u; \tilde{v}_k - u)\, d\Omega \geqslant 0,$$

we get

$$\int_\Omega j^0(u; \tilde{v}_k - u)\, d\Omega \geqslant \langle Au, -\tilde{v}_k + u\rangle_{W_0^{1,p}(\Omega)} \geqslant -C, \quad C = \text{const},$$

and consequently

$$\int_\Omega \left[j^0(u; \tilde{v}_k - u)\right]^+ d\Omega \geqslant \int_\Omega \left[j^0(u; \tilde{v}_k - u)\right]^- d\Omega - C. \tag{4.24}$$

By the hypothesis we have $\int_\Omega [j^0(u; v-u)]^- \, d\Omega = +\infty$ and $\int_\Omega [j^0(u; v-u)]^+ \, d\Omega < +\infty$. Since

$$\begin{aligned} j^0(u; \tilde{v}_k - u) &\leqslant (1-\epsilon_k) j^0(u; v-u) + \epsilon_k j^0(u; -u) \\ &\leqslant (1-\epsilon_k) j^0(u; v-u) + \kappa\left(1 + |u|^q\right), \end{aligned}$$

so we obtain

$$\begin{aligned} \int_\Omega \left[j^0(u; v_k - u)\right]^+ d\Omega &\leqslant \int_\Omega \left[j^0(u; v-u)\right]^+ d\Omega + \int_\Omega \kappa\left(1 + |u|^q\right) d\Omega \\ &\leqslant D, \quad D = \text{const}, \end{aligned}$$

which combined with (4.24) yields

$$\int_\Omega \left[j^0(u; \tilde{v}_k - u)\right]^- d\Omega \leqslant C + D.$$

The application of Fatou's lemma concludes

$$\int_\Omega \left[j^0(u; v - u)\right]^- d\Omega \leqslant C + D,$$

which is a contradiction with the assumption that $\int_\Omega j^0(u; v - u)\, d\Omega = -\infty$. This contradiction completes the proof of (4.1).

Step 8. In order to show that $j(u) \in L^1(\Omega)$ it is enough to use (2.14) and (3.7) to get

$$\int_\Omega j(u_n)\, d\Omega \leqslant \gamma_2 - \frac{1}{p}\|Du_n\|^p_{L^p(\Omega;\mathbb{R}^N)} + \frac{\lambda_1}{p}\|u_n\|^p_{L^p(\Omega)} \leqslant \gamma_2$$

and

$$j(u_n) \geqslant -\kappa_0\left(1 + |u_n|^q\right).$$

Since $j(u_n) \to j(u)$ a.e. in Ω as $n \to \infty$, we are allowed to apply Fatou's lemma which yields the assertion.

Step 9. The existence of a nontrivial solution $u \neq 0$ follows from (H6). Indeed, if we suppose that $u = 0$ then we have $\{u_n\} \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $u_n \to 0$ strongly in $W_0^{1,p}(\Omega)$. By making use of (4.6) with $v = 2u_n$ and the Rayleigh quotient characterization of λ_1, it follows

$$\int_\Omega \min\left\{\psi u_n\colon\ \psi \in \partial j(u_n)\right\} d\Omega \leqslant \int_\Omega \chi_n u_n\, d\Omega \leqslant 0.$$

Hence, by (H6),

$$\limsup_{n\to\infty} \int_\Omega j(u_n)\, d\Omega \leqslant 0$$

and consequently

$$\limsup_{n\to\infty} \mathcal{R}(u_n) \leqslant 0,$$

which contradicts to (3.7). This contradiction yields the assertion. The proof of Theorem 4.1 is complete. □

From (4.2) and (4.3) we obtain the result.

Corollary 4.1. *Assume the hypotheses* (H0)–(H6). *Then the problem*: *find* $u \in W_0^{1,p}(\Omega)$ *and* $\chi \in L^1(\Omega)$ *such that*

$$\begin{cases} \Delta_p u + \lambda_1 |u|^{p-2} u = \chi & \textit{in the distributional sense}, \\ \chi \in \partial j(u) & \textit{a.e. in } \Omega, \\ \chi u \in L^1(\Omega), \\ j(u) \in L^1(\Omega), \\ u = 0 & \textit{on } \partial\Omega \ (\textit{in the sense of traces}) \end{cases} \tag{P}$$

has at least one nontrivial solution ($u \neq 0$).

Remark 4.1. The energy functional $\mathcal{R}$ is finite at a solution u of (P), i.e., $\mathcal{R}(u) = \|Du\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1 \|u\|^p_L(\Omega)^p + \int_\Omega j(u)\,d\Omega \in \mathbb{R}$.

Remark 4.2. In the case of the unilateral growth condition as formulated in (H3), the function $J(v) = \int_\Omega j(v)\,d\Omega$, $v \in W_0^{1,p}(\Omega)$, is not upper semicontinuous. Thus the problem concerning the existence of a nontrivial solution of (P) arises because we are not allowed to conclude by making use of the estimate (3.7) that $\mathcal{R}(u) \geqslant \eta_1 > 0$. To overcome this difficulty the hypothesis (H6) has been introduced.

Note that when the classical growth condition $|\partial j(\xi)| \leqslant c(1 + |\xi|^{q-1})$, $\forall \xi \in \mathbb{R}$, holds then the upper semicontinuity of J is ensured.

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