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On Analytic Extension of Semigroups of Operators

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INTRODUCTION

This paper will deal with one-parametric families $\{S(\xi); \xi > 0\}$ of linear bounded operators having the semigroup property

$$S(\xi_1)\, S(\xi_2) = S(\xi_1 + \xi_2), \qquad \xi_1, \xi_2 > 0, \tag{1}$$

and taking a Banach space B to itself. In order to state our problem we first recall some pertinent notions and definitions.

The weak operator topology is defined by means of the weak topology in the space B. In particular, $S(\xi)$ is said to be measurable, continuous, differentiable or infinitely differentiable in the weak operator topology if for each $X \in B$ and for each Y belonging to the dual space B^*, the numeric function

$$(Y, S(\xi)\, X) = f(\xi) \tag{2}$$

possesses the corresponding property in the usual sense. Analyticity of $S(\xi)$ can also be defined by means of the weak topology. Thus $S(\xi)$ is analytic at a point $\xi_0 > 0$ if each f of the form (2) is analytic there. Analytic continuation of these functions induces analytic continuation of the operator if B is a complex Banach space and the semigroup property is not violated.

If the operator norm $\| S(\xi)\|$ is bounded in an interval $(0, \epsilon]$ then there exist constants $c_1 \geqslant 1$ and c_2 such that $\| S(\xi)\| \leqslant c_1 \exp(c_2 \xi)$, $\xi > 0$, and $S(\xi)$ will be called exponentially bounded. Our objective is to find a complete characterization of those weakly measurable and exponentially bounded semigroups which possess an analytic extension to some angle $V_\varphi = \{\zeta; \mid \arg \zeta \mid < \varphi\}$ $0 < \varphi \leqslant \pi/2$ and remain exponentially bounded there in the sense that

$$\| S(\zeta)\| \leqslant c_1 e^{c_2 |\zeta|}, \qquad \zeta \in V_\varphi,$$

for suitable c_1 and c_2. In order to establish such an extension it is sufficient to show that the functions f are analytic and uniformly bounded in a fix sector $V_{\varphi,\epsilon} = \{\zeta; |\arg \zeta| < \varphi, |\zeta| < \epsilon\}$ for X and Y bounded. Since V_φ is additively generated by $V_{\varphi,\epsilon}$ it follows by a general theorem of Hille [1] that $S(\xi)$ can be extended to the whole angle. There seems to be no known result about analytic continuations of the type described unless $S(\xi)$ is assumed to converge to the identity I in some topology as $\xi \to +0$.

In connection with problems on Markov processes D. G. Kendall [4] has raised the question whether the condition

$$\limsup_{\xi\to+0} \| S(\xi) - I \| = \rho < 2 \tag{3}$$

implies that a weakly continuous semigroup is analytic on the positive real axis. The problem originated in some quasianalytic classes considered by J. W. Neuberger [3], to which we shall return later. We shall prove

THEOREM I. *For a weakly measurable semigroup the condition* (3) *implies the existence of an exponentially bounded analytic extension to an angle* V_φ, *where*

$$\varphi \geqslant k(2 - \rho)^2 \tag{4}$$

k being an absolute constant.

In spite of its simplicity the previous theorem is unsatisfactory in at least one respect. Condition (3) is merely sufficient and the stated analytic extension can be established under much weaker assumptions. In this direction we shall prove

THEOREM II. *The qualitative conclusion in Theorem* I *remains true if* (3) *is replaced by*

$$\limsup_{\substack{n\to\infty \\ \alpha\to+0}} \left\| \left(S\left(\frac{\alpha}{n}\right) - I\right)^n \right\|^{1/n} < 2, \tag{5}$$

if in addition $S(\xi)$ *is assumed exponentially bounded.*

Both theorems are, however, afflicted by another deficiency in that neither version throws any light on the role played by the constant 2. Nor is it obvious that (5) is a necessary condition. In order to resolve these questions we shall consider polynomial operators of the form

$P^n(S(\alpha/n))$, where P is a polynomial of arbitrary degree $m \geqslant 1$ and the power n tends to infinity. Set

$$P^n(z) = \sum_{\nu=0}^{mn} c_{\nu,n} z^\nu.$$

In the expression for $P(S(\xi))$, as in similar cases henceforth, the symbols $S(\xi)^0$ and $S(0)$ do appear and will then always be interpreted as the identity operator I. This is a notational convention and does not imply that $S(\xi)$ is assumed to converge to I in any topology as $\xi \to +0$. The following definitions and properties about polynomials will be of importance later on. By $\| P \|$ we shall denote the norm of $P(e^{ix})$ as an element in the Banach algebra of absolutely convergent Fourier series. Thus,

$$\| P^n \| = \sum_\nu | c_{\nu,n} |, \qquad n = 1, 2, \ldots .$$

On defining

$$\| P \|_\infty = \max_{|z|=1} | P(z) |, \qquad C(P^n) = \max_\nu | c_{\nu,n} |$$

we shall have

$$\| P \|_\infty^n \leqslant \| P^n \| \leqslant \sqrt{mn+1}\, \| P \|_\infty^n$$

and

$$C(P^n) \leqslant \| P^n \| \leqslant mn + 1\, C(P^n).$$

Hence

$$\lim_{n\to\infty} \| P^n \|^{1/n} = \lim_{n\to\infty} C^{1/n}(P^n) = \| P \|_\infty . \tag{6}$$

For polynomials of several variables the previous definitions and relations are still valid. For later use we also note that

$$\frac{1}{2\pi} \int_0^{2\pi} | P(e^{ix}) |^{2n} \, dx \geqslant \frac{c}{\sqrt{n}} \| P \|_\infty^{2n}, \tag{7}$$

where c is a constant depending on P but not on n.

If $\| S(\eta) \| \leqslant M$ for $0 < \eta \leqslant \xi + m\alpha$, then

$$\left\| P^n \left(S\left(\frac{\alpha}{n}\right)\right) S(\xi) \right\| \leqslant M \| P^n \|,$$

and it follows by (6) that

$$\limsup_{n\to\infty} \left\| P^n \left(S\left(\frac{\alpha}{n}\right)\right) S(\xi) \right\|^{1/n} \leqslant \| P \|_\infty ,$$

with uniformity for bounded α and ξ. We now introduce a quantity depending on the semigroup and the polynomial P, defined by the relation

$$\limsup_{\xi+\frac{\alpha}{\xi}+\frac{1}{n}\to 0} \left\| P^n\left(S\left(\frac{\alpha}{n}\right) S(\xi)\right\|^{1/n} = \vartheta(P) \, \| P \|_\infty , \tag{8}$$

where α, ξ are assumed > 0. We already know that $\vartheta(P) \leqslant 1$. It is easy to see that $| P(1)| = \| P \|_\infty$ implies $\vartheta(P) = 1$ unless $S(\xi)$ vanish identically. It is therefore convenient to introduce the set $\{P_1\}$ consisting of all polynomials normalized by the condition $\| P \|_\infty = 1$ and restricted by the property $| P(1)| < 1$. The solution to the angular extension problem can now be expressed in terms of $\vartheta(P)$:

THEOREM III. *Let $\{S(\xi);\ \xi > 0\}$ be weakly measurable and exponentially bounded. Then one of these two alternative holds: Either $\vartheta(P) < 1$ throughout $\{P_1\}$ and $S(\xi)$ possesses an exponentially bounded analytic extension to some angle V_φ, $\varphi > 0$, or $\vartheta(P) = 1$ for each $P \in \{P_1\}$ and no such extension exists.*

It is interesting to note that in the angular extension problem for $S(\xi)$ the quantity $1 - \vartheta(P)$ plays a role analogous to the radius of convergence in classical function theory: Analytic continuation is possible if and only if the quantity in question is positive.

The main part of Theorem III states that the requested extension exists if $\vartheta(P) < 1$ for at least one $P \in \{P_1\}$. The proof hereof is lengthy and consists of a number of elements of different natures. We therefore include in this introduction a brief outline of the proof.

The function $f(\xi)$ defined by (2) is measurable, well-defined at each point $\xi > 0$ and bounded for bounded ξ. Writing $S(\xi)\, X = X(\xi)$ we shall have

$$\begin{aligned}\left(Y, P^n\left(S\left(\frac{\alpha}{n}\right)\right) X(\xi)\right) &= \sum_{\nu=0}^{mn} c_{\nu,n} f\left(\xi + \nu\left(\frac{\alpha}{n}\right)\right)\\ &\equiv P^n\left(T\left(\frac{\alpha}{n}\right)\right) f(\xi),\end{aligned} \tag{9}$$

where $T(\eta)$ denotes the shift operator: $f(\xi) \to f(\xi + \eta)$. The original problem is thus transformed to the question whether the condition

$$\limsup_{\xi+\frac{\alpha}{\xi}+\frac{1}{n}\to 0} \left\| P^n\left(T\left(\frac{\alpha}{n}\right)\right) f(\xi) \right\|^{1/n} = \vartheta(P) < 1 \tag{10}$$

implies that $f(\xi)$ is analytic and exponentially bounded in an angle V_φ.

The first step in the proof consists of a localizing procedure. We replace f by a function f_n which coincides with f on an interval γ_0 and vanishes off another interval $\gamma_1 \supset \gamma_0$. It will be proved that for each $\vartheta_1 > \vartheta(P)$, a sequence $\{f_n\}_1^\infty$ can be constructed such that $P^n(T(\alpha/n)) f_n(\xi)$ tends exponentially to 0 as $n \to \infty$ if α is small enough. The f_n will be defined as products fk_n where the multipliers k_n equal 1 on γ_0 and vanish off γ_1. The main difficulty here is finding a suitable expression for the operator $P^n(T(\alpha/n))$ applied to a product of two functions. This problem will be solved by means of certain polynomial identities.

In the next step in the proof harmonic analysis is applied to f_n. If $\hat{f}_n(t)$ is its Fourier transform then the transform of $P^n(T(\alpha/n)) f_n(\xi)$ is $\hat{f}_n(t) P^n(e^{it(\alpha/n)})$ and Parseval relation yields

$$\int_{-\infty}^{\infty} |\hat{f}_n(t)|^2 \, | P(e^{it(\alpha/n)})|^{2n} = O(e^{-2\theta n}), \tag{11}$$

for some $\theta > 0$. It is possible to deduce from these inequalities and the fact that all the f_n coincide with f on γ_0, that f coincide a.e. on this interval with an analytic function g. That $f(\xi) = g(\xi)$ everywhere for $\xi > 0$ is proved separately.

The proof of the remaining part of Theorem III is straightforward and does not present any complications.

Proofs

The Localizing.

Let J_n denote the integral of $\sin^n \pi\xi$ extended over [0, 1] and define for $n = 1, 2, \ldots$

$$K_n(\xi) = \begin{cases} 0, & \xi \leqslant 0, \\ J_n^{-1} \int_0^\xi \sin^n \pi\xi \, d\xi, & 0 \leqslant \xi \leqslant 1, \\ 1, & \xi \geqslant 1. \end{cases}$$

By an inequality of S. Bernstein, $| D^\nu \sin^n \pi\xi | \leqslant (\pi n)^\nu$. We have $J_n > (\pi n)^{-\frac{1}{2}}$ and consequently

$$| D^\nu K_n(\xi)| < (\pi n)^{\nu - \frac{1}{2}}, \qquad 1 \leqslant \nu \leqslant n,$$

and the first n derivatives of K_n are continuous on $(-\infty, \infty)$. Multipliers of this type have been used previously by the author in the proof of a theorem on quasianalyticity [2, Theorem IV, Lecture 3] and will in this paper serve a similar purpose. Let γ_0 and γ_1 be two concentric intervals of the real axis and assume $|\gamma_1| = |\gamma_0| + 2\lambda$. By proper choice of ξ_1 and ξ_2 the function

$$k_n(\xi) = K_n\left(\frac{\xi - \xi_1}{\lambda}\right) - K_n\left(\frac{\xi - \xi_2}{\lambda}\right)$$

will equal 1 on γ_0 and vanish off γ_1. Furthermore,

$$|D^\nu k_n(\xi)| < \left(\frac{\pi n}{\lambda}\right)^{\nu - \frac{1}{2}}, \qquad 1 \leqslant \nu \leqslant n.$$

If $P_0(z) = z - 1$, then for $1 \leqslant \nu \leqslant n$,

$$\begin{aligned} \left| P_0^\nu\left(T\left(\frac{\alpha}{n}\right)\right) k_n(\xi) \right| &= \left| \sum_{i=0}^{\nu} (-1)^i \binom{\nu}{i} k_n\left(\xi + (\nu - i)\left(\frac{\alpha}{n}\right)\right) \right| \\ &\leqslant \left(\frac{\alpha}{n}\right)^\nu \left(\frac{\pi n}{\lambda}\right)^{\nu - \frac{1}{2}} < \left(\frac{\pi\alpha}{\lambda}\right)^\nu, \end{aligned} \tag{12}$$

where the last inequality holds also for $\nu = 0$.

Assume $P \in \{P_1\}$ and set $\vartheta(P) = 1 - 3\theta < 1$. According to the definition (10) there exists a positive integer n_0 such that the conditions

$$0 < \xi \leqslant \frac{1}{n_0}, \qquad 0 < \frac{\alpha}{\xi} \leqslant \frac{1}{n_0}, \qquad n \geqslant n_0 \tag{13}$$

imply

$$\left| P^n\left(T\left(\frac{\alpha}{n}\right)\right) f_n(\xi) \right| \leqslant (1 - 2\theta)^n. \tag{14}$$

Let a be a positive number and set $\gamma_0 = [4a, 12a]$, $\gamma_1 = [3a, 13a]$, $\gamma_2 = [2a, 13a]$. There exist positive quantities a and $\beta_0 \leqslant 2a/n_0$ such that (13), and therefore also (14), is satisfied for $\xi \in \gamma_2$ and for $0 < \alpha \leqslant \beta_0$. If $n_0 \geqslant 2m$, which we assume, then the function in (14) vanishes off γ_2.

Our next objective is to express the operator $P^n(T(\alpha/n))$, applied to a product fk_n, in terms of $P^\nu(T(\alpha/n))f(\xi)$ and $P_0^\nu(T(\alpha/n))\, k_n(\xi)$, $\nu \leqslant n$, thus making (12) and (14) useful in estimating $P^n fk_n$. To this purpose we consider P as a polynomial of the product xy of two independent variables x and y. Writing

$$P(xy) = P(x) + (y-1)\,Q(x, y)$$

and

$$Q(x, y) = \sum_{\nu=1}^{m} c_{\nu,1} x^{\nu}(1 + y + \cdots + y^{\nu-1})$$

and setting $P_0(y) = y - 1$ we shall have

$$P^n(xy) = \sum_{\nu=0}^{n} \binom{n}{\nu} P^{n-\nu}(x)\, P_0^{\nu}(y)\, Q^{\nu}(x, y). \tag{15}$$

The function $f(\xi)\, k_n(\xi)$ will now be replaced by $f(\xi)\, k_n(\eta)$ where ξ and η are independent real variables. At the same time we let x symbolize a shift operator acting only on the ξ axis and taking $f(\xi)$ to $f(\xi + \alpha/n)$, α and n being fixed. Similarly y will denote the shift acting only on the η axis, taking $k_n(\eta)$ to $k_n(\eta + \alpha/n)$. In particular

$$x^i y^j f(\xi)\, k_n(\eta) = f\left(\xi + i\left(\frac{\alpha}{n}\right)\right) k_n\left(\eta + j\left(\frac{\alpha}{n}\right)\right).$$

Writing

$$Q^{\nu}(x, y) = \sum_{i,j} e_{i,j,\nu} x^i y^j$$

we obtain

$$P^n(xy)\, f(\xi)\, k_n(\eta) = \sum_{\nu=0}^{n} \binom{n}{\nu} \sum_{i,j} P^{n-\nu}(x)\, P_0^{\nu}(y)\, e_{i,j,\nu} f\left(\xi + i\left(\frac{\alpha}{n}\right)\right) k_n\left(\eta + j\left(\frac{\alpha}{n}\right)\right). \tag{16}$$

For the k_n corresponding to the intervals γ_0, γ_1, we have $\lambda = a$. We also observe that (14) holds for $n \geqslant \nu \geqslant n_0$, if the power n on both sides is replaced by ν, but α/n is unchanged. If therefore $n - \nu \geqslant n_0$ the second series in (16) is majorized by

$$(1 - 2\theta)^{n-\nu} \left(\frac{\pi\alpha}{a}\right)^{\nu} \sum_{i,j} | e_{i,j,\nu} |$$

for $\xi \in \gamma_2$, $0 < \alpha \leqslant \beta_0$. If $n - \nu < n_0$ the first factor above has to be replaced by $\| P \|^{n-\nu}$. This gives the following estimate of (16),

$$\sum_{\nu=0}^{n} \binom{n}{\nu} (1 - 2\theta + \delta_{\nu})^{n-\nu} \left(\frac{\pi\beta_0}{a}\right)^{\nu} \| Q^{\nu} \|$$

where $\delta_{\nu} = 0$ for $\nu \leqslant n - n_0$ and $< \| P \|^{n_0}$ for $\nu > n - n_0$.

Taking into account that $\| Q^\nu \|^{1/\nu} \to \| Q \|_\infty$ we obtain on the diagonal $\xi = \eta$,

$$\left| P^n \left(T \left(\frac{\alpha}{n} \right) \right) f_n(\xi) \right| \leqslant \left(1 - 2\theta + \frac{\pi \beta_0}{a} \| Q \|_\infty + \epsilon_n \right)^n$$

with $\epsilon_n \to 0$ as $n \to \infty$. We have trivially $\| Q \|_\infty \leqslant \| P' \|$ where P' stands for the derivative of P. The previous results may now be summarized.

LEMMA I. *Let a and β_0 be defined as above. Then there exists an integer n_1 such that the inequality*

$$\left| P^n \left(T \left(\frac{\alpha}{n} \right) \right) f_n(\xi) \right| \leqslant (1 - \theta)^n \tag{17}$$

holds for $n \geqslant n_1$, $\xi \in \gamma_2$, $0 < \alpha \leqslant \beta < \min(\beta_0, \frac{\theta a}{\pi \| P' \|}$.

It is important to note at this instance that all the inequalities satisfied by $f(\xi)$ and $f_n(\xi)$ and leading to (17) are also satisfied by $f(t\xi)$ and $f_n(t\xi)$ if $0 < t \leqslant 1$. This provides us with the uniformity at the origin which is necessary to establish the extension to the whole angle V_φ.

The Analyticity.

With the notations used in the introduction we have

$$\int_{-\infty}^{\infty} | \hat{f}_n(t) |^2 \, | P(e^{it\alpha/n}) |^{2n} \, dt \leqslant \text{const}(1 - \theta)^{2n}, \qquad n \geqslant n_1 . \tag{18}$$

Here and forthwith unnamed constants will always be independent of n. By virtue of (7)

$$\frac{1}{\beta} \int_0^\beta | P(e^{it\alpha/n}) |^{2n} \, d\alpha = \frac{n}{\beta t} \int_0^{\beta t/n} | P(e^{ix}) |^{2n} \, dx \geqslant \frac{\text{const}}{\sqrt{n}} \tag{19}$$

provided $| t | \geqslant t_n = 2\pi n/\beta$. Hence,

$$\int_{|t| \geqslant t_n} | \hat{f}_n(t) |^2 \, dt \leqslant \text{const} \sqrt{n} \, (1 - \theta)^{2n}, \qquad n \geqslant n_1 . \tag{20}$$

Since $f_n = f$ on γ_0, we shall have

$$\int_{\gamma_0} | f(\xi) - g_n(\xi) |^2 \leqslant \text{const} \sqrt{n} \, e^{-2\theta n}, \qquad n \geqslant n_1 , \tag{21}$$

with

$$g_n(\xi) = \frac{1}{\sqrt{2\pi}} \int_{|t| \leqslant t_n} \hat{f}_n(t)\, e^{it\xi}\, dt. \tag{22}$$

The g_n are entire functions satisfying the inequalities

$$| g_n(\xi + i\eta)| \leqslant \text{const} \sqrt{n}\, e^{t_n|\eta|}. \tag{23}$$

LEMMA II. *Let A_λ, denote the set of functions $g(\zeta)$ analytic in the unit disk D and satisfying there,* $\log | g | \leqslant \lambda$. *Let $f(\xi)$ belong to a space $L^p(-1, 1)$, $1 \leqslant p \leqslant \infty$, and let $a(\lambda, p, f) = a(\lambda)$ be the approximation index of f in the L^p-topology defined by the relation*

$$e^{-a(\lambda)} = \inf_{g \in A_\lambda} \| f - g \|_p . \tag{24}$$

Then the following holds:

(a) *Under the condition*

$$\int_1^\infty \frac{a(\lambda)}{\lambda^2}\, d\lambda = \infty \tag{25}$$

f is quasianalytic in the sense that f vanishes a.e. on $(-1, 1)$ if $f = 0$ on a set of positive measure.

(b) *Under the stronger condition*

$$\liminf_{\lambda \to \infty} \frac{a(\lambda)}{\lambda} = \delta > 0 \tag{25'}$$

f coincide a.e. on $(-1, 1)$ with a function g which is analytic in a region D_δ formed by the intersection of the two circles which cut the real axis at ± 1 *under the angle* $\dfrac{\pi\delta}{2(1+\delta)}$.

The first part of the lemma is a special case of a more general theorem [2, Theorem III, Lecture 2]. It has no application to the problem considered in this paper but is included here because it clarifies questions raised concerning quasianalyticity in connection with condition (3). Part (b) is basically equivalent to a theorem on polynomial approximation by S. Bernstein. The proof follows trivially by current function theoretic methods.

Let $g_\lambda \in A_\lambda$ and minimize $\| f - g_\lambda \|_p$. If F and G_λ are the primitive of f and g_λ respectively, normalized by the condition $F(0) = G_\lambda(0) = 0$, then we shall have in the uniform norm: $\| F - G_\lambda \| \leqslant \exp(-a(\lambda))$.

Since $G_\lambda \in A_\lambda$, we conclude that the inequality (25′) holds for the approximation index of F in the uniform topology. For each $\lambda = n$, $n = 1, 2, \ldots$, there exists thus a $G_n \in A_n$ such that

$$\log \| F - G_n \| \leqslant - \delta n + o(n).$$

Consequently,

$$\log \| G_{n+1} - G_n \| \leqslant - \delta n + o(n), \tag{26}$$

$$\log | G_{n+1}(\zeta) - G_n(\zeta)| \leqslant n + O(1), \qquad \zeta \in D. \tag{27}$$

On the interval $(-1, 1)$ we have

$$F(\xi) = G_{n_1}(\xi) + \sum_{n_1}^{\infty} (G_{n+1}(\xi) - G_n(\xi)), \tag{28}$$

where the series converges uniformly. By applying harmonic majorization to (27) in the upper half-circle D^+ of D we obtain

$$\log | G_{n+1}(\zeta - G_n(\zeta)| \leqslant \omega(\zeta) (- \delta n + o(n)) + (1 - \omega(\xi)) (n + O(1)),$$

where $\omega(\zeta)$ denote the harmonic measure of the arc $(-1, 1)$ of the boundary ∂D^+. In order to compute ω we recall that $1 - \omega$ equals $2\psi(\zeta)/\pi$ where ψ is the angle under which the circle passing the points ± 1 and ζ cuts the real axis. Together with a similar result for the lower half circle D^- we obtain in D

$$\log | G_{n+1}(\zeta) - G_n(\zeta)| \leqslant n \left(\frac{2\psi(\zeta)}{\pi} (1 + \delta) - \delta \right) + 0(n) \tag{29}$$

proving that the series (28) converge uniformly on closed sets in D_δ to a function $G(\zeta)$ holomorphic there. The lemma follows on taking the derivative of F and G.

If this lemma is applied to γ_0 and the circle D having γ_0 as diameter we conclude by (21) and (23) that the approximation index of f in $L^2(\gamma_0)$ satisfies (25) with

$$\delta = \frac{\theta\beta}{8\pi a}. \tag{30}$$

In the present case the region of analyticity of $g(\zeta)$ contains the circle

$$| \zeta - 8a | < 4a \operatorname{tg} \frac{\pi}{2} \frac{\delta}{1 + \delta},$$

and g is therefore regular and bounded by some constant M in the smaller disk

$$D_a = \{\zeta; \mid \zeta - 8a \mid < 8a \operatorname{tg} \varphi\}, \qquad \varphi = \frac{\pi}{4} \frac{\delta}{1+\delta},$$

provided $\| X \|$, $\| Y \| \leqslant 1$. According to the remark following Lemma I, g is bounded by the same constant on the disk D_b, $0 < b \leqslant a$, the union of which contains a sector $V_{\varphi,\epsilon}$.

This would finish the proof of the main part of Theorem III if f were continuous on $(0, \epsilon]$ and consequently $f \equiv g$. As will be shown later, the analyticity of g implies that $P^n(T(\alpha/n))\, g(\xi)$ tends exponentially to 0 with increasing n. Setting $h = f - g$ we will have on $(0, \epsilon]$, for some $\vartheta < 1$, if n is sufficiently large and α sufficiently small,

$$\left| P^n \left(T\left(\frac{\alpha}{n}\right)\right) h(\xi) \right| < \vartheta^n. \tag{31}$$

Let ξ' be some fixed point and let ν' be the index of the maximal coefficient of P^n. We can choose ξ and α so that $\xi + \nu'(\alpha/n) = \xi'$, and $h(\xi + \nu(\alpha/n)) = 0$ for $\nu' \neq \nu \leqslant mn$, since h vanishes a.e. Then (31) reduces to $| c_{\nu',n} h(\xi')| < \vartheta^n$ and $h(\xi') = 0$ follows by (6). Hence, the analytic extension of $S(\xi)$ to V_φ is established. By virtue of the implications (3) $\Rightarrow$ (5) $\Rightarrow \vartheta(P_0) < 1$, Theorem II as well as the qualitative part of Theorem I are also established.

As for the proof of (4) we note that condition (3) permits us to take for a any positive number, for example $a = 1$ making $\gamma_0 = [4, 12]$ and $\gamma_1 = [3, 13]$. The restrictions on β are now two: The support of $P_0{}^n(T(\alpha/n)) f_n(\xi)$ must not enter the negative real axis, hence $\beta \leqslant 3$. Secondly, according to Lemma I we must have $\beta \leqslant c_1(1 - \vartheta(P_0))$, where c_1 is a certain numeric constant. This implies $\delta \geqslant c_2(1 - \vartheta(P_0))^2$, proving (4). It would be futile to try to find the best value of k since the order of magnitude $(2 - \rho)^2$ is unlikely to be the right one.

The quasianalytic classes introduced by Neuberger consisted of continuous functions on an interval (a, b) satisfying the inequalities

$$\Delta^n_{u,v} f \mid = \left| \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} f\left(u + (v-u)\frac{\nu}{n}\right) \right| \leqslant M\rho^u, \qquad \rho < 2, \tag{32}$$

for $0 < u < v < b$. Since the difference above can be written $P_0{}^n(T((v-u)/n)) f(u)$ the previous results apply and show that f is analytic in an open rhombus having (a, b) as diagonal and with sides forming an angle φ with the real axis, φ being the angle occuring in

Theorem I. Modifications of the condition (32) are leading to similar results.

In order to establish the remaining part of Theorem III it is sufficient to show that (10) holds for functions f analytic and exponentially bounded in an angle V_φ. In the proof we need this corollary of an inequality by S. Bernstein. Let

$$h(x) = \sum_{-m}^{m} c_\nu e^{i\nu x}$$

be a trigonometric polynomial with $|h(x)| \leqslant 1$ for real x. Then

$$|D^\nu h^n(x)| \leqslant \begin{cases} (mn)^\nu \,|h(x)|^{n-\nu}, & \nu \leqslant n \\ (mn)^\nu, & \nu > n. \end{cases} \tag{33}$$

The proof is a simple case of induction. In the relation $D^\nu h^n = h^{n-\nu} k_\nu$, k_ν is a trigonometric polynomial of degree $m\nu$. If $\| k_\nu \|$ is its supremum norm then $\| k_\nu' \| \leqslant m\nu \| k_\nu \|$. Furthermore, $k_{\nu+1} = (n-\nu)\, k_\nu h' + k_\nu' h$. Hence, $\| k_{\nu+1} \| \leqslant mn \| k_\nu \|$, and (33) follows.

Assume now that f is analytic and bounded by M in the circle $|\zeta - \xi| < \xi \sin\varphi$. Then trivially

$$\frac{|D^p f(\xi)|}{p!} \leqslant \frac{M}{(\xi \sin\varphi)^p}.$$

For $0 < \alpha < (\xi \sin\varphi)/m$ we shall have

$$P^n\left(T\left(\frac{\alpha}{n}\right)\right) f(\xi) = \sum_{\nu=0}^{mn} c_{\nu,n} \sum_{p=0}^{\infty} \frac{f^{(p)}(\xi)}{p!} \left(\frac{\nu\alpha}{n}\right)^p. \tag{34}$$

By applying (33) to $h(x) = P(e^{ix})$ we obtain

$$\left|\sum_\nu c_{\nu,n}\nu^p\right| \leqslant \begin{cases} (mn)^p \,|P(1)|^{n-p}, & p \leqslant n \\ (mn)^p, & p > n. \end{cases} \tag{35}$$

Inserted in (34), the previous majorations yield

$$\left| P^n\left(T\left(\frac{\alpha}{n}\right)\right) f(\xi) \right| \leqslant M \left\{ |P(1)|^n \frac{1-\delta_1^{n+1}}{1-\delta_1} + \frac{\delta_2^{n+1}}{1-\delta_2} \right\}$$

with

$$\delta_1 = \frac{\alpha m}{\xi\,|P(1)| \sin\varphi}, \qquad \delta_2 = \frac{\alpha m}{\xi \sin\varphi}.$$

The existence of an exponentially bounded analytic extension to V_φ therefore implies $\vartheta(P) = |P(1)|$ for $P \in \{P_1\}$ and this ends the proof.

Concluding Remarks

The analyticity problem can be approached also by an elementary method relying exclusively on real analysis. In order to present the basic idea in its simplest form we assume that a function $f(\xi)$ is continuous on the real axis and that for some given $P \in \{P_1\}$ we have

$$\left| P^n\left(T\left(\frac{\alpha}{n}\right)\right) f(\xi) \right| \leqslant A(n), \qquad n \geqslant 1, \qquad |f(\xi)| \leqslant A(0), \tag{36}$$

for $0 < \alpha \leqslant \beta$; $A(n)$ being given positive numbers with the property

$$\limsup_{n\to\infty} \sqrt[n]{A(n)} < 1.$$

The method used previously ascertains the existence of two positive constants M_0 and η_0, depending only on $\{A(n)\}$, β and P and such that (36) implies that f is analytic and bounded by M_0 in the strip $\{\zeta = \xi + i\eta;\ |\ \eta\ | < \eta_0\}$.

To the operator in (36) we associate a measure $\mu(\xi) = \mu(\xi, P, n, \alpha)$ by the relation

$$P^n\left(T\left(\frac{\alpha}{n}\right)\right) f(\xi) = \int f(\xi)\, d\mu(\xi).$$

A set $\mathfrak{M}(P)$ of measures can now be defined as all weak limits τ of sequences $\{\sigma_j\}$ of linear combinations

$$\sigma_j(\xi) = \sum_i \lambda_{i,j} \mu(\xi, P, n_{i,j}, \alpha_{i,j})$$

with $n_{i,j} \geqslant 1$, $0 < \alpha_{i,j} \leqslant \beta$, satisfying in addition the condition that

$$\sum_i |\ \lambda_{i,j}\ |\ A(n_{i,j}) \tag{37}$$

is uniformly bounded within the sequence. To each $\tau \in \mathfrak{M}(P)$ we assign a norm $\|\ \tau\ \|$ defined as the infimum of (37) for sequences converging weakly to τ. Thus, for continuous functions (36) implies

$$\left| \int f(\xi)\, d\tau(\xi) \right| \leqslant \| f \| \, \| \tau \| .$$

In the sequel δ_ξ will denote the Dirac measure centered at ξ, and we will write $\Delta_{n,\alpha}$ for the measure

$$\Delta_{n,\alpha} = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \delta_{\nu\alpha/n}, \qquad \Delta_{0,\alpha} = \delta_0 .$$

By a duality argument we conclude that the previous statement relating to the analyticity and boundedness of f is equivalent with the following: There exist constants M_0 and c_0 depending only on $\{A(n)\}$, β and P, and such that (36) implies $\{\Delta_{n,\alpha};\, n \geqslant 1,\, 0 \leqslant \alpha \leqslant \beta\} \in \mathfrak{M}(P)$ with norms satisfying

$$\| \Delta_{n,\alpha} \| \leqslant M_0(c_0\alpha)^n, \qquad \| \delta_0 \| \leqslant M_0 .$$

A successful proof along this line will yield the maximal region of analyticity and is therefore likely to give an improved lower estimate of the angle φ in Theorem I.

Upper estimates of φ can always be obtained by considering special cases. Of particular interest is the space $L^2(\gamma)$ with respect to the linear measure on the boundary γ of an angular region V_φ, $0 < \varphi \leqslant \pi/2$. A semi-group $\{T(\xi);\, \xi > 0\}$ is defined by the Cauchy integral

$$T(\xi)f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(t)\, dt}{t - z - \xi}, \qquad \xi > 0,$$

where γ is described in the positive direction in relation to V_φ. According to well-known properties of the Hilbert transform, the operators $T(\xi)$ are bounded. Furthermore, since the transformation $f(z) \to \sqrt{k}\, f(kz)$, $k > 0$, is isometric in $L^2(\gamma)$ and the Cauchy kernel is homogeneous it follows that the norms $\| T(\xi)\|$ and $\| T(\xi) - I \|$ are independent of ξ for $\xi > 0$ and hence depending only on φ. An upper estimate of $\rho(\varphi) = \| T(\xi) - I \|$ will therefore yield an upper bound for φ.

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