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# Gorenstein homology, relative pure homology and virtually Gorenstein rings ☆

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ABSTRACT

We consider the following question: Is Gorenstein homology a $\mathscr{X}$-pure homology, in the sense defined by Warfield, for a class $\mathscr{X}$ of modules? Let $\mathcal{GP}$ denote the class of Gorenstein projective modules. We prove that over a commutative Noetherian ring $R$ of finite Krull dimension, Gorenstein homology is a $\mathcal{GP}$-pure homology if and only if $R$ is virtually Gorenstein.



## 1. Introduction

Throughout this paper, $R$ will be a commutative ring with identity. Let $\mathcal{GP}$ and $\mathcal{GI}$ denote the classes of Gorenstein projective and Gorenstein injective $R$-modules, respectively. We refer the reader to [1,6,7,9] for all unexplained definitions in the sequel. Warfield [20] has introduced a notion of $\mathscr{X}$-purity for any class $\mathscr{X}$ of $R$-modules. Recall from [20] that an exact sequence $0 \to A \to B \to C \to 0$ of $R$-modules is called *$\mathscr{X}$-pure exact* if for any $U \in \mathscr{X}$ the induced $R$-homomorphism $\operatorname{Hom}_R(U, B) \to \operatorname{Hom}_R(U, C)$ is surjective. An $R$-module $M$ is called *$\mathscr{X}$-pure projective* (respectively *$\mathscr{X}$-pure injective*; *$\mathscr{X}$-pure flat*) if the functor $\operatorname{Hom}_R(M, -)$ (respectively $\operatorname{Hom}_R(-, M)$; $M \otimes_R -$) leaves any $\mathscr{X}$-pure exact sequence exact. For a survey of results on purity, we refer the reader to [14,19,20].

Let $\mathscr{X}$ be a class of $R$-modules. We say a homology theory $\mathcal{T}$ is a *$\mathscr{X}$-pure homology* if an $R$-module $M$ is projective (respectively injective; flat) in $\mathcal{T}$ if and only if it is $\mathscr{X}$-pure projective (respectively $\mathscr{X}$-pure injective; $\mathscr{X}$-pure flat). In this paper, we investigate the question: Is Gorenstein homology a $\mathscr{X}$-pure homology for an appropriate class $\mathscr{X}$ of $R$-modules? Our candidate for a such class $\mathscr{X}$ is $\mathcal{GP}$. To treat this question,

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we focus on Noetherian rings of finite Krull dimension. So, from now to the end of the introduction, assume that $R$ is Noetherian of finite Krull dimension. It is not hard to verify that

$$\mathcal{GP} = \{M \in R\text{-Mod} | M \text{ is } \mathcal{GP}\text{-pure projective}\};$$

see Proposition 2.5 and Remark 2.6(iv) below. On the other hand, we show that if the classes of Gorenstein injectives and $\mathcal{GP}$-pure injectives are the same, then also the classes of Gorenstein flats and $\mathcal{GP}$-pure flats are the same; see Corollary 3.2 below. Therefore, our question reduces to: Are the classes of Gorenstein injectives and $\mathcal{GP}$-pure injectives the same? In Lemma 3.7, we show that ${}^{\perp}\mathcal{GI} \subseteq \mathcal{GP}^{\perp}$; see the beginning of the next section for the definitions of ${}^{\perp}\mathcal{GI}$ and $\mathcal{GP}^{\perp}$. We call $R$ *virtually Gorenstein* if ${}^{\perp}\mathcal{GI} = \mathcal{GP}^{\perp}$. This generalizes the notion of virtually Gorenstein Artin algebras which was introduced by Beligiannis and Reiten in [3]. Our main results make such algebras relevant also in commutative ring theory. We prove that Gorenstein homology is a $\mathcal{GP}$-pure homology if and only if the functor $\mathrm{Hom}_R(-, \sim)$ is right balanced by $\mathcal{GP} \times \mathcal{GI}$ and if and only if $R$ is virtually Gorenstein; see Theorems 3.10 and 3.12 below.

## 2. Gorenstein and relative pure projectivity

Recall from the introduction that $\mathcal{GP}$ and $\mathcal{GI}$ denote the classes of Gorenstein projective and Gorenstein injective $R$-modules, respectively. Our main result in this section is that when $\mathcal{GP}$ is precovering, then

$$\mathcal{GP} = \{M \in R\text{-Mod} | M \text{ is } \mathcal{GP}\text{-pure projective}\}.$$

For any class $\mathscr{X}$ of $R$-modules, let $\mathscr{X}^{\perp}$ (respectively ${}^{\perp}\mathscr{X}$) denote the class of $R$-modules $M$ with the property that $\mathrm{Ext}^1_R(X, M) = 0$ (respectively $\mathrm{Ext}^1_R(M, X) = 0$) for all $R$-modules $X \in \mathscr{X}$. Notice that our definitions of $\mathscr{X}^{\perp}$ and ${}^{\perp}\mathscr{X}$ are not the same as [3] and [2]. Nevertheless, the following result indicates that our definitions of $\mathcal{GP}^{\perp}$ and ${}^{\perp}\mathcal{GI}$ coincide with those in [3] and [2].

**Lemma 2.1.** *The following statements hold.*

(i) *If $M \in \mathcal{GP}^{\perp}$, then $\mathrm{Ext}^i_R(Q, M) = 0$ for all $i \geqslant 1$ and all $Q \in \mathcal{GP}$.*
(ii) *If $M \in {}^{\perp}\mathcal{GI}$, then $\mathrm{Ext}^i_R(M, E) = 0$ for all $i \geqslant 1$ and all $E \in \mathcal{GI}$.*

**Proof.** (i) Let $Q$ be a Gorenstein projective $R$-module and $i \geqslant 2$. Then by the definition, there exists an exact sequence

$$0 \longrightarrow \widetilde{Q} \longrightarrow P_{i-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow Q \longrightarrow 0$$

of $R$-modules, where $P_0, \ldots, P_{i-2}$ are projective and $\widetilde{Q}$ is Gorenstein projective. Now for any $M \in \mathcal{GP}^{\perp}$, as $\widetilde{Q}$ is Gorenstein projective, one has $\mathrm{Ext}^i_R(Q, M) \cong \mathrm{Ext}^1_R(\widetilde{Q}, M) = 0$.

(ii) The proof is similar to the proof of (i), and so we leave it to the reader. □

We need the following lemma in the proof of Proposition 2.5.

**Lemma 2.2.** *The following statements hold.*

(i) *We have $\mathcal{GP} \subseteq \{M \in R\text{-Mod} | M \text{ is } \mathcal{GP}\text{-pure projective}\} \subseteq {}^{\perp}(\mathcal{GP}^{\perp})$.*
(ii) *Every $\mathcal{GP}$-pure projective $R$-module of finite Gorenstein projective dimension is Gorenstein projective.*

**Proof.** (i) The left containment is trivial by the definition. Now, let $P$ be a $\mathcal{GP}$-pure projective $R$-module and $M \in \mathcal{GP}^{\perp}$. One could have an exact sequence

$$\mathbf{X} = 0 \longrightarrow M \longrightarrow E \longrightarrow C \longrightarrow 0,$$

of $R$-modules in which $E$ is injective. Then for any $Q \in \mathcal{GP}$, as $\mathrm{Ext}^1_R(Q, M) = 0$, we deduce that the sequence

$$0 \longrightarrow \mathrm{Hom}_R(Q, M) \longrightarrow \mathrm{Hom}_R(Q, E) \longrightarrow \mathrm{Hom}_R(Q, C) \longrightarrow 0$$

is exact. So, $\mathbf{X}$ is $\mathcal{GP}$-pure exact. Then, as $P$ is $\mathcal{GP}$-pure projective, $\mathrm{Hom}_R(P, \mathbf{X})$ is exact, and so the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(P, E) \longrightarrow \mathrm{Hom}_R(P, C) \longrightarrow \mathrm{Ext}^1_R(P, M) \longrightarrow 0$$

implies that $\mathrm{Ext}^1_R(P, M) = 0$. Thus $P \in {}^{\perp}(\mathcal{GP}^{\perp})$, as required.

(ii) Let $M$ be a $\mathcal{GP}$-pure projective $R$-module of finite Gorenstein projective dimension and let $\mathrm{Gpd}_R M$ denote the Gorenstein projective dimension of $M$. Clearly, we may assume that $M$ is nonzero. It is immediate from the definition of Gorenstein projective modules that every projective $R$-module is contained in $\mathcal{GP}^{\perp}$, and so by (i), one has $\mathrm{Ext}^i_R(M, P) = 0$ for all projective $R$-modules $P$ and all $i > 0$. Thus, by [5, Theorem 3.1] one has

$$\mathrm{Gpd}_R M = \sup\{i \in \mathbb{N}_0 \mid \mathrm{Ext}^i_R(M, P) \neq 0 \text{ for some projective } R\text{-module } P\} = 0. \qquad \square$$

**Corollary 2.3.** *Assume that $R$ is a coherent ring of finite Krull dimension. Then every $\mathcal{GP}$-pure projective $R$-module is Gorenstein flat.*

**Proof.** Let $\mathcal{GF}$ denote the class of Gorenstein flat $R$-modules. By [8, Theorem 2.11], we have ${}^{\perp}(\mathcal{GF}^{\perp}) = \mathcal{GF}$. Since $R$ is coherent and $\dim R < \infty$, by [5, Proposition 3.7] every Gorenstein projective $R$-module is Gorenstein flat. Thus, ${}^{\perp}(\mathcal{GP}^{\perp}) \subseteq {}^{\perp}(\mathcal{GF}^{\perp})$, and so Lemma 2.2(i) completes the proof. $\square$

Next, we recall the definitions of special precovers and preenvelopes.

**Definition 2.4.** Let $M$ be an $R$-module and $\mathscr{X}$ be a class of $R$-modules.

(i) A $\mathscr{X}$-precover $\varphi : X \to M$ is called *special* if $\varphi$ is surjective and $\mathrm{Ker}\,\varphi \in \mathscr{X}^{\perp}$. Also, a $\mathscr{X}$-*preenvelope* $\psi : M \to X$ is called *special* if $\psi$ is injective and $\mathrm{Coker}\,\psi \in {}^{\perp}\mathscr{X}$.

(ii) A pair $(\mathscr{X}, \mathscr{Y})$ of $R$-modules is called a *cotorsion theory* if $\mathscr{X}^{\perp} = \mathscr{Y}$ and ${}^{\perp}\mathscr{Y} = \mathscr{X}$. A cotorsion theory $(\mathscr{X}, \mathscr{Y})$ is said to be *complete* if every $R$-module has a special $\mathscr{X}$-precover; or equivalently if every $R$-module has a special $\mathscr{Y}$-preenvelope.

A class $\mathscr{X}$ of $R$-modules is called *precovering* (respectively *preenveloping*) if every $R$-module possesses a $\mathscr{X}$-precover (respectively $\mathscr{X}$-preenvelope).

**Proposition 2.5.** *Assume that every $R$-module in ${}^{\perp}(\mathcal{GP}^{\perp})$ has a Gorenstein projective precover (this is the case if for instance the class $\mathcal{GP}$ is precovering). Then $\mathcal{GP}$ coincides with the class of $\mathcal{GP}$-pure projective $R$-modules.*

**Proof.** Obviously, any Gorenstein projective $R$-module is $\mathcal{GP}$-pure projective. Assume that $P$ is a $\mathcal{GP}$-pure projective $R$-module. Then, Lemma 2.2(i) yields that $P \in {}^{\perp}(\mathcal{GP}^{\perp})$. Thus, by the hypothesis $P$ admits a Gorenstein projective precover. Hence, there is an exact sequence

$$\mathbf{X} = 0 \longrightarrow K \overset{i}{\hookrightarrow} G \overset{\varphi}{\longrightarrow} P \longrightarrow 0,$$

in which $G$ is Gorenstein projective and $\mathrm{Hom}_R(\widetilde{G}, \mathbf{X})$ is exact for all Gorenstein projective $R$-modules $\widetilde{G}$. Equivalently, $\mathbf{X}$ is $\mathcal{GP}$-pure exact. So, as $P$ is $\mathcal{GP}$-pure projective, the sequence

$$0 \longrightarrow \mathrm{Hom}_R(P, K) \xrightarrow{i_*} \mathrm{Hom}_R(P, G) \xrightarrow{\varphi_*} \mathrm{Hom}_R(P, P) \longrightarrow 0$$

is exact. Consequently, $\mathbf{X}$ splits. Since, by [12, Theorem 2.5] the class $\mathcal{GP}$ is closed under direct summands, we deduce that $P$ is Gorenstein projective. □

We end this section by the following useful remark.

**Remark 2.6.**

(i) Each of the theories of classical homology, pure homology, RD-pure homology and cyclically pure homology is a relative pure homology that its corresponding class $\mathscr{X}$ is a class of finitely presented $R$-modules; see [20].

(ii) There is no class $\mathscr{X}$ of finitely presented $R$-modules such that Gorenstein homology is a $\mathscr{X}$-pure homology. To this end, let $R$ be an Artinian Gorenstein local ring which is not a principal ideal ring. (The ring $\mathbb{R}[x, y]/\langle x^2, y^2\rangle$ is an instance of such a ring.) Assume in the contrary that there is a class $\mathscr{X}$ of finitely presented $R$-modules such that Gorenstein homology is a $\mathscr{X}$-pure homology. Then, every $R$-module is $\mathscr{X}$-pure projective. Note that every $R$-module is Gorenstein projective, because $R$ is assumed to be an Artinian Gorenstein local ring. As any pure exact sequence is also $\mathscr{X}$-pure exact, it follows that every $R$-module is pure projective. Hence, the global pure projective dimension of $R$ is zero. Now, [11, Theorem 4.3] yields that $R$ is a principal ideal ring. This contradicts our assumption on $R$.

(iii) Enochs, Jenda and López-Ramos [8, Theorem 2.12] have proved that over a coherent ring $R$, every $R$-module possesses a Gorenstein flat cover. Assume that $R$ is an arbitrary ring admitting a class $\mathscr{X}$ of $R$-modules such that the class of Gorenstein flat $R$-modules coincides with the class of $\mathscr{X}$-pure flat $R$-modules. Then [21, Corollary 2.3] implies that every $R$-module possesses a Gorenstein flat cover.

(iv) Assume that $R$ is such that $\mathcal{GP}$, the class of Gorenstein projective $R$-modules, is precovering. Then for any given $R$-module $M$, we have a complex $\mathbf{Q}_\bullet = \cdots \to Q_i \xrightarrow{d_i} \cdots \xrightarrow{d_1} Q_0 \to 0$ of Gorenstein projective $R$-modules and an $R$-homomorphism $\varphi : Q_0 \to M$ such that the complex $\cdots \to Q_i \xrightarrow{d_i} \cdots \xrightarrow{d_1} Q_0 \xrightarrow{\varphi} M \to 0$ is $\mathcal{GP}$-pure exact; see Definition 3.5 below. We define $\mathrm{Ext}^i_{\mathcal{GP}}(M, N) := H^i(\mathrm{Hom}_R(\mathbf{Q}_\bullet, N))$ for all $R$-modules $N$ and all $i \geqslant 0$. Definition of $\mathrm{Ext}^i_{\mathcal{GP}}(M, N)$ is independent of the choose of $\mathbf{Q}_\bullet$. Jørgensen [16, Corollary 2.13] proved that over any Noetherian ring with dualizing complex, the class $\mathcal{GP}$ is precovering. Then, Murfet and Salarian [18, Theorem A.1] extended his result to Noetherian rings of finite Krull dimension. More recently, it was shown that if $R$ is coherent and every flat $R$-module has finite projective dimension, then

$(\mathcal{GP}, \mathcal{GP}^{\perp})$ is a complete cotorsion theory. This is part of the ongoing work [4] which is also reported in [10].

(v) Let $R$ be such that $\mathcal{GI}$, the class of Gorenstein injective $R$-modules, is preenveloping. Then for any given $R$-module $N$, there is a complex $\mathbf{E}^\bullet = 0 \to E^0 \xrightarrow{d^0} \cdots \to E^i \xrightarrow{d^i} \cdots$ of Gorenstein injective $R$-modules and an $R$-homomorphism $\psi : N \to E^0$ such that the complex $0 \to N \xrightarrow{\psi} E^0 \xrightarrow{d^0} \cdots \to E^i \xrightarrow{d^i} \cdots$ is $\mathcal{GI}$-copure exact; see Definition 3.5 below. We define $\mathrm{Ext}^i_{\mathcal{GI}}(M, N) := H^i(\mathrm{Hom}_R(M, \mathbf{E}^\bullet))$ for all $R$-modules $M$ and all $i \geqslant 0$. This definition is well-defined. Let $R$ be a Noetherian ring. Then [17, Theorem 7.12] yields that $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ is a complete cotorsion theory. In particular, $\mathcal{GI}$ is a preenveloping class.

## 3. Gorenstein and relative pure injectivity and flatness

In what follows, we denote the Pontryagin duality functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ by $(-)^+$. We summarize [21, Lemma 2.2] and [12, Theorem 3.6] in the following lemma.

**Lemma 3.1.** *Let $R$ be a coherent ring, $M$ an $R$-module and $\mathscr{X}$ a class of $R$-modules.*

(i) *$M$ is $\mathscr{X}$-pure flat if and only if $M^+$ is $\mathscr{X}$-pure injective.*
(ii) *$M$ is Gorenstein flat if and only if $M^+$ is Gorenstein injective.*

We record the following immediate corollary.

**Corollary 3.2.** *Let $R$ be a coherent ring. Assume that the class of $\mathcal{GP}$-pure injective $R$-modules coincides with the class of Gorenstein injective $R$-modules. Then the class of $\mathcal{GP}$-pure flat $R$-modules coincides with the class of Gorenstein flat $R$-modules.*

The next result implies the Gorenstein injective and Gorenstein flat counterparts of Lemma 2.2(ii).

**Lemma 3.3.** *The following statements hold.*

(i) *Every $\mathcal{GP}$-pure injective $R$-module of finite Gorenstein injective dimension is Gorenstein injective.*
(ii) *Assume that $R$ is coherent. Every $\mathcal{GP}$-pure flat $R$-module of finite Gorenstein flat dimension is Gorenstein flat.*

**Proof.** (i) Let $M$ be a $\mathcal{GP}$-pure injective $R$-module of finite Gorenstein injective dimension. In view of [12, Theorem 2.15] and [13, Lemma 3.5], there exists a $\mathcal{GP}$-pure exact sequence

$$0 \longrightarrow M \xrightarrow{f} E \longrightarrow C \longrightarrow 0, \qquad (\dagger)$$

where $E$ is Gorenstein injective. Since $M$ is $\mathcal{GP}$-pure injective, we obtain the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(C, M) \longrightarrow \mathrm{Hom}_R(E, M) \xrightarrow{f^*} \mathrm{Hom}_R(M, M) \longrightarrow 0.$$

Hence (†) splits, and so by [12, Theorem 2.6], $M$ is Gorenstein injective.

(ii) Let $M$ be a $\mathcal{GP}$-pure flat $R$-module of finite Gorenstein flat dimension. Since $M$ has a finite Gorenstein flat resolution, Lemma 3.1 yields that $M^+$ is $\mathcal{GP}$-pure injective and it has finite Gorenstein injective dimension. Thus by (i), we obtain that $M^+$ is Gorenstein injective. By using Lemma 3.1 again, we deduce that $M$ is Gorenstein flat. □

Let $\mathscr{X}$ be a class of $R$-modules and $\mathbf{Y} := 0 \to L \to M \to N \to 0$ an exact sequence of $R$-modules. We say $\mathbf{Y}$ is *$\mathscr{X}$-copure exact* if $\mathrm{Hom}_R(\mathbf{Y}, V)$ is exact for all $V \in \mathscr{X}$. In what follows, we will use the following standard fact:

**Lemma 3.4.** *Let $\mathscr{X}$ be a class of $R$-modules and $\mathbf{X} = \cdots \to X_{i+1} \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \to \cdots$ an exact complex of $R$-modules. For each $i \in \mathbb{Z}$, set $\mathbf{X_i} := 0 \to \mathrm{im}\, d_{i+1} \hookrightarrow X_i \to \mathrm{im}\, d_i \to 0$. Then*

(i) $\mathrm{Hom}_R(U, \mathbf{X})$ *is exact for all $U \in \mathscr{X}$ if and only if $\mathbf{X_i}$ is $\mathscr{X}$-pure exact for all $i \in \mathbb{Z}$.*
(ii) $\mathrm{Hom}_R(\mathbf{X}, V)$ *is exact for all $V \in \mathscr{X}$ if and only if $\mathbf{X_i}$ is $\mathscr{X}$-copure exact for all $i \in \mathbb{Z}$.*

**Definition 3.5.** Let $\mathscr{X}$ be a class of $R$-modules. An exact complex $\mathbf{X}$ of $R$-modules is said to be *$\mathscr{X}$-pure exact* (respectively *$\mathscr{X}$-copure exact*) if it satisfies the equivalent conditions of Lemma 3.4(i) (respectively Lemma 3.4(ii)).

The following lemma will be used several times in the rest of the paper. Its second part is also followed by [17, Theorem 7.12(4)] when $R$ is assumed to be Noetherian.

**Lemma 3.6.** *Let $\mathcal{P}$ and $\mathcal{I}$ be the classes of projective and injective $R$-modules, respectively. Then the following assertions hold.*

(i) $\mathcal{GP} \cap \mathcal{GP}^{\perp} = \mathcal{P}$.
(ii) $\mathcal{GI} \cap^{\perp} \mathcal{GI} = \mathcal{I}$.

**Proof.** The proofs of (i) and (ii) are similar, and so we only prove (i).

Clearly, any projective $R$-module is Gorenstein projective. On the other hand, from the definition of Gorenstein projective modules, it is immediate that $\mathrm{Ext}^i_R(Q, P) = 0$ for all $Q \in \mathcal{GP}$, $P \in \mathcal{P}$ and $i > 0$. Hence, $\mathcal{P} \subseteq \mathcal{GP} \cap \mathcal{GP}^{\perp}$. Next, let $M \in \mathcal{GP} \cap \mathcal{GP}^{\perp}$. As $M$ is Gorenstein projective, there exists an exact sequence

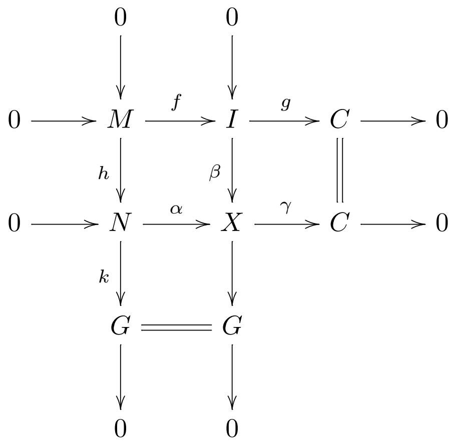
$$0 \longrightarrow M \longrightarrow P \longrightarrow \widetilde{M} \longrightarrow 0 \qquad (*)$$

of $R$-modules, in which $P$ is projective and $\widetilde{M}$ is Gorenstein projective. Since $M \in \mathcal{GP}^{\perp}$, we have $\mathrm{Ext}^1_R(\widetilde{M}, M) = 0$, and so $(*)$ splits. Thus $M$ is projective. □

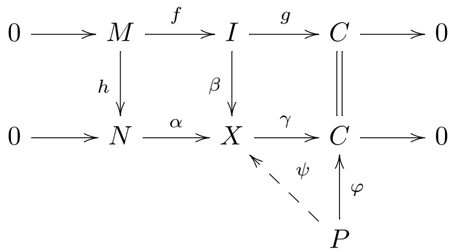
**Lemma 3.7.** *Assume that $R$ is a Noetherian ring of finite Krull dimension. Then ${}^{\perp}\mathcal{GI} \subseteq \mathcal{GP}^{\perp}$.*

**Proof.** Let $M \in^{\perp} \mathcal{GI}$. By Remark 2.6(v), there exists a $\mathcal{GI}$-copure exact sequence $0 \to M \xrightarrow{f} I \xrightarrow{g} C \to 0$, where $I$ is Gorenstein injective and $C \in^{\perp} \mathcal{GI}$. Since both $M$ and $C$ belong to ${}^{\perp}\mathcal{GI}$, it follows that $I \in^{\perp} \mathcal{GI}$. So, $I$ is injective by Lemma 3.6(ii). Next, as by Remark 2.6(iv) the pair $(\mathcal{GP}, \mathcal{GP}^{\perp})$ is a complete cotorsion theory, we can find an exact sequence $0 \to M \xrightarrow{h} N \xrightarrow{k} G \to 0$, where $N \in \mathcal{GP}^{\perp}$ and $G \in \mathcal{GP}$.

Consider the following pushout diagram




where $X := (N \oplus I)/S$ with $S = \{(h(m), -f(m)) \mid m \in M\}$ and $\gamma((x, y) + S) = g(y)$ for all $(x, y) + S \in X$. We intend to prove that the upper short exact sequence is $\mathcal{GP}$-pure exact. To this end, we have to show that for every Gorenstein projective $R$-module $P$, the induced $R$-homomorphism $\mathrm{Hom}_R(P, I) \longrightarrow \mathrm{Hom}_R(P, C)$ is surjective. Let $P$ be a Gorenstein projective $R$-module and $\varphi : P \longrightarrow C$ be an $R$-homomorphism. As $N \in \mathcal{GP}^{\perp}$, the lower short exact sequence is $\mathcal{GP}$-pure exact. So, there exists an $R$-homomorphism $\psi : P \to X$ such that $\gamma\psi = \varphi$. Thus, we have the following commutative diagram:




Because $\beta : I \longrightarrow X$ is one-to-one and the $R$-module $I$ is injective, there is an $R$-homomorphism $\rho : X \longrightarrow I$ such that $\rho\beta = \mathrm{id}_I$. Set $\theta := \rho\psi$. Then $\theta \in \mathrm{Hom}_R(P, I)$, and it can be easily verified that $g\theta = \varphi$. Thus the upper short exact sequence is $\mathcal{GP}$-pure exact, as required.

Now, let $Q$ be a Gorenstein projective $R$-module. As $I$ is injective, one has $\mathrm{Ext}^1_R(Q, I) = 0$. Hence, we have the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(Q, M) \longrightarrow \mathrm{Hom}_R(Q, I) \longrightarrow \mathrm{Hom}_R(Q, C) \longrightarrow \mathrm{Ext}^1_R(Q, M) \longrightarrow 0.$$

This implies that $\mathrm{Ext}^1_R(Q, M) = 0$, and so $M \in \mathcal{GP}^{\perp}$. □

The following corollary will be needed in the proof of Theorem 3.10.

**Corollary 3.8.** *Assume that $R$ is a Noetherian ring of finite Krull dimension. Then every $R$-module $M$ admits a $\mathcal{GI}$-copure exact complex $0 \to M \xrightarrow{\psi} E^0 \xrightarrow{d^0} \cdots \to E^i \xrightarrow{d^i} \cdots$ which is $\mathcal{GP}$-pure exact and each $E^i$ is Gorenstein injective.*

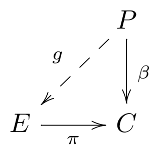
**Proof.** Let $M$ be an $R$-module. By Remark 2.6(v), $M$ possesses a Gorenstein injective preenvelope $\psi : M \to E$ with $C := \operatorname{Coker}\psi \in {}^{\perp}\mathcal{GI}$. Then, the sequence

$$\mathbf{X} = 0 \longrightarrow M \xrightarrow{\psi} E \xrightarrow{\pi} C \longrightarrow 0$$

is $\mathcal{GI}$-copure exact. Let $Q$ be a Gorenstein projective $R$-module. We show that $\operatorname{Hom}_R(Q, \mathbf{X})$ is exact. Equivalently, we prove that $\pi_* : \operatorname{Hom}_R(Q, E) \to \operatorname{Hom}_R(Q, C)$ is surjective. Let $\alpha : Q \to C$ be an $R$-homomorphism. There exists an exact sequence

$$0 \longrightarrow Q \xrightarrow{f} P \longrightarrow \widetilde{Q} \longrightarrow 0,$$

where $P$ is projective and $\widetilde{Q}$ is Gorenstein projective. Lemma 3.7 implies that $C \in \mathcal{GP}^{\perp}$, and so $\operatorname{Ext}^1_R(\widetilde{Q}, C) = 0$. Hence, $f^* : \operatorname{Hom}_R(P, C) \to \operatorname{Hom}_R(Q, C)$ is surjective, and so there exists an $R$-homomorphism $\beta : P \to C$ such that $\alpha = f^*(\beta) = \beta f$. Since $P$ is projective, we have an $R$-homomorphism $g : P \to E$ making the following diagram commutative




Now, for the $R$-homomorphism $gf : Q \to E$, one has

$$\pi_*(gf) = \pi(gf) = \beta f = \alpha.$$

By continuing the above argument and applying Lemma 3.4, we obtain a $\mathcal{GI}$-copure exact complex $0 \to M \xrightarrow{\psi} E^0 \xrightarrow{d^0} \cdots \to E^i \xrightarrow{d^i} \cdots$ which is $\mathcal{GP}$-pure exact and each $E^i$ is Gorenstein injective. □

The notion of virtually Gorenstein Artin algebras was introduced by Beligiannis and Reiten in [3]; see also [2]. Next, we extend this notion to commutative rings.

**Definition 3.9.** A Noetherian ring $R$ of finite Krull dimension is called *virtually Gorenstein* if $\mathcal{GP}^{\perp} = {}^{\perp}\mathcal{GI}$.

Next, we present a characterization of virtually Gorenstein rings. Since in its statement the phrase "right balanced" is appeared, we recall the meaning of this phrase here. Let $\mathcal{F}$ and $\mathcal{G}$ be two classes of $R$-modules. The functor $\operatorname{Hom}_R(-, \sim)$ is said to be *right balanced* by $\mathcal{F} \times \mathcal{G}$ if for every $R$-module $M$, there are complexes

$$\mathbf{F}_\bullet = \cdots \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

and

$$\mathbf{G}^\bullet = 0 \to M \to G^0 \to \cdots \to G^n \to G^{n+1} \to \cdots$$

in which $F_n \in \mathcal{F}, G^n \in \mathcal{G}$ for all $n \geqslant 0$, such that for any $F \in \mathcal{F}$ and any $G \in \mathcal{G}$, the two complexes $\operatorname{Hom}_R(\mathbf{F}_\bullet, G)$ and $\operatorname{Hom}_R(F, \mathbf{G}^\bullet)$ are exact.

**Theorem 3.10.** *Let $R$ be a Noetherian ring of finite Krull dimension and let $\mathcal{GP}$ and $\mathcal{GI}$ be the classes of Gorenstein projective and Gorenstein injective $R$-modules, respectively. The following are equivalent:*

(i) $\operatorname{Hom}_R(-, \sim)$ *is right balanced by* $\mathcal{GP} \times \mathcal{GI}$.

(ii) *A short exact sequence* $\mathbf{X} = 0 \to X_1 \to X_2 \to X_3 \to 0$ *of* $R$*-modules is* $\mathcal{GP}$*-pure exact if and only if it is* $\mathcal{GI}$*-copure exact.*

(iii) $R$ *is virtually Gorenstein.*

**Proof.** (i) $\Rightarrow$ (ii) Assume that $\mathbf{X}$ is $\mathcal{GP}$-pure exact and $E$ is a Gorenstein injective $R$-module. Then, by [7, Theorem 8.2.3(2)] we obtain the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(X_3, E) \longrightarrow \mathrm{Hom}_R(X_2, E) \longrightarrow \mathrm{Hom}_R(X_1, E) \longrightarrow \mathrm{Ext}^1_{\mathcal{GP}}(X_3, E).$$

As $E$ is Gorenstein injective, by [7, Theorem 8.2.14] we conclude that

$$\mathrm{Ext}^1_{\mathcal{GP}}(X_3, E) \cong \mathrm{Ext}^1_{\mathcal{GI}}(X_3, E) = 0.$$

Hence, $\mathbf{X}$ is $\mathcal{GI}$-copure exact. Similarly if $\mathbf{X}$ is $\mathcal{GI}$-copure exact, then by using [7, Theorem 8.2.5(1)], we can deduce that $\mathbf{X}$ is $\mathcal{GP}$-pure exact.

(ii) $\Rightarrow$ (iii) In view of Lemma 3.7, it is enough to show that $\mathcal{GP}^{\perp} \subseteq {}^{\perp}\mathcal{GI}$. Let $M \in \mathcal{GP}^{\perp}$. Since, by Remark 2.6(iv), $M$ admits a special Gorenstein projective precover, we have an exact sequence

$$0 \longrightarrow K \longrightarrow Q \longrightarrow M \longrightarrow 0, \qquad (*)$$

in which $Q$ is Gorenstein projective and $K \in \mathcal{GP}^{\perp}$. Then $(*)$ is $\mathcal{GP}$-pure exact, and so by the assumption it is also $\mathcal{GI}$-copure. From $(*)$, we can see that $Q \in \mathcal{GP}^{\perp}$. But, then Lemma 3.6(i) yields that $Q$ is projective. Let $E$ be a Gorenstein injective $R$-module. Since $\mathrm{Ext}^1_R(Q, E) = 0$, we deduce the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, E) \longrightarrow \mathrm{Hom}_R(Q, E) \longrightarrow \mathrm{Hom}_R(K, E) \longrightarrow \mathrm{Ext}^1_R(M, E) \longrightarrow 0.$$

But, the functor $\mathrm{Hom}_R(-, E)$ leaves $(*)$ exact, and so $\mathrm{Ext}^1_R(M, E) = 0$. Thus, $M \in {}^{\perp}\mathcal{GI}$.

(iii) $\Rightarrow$ (i) Let $M$ be an $R$-module. By an argument dual to the proof of Corollary 3.8, we can construct a $\mathcal{GP}$-pure exact complex $\cdots \to Q_i \xrightarrow{d_i} \cdots \xrightarrow{d_1} Q_0 \xrightarrow{\varphi} M \to 0$ which is $\mathcal{GI}$-copure exact and each $Q_i$ is Gorenstein projective. Thus, in view of Corollary 3.8, it turns out that $\mathrm{Hom}_R(-, \sim)$ is right balanced by $\mathcal{GP} \times \mathcal{GI}$. $\square$

In view of [7, Theorem 8.2.14], we record the following immediate corollary.

**Corollary 3.11.** *Let* $R$ *be a virtually Gorenstein ring. Then* $\mathrm{Ext}^i_{\mathcal{GP}}(M, N) \cong \mathrm{Ext}^i_{\mathcal{GI}}(M, N)$ *for all* $R$*-modules* $M$ *and* $N$ *and all* $i \geqslant 0$.

The next result provides an answer to the main question of this investigation in the case the ground ring is Noetherian of finite Krull dimension.

**Theorem 3.12.** *Let* $R$ *be a Noetherian ring of finite Krull dimension and* $\mathcal{GP}$ *denote the class of Gorenstein projective* $R$*-modules. The following are equivalent:*

(i) $R$ *is virtually Gorenstein.*

(ii) *the classes of Gorenstein injective and* $\mathcal{GP}$*-pure injective* $R$*-modules are the same.*

(iii) *Gorenstein homology is a* $\mathcal{GP}$*-pure homology.*

**Proof.** (i) $\Rightarrow$ (ii) Let $E$ be a $\mathcal{GP}$-pure injective $R$-module. Then $E$ has a Gorenstein injective preenvelope $\psi : E \to \widetilde{E}$ with $C := \mathrm{Coker}\,\psi \in^{\perp} \mathcal{GI}$; see Remark 2.6(v). One can easily see that the sequence

$$\mathbf{X} = 0 \longrightarrow E \xrightarrow{\psi} \widetilde{E} \longrightarrow C \longrightarrow 0$$

is $\mathcal{GI}$-copure exact, and so by Theorem 3.10 it is also $\mathcal{GP}$-pure exact. Hence, as $E$ is $\mathcal{GP}$ -pure injective, the sequence $\mathrm{Hom}_R(\mathbf{X}, E)$ is exact, and so $\mathbf{X}$ splits. Thus $E$ is a direct summand of $\widetilde{E}$, and so it is Gorenstein injective by [12, Theorem 2.6].

Conversely, let $E$ be a Gorenstein injective $R$-module. Let $\mathbf{X} = 0 \to X_1 \to X_2 \to X_3 \to 0$ be a $\mathcal{GP}$-pure exact sequence. Our assumptions on $R$ implies that the class $\mathcal{GP}$ is precovering, and so by [7, Theorem 8.2.3(2)] we have the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(X_3, E) \longrightarrow \mathrm{Hom}_R(X_2, E) \longrightarrow \mathrm{Hom}_R(X_1, E) \longrightarrow \mathrm{Ext}^1_{\mathcal{GP}}(X_3, E)$$

Now, Corollary 3.11 yields that

$$\mathrm{Ext}^1_{\mathcal{GP}}(X_3, E) \cong \mathrm{Ext}^1_{\mathcal{GI}}(X_3, E) = 0,$$

and so $\mathrm{Hom}_R(\mathbf{X}, E)$ is exact. Thus, $E$ is $\mathcal{GP}$-pure injective.

(ii) $\Rightarrow$ (i) By Lemma 3.7, it suffices to show that $\mathcal{GP}^\perp \subseteq {}^\perp\mathcal{GI}$. Let $M \in \mathcal{GP}^\perp$. Then, as by Remark 2.6(iv) $M$ has a special Gorenstein projective precover, we have a $\mathcal{GP}$-pure exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0, \qquad (*)$$

where $P$ is Gorenstein projective and $K \in \mathcal{GP}^\perp$. As $M$ and $K$ belong to $\mathcal{GP}^\perp$, we deduce that $P \in \mathcal{GP} \cap \mathcal{GP}^\perp$. Hence, $P$ is projective by Lemma 3.6(i). Since every Gorenstein injective $R$-module is $\mathcal{GP}$-pure injective, it follows that $(*)$ is also $\mathcal{GI}$-copure exact. Let $E$ be a Gorenstein injective $R$-module. Then the functor $\mathrm{Hom}_R(-, E)$ leaves $(*)$ exact and $\mathrm{Ext}^1_R(P, E) = 0$. Thus, from the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, E) \longrightarrow \mathrm{Hom}_R(P, E) \longrightarrow \mathrm{Hom}_R(K, E) \longrightarrow \mathrm{Ext}^1_R(M, E) \longrightarrow 0,$$

we conclude that $\mathrm{Ext}^1_R(M, E) = 0$. Equivalently, $M \in {}^\perp\mathcal{GI}$.

(ii) $\Longleftrightarrow$ (iii) follows in view of Remark 2.6(iv), Proposition 2.5, and Corollary 3.2. □

We conclude the paper with the following example which, in particular shows that there exists a ring such that the functor $\mathrm{Hom}_R(-, \sim)$ is not right balanced by $\mathcal{GP} \times \mathcal{GI}$.

**Example 3.13.**

(i) By [7, Remarks 11.2.3 and 11.5.10], every finite Krull dimensional Gorenstein ring is virtually Gorenstein.

(ii) Let $(R, \mathfrak{m})$ be any local ring with $\mathfrak{m}^2 = 0$. Then by [15, Proposition 6.1 and Remark 6.5] the only Gorenstein projectives are the projectives and the only Gorenstein injectives are the injectives. Thus both classes $\mathcal{GP}^\perp$ and ${}^\perp\mathcal{GI}$ are equal to the class of all $R$-modules, and so $R$ is virtually Gorenstein. So, we have plenty examples of virtually Gorenstein local rings which are not Gorenstein.

(iii) Let $k$ be a field. Then $R := k[x, y, z]/\langle x^2, yz, y^2 - xz, z^2 - yx\rangle$ is not virtually Gorenstein by [2, Proposition 4.3]. Hence, $\mathrm{Hom}_R(-, \sim)$ is not right balanced by $\mathcal{GP} \times \mathcal{GI}$.

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