Journal of Algebra 357 (2012) 69–93

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Multiparameter twisted Weyl algebras

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ARTICLE INFO

Article history:
Received 29 March 2011
Available online 2 March 2012
Communicated by Vera Serganova

Keywords:
Quantized Weyl algebra
Generalized Weyl algebra
Simple ring
Whittaker module
Irreducible representation

ABSTRACT

We introduce a new family of twisted generalized Weyl algebras, called multiparameter twisted Weyl algebras, for which we parametrize all simple quotients of a certain kind. Both Jordan's simple localization of the multiparameter quantized Weyl algebra and Hayashi's q-analog of the Weyl algebra are special cases of this construction. We classify all simple weight modules over any multiparameter twisted Weyl algebra. Extending results by Benkart and Ondrus, we also describe all Whittaker pairs up to isomorphism over a class of twisted generalized Weyl algebras which includes the multiparameter twisted Weyl algebras.

Contents

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.11.004

1. Introduction

Let R be an algebra, $\sigma_1,\dots,\sigma_n$ commuting algebra automorphisms of R, $t_1,\dots,t_n$ elements from the center of R, and μ_{ij} an $n\times n$ matrix of invertible scalars. To these data one associates a *twisted generalized Weyl algebra* $\mathcal{A}_\mu(R,\sigma,t)$, an associative $\mathbb{Z}^n$-graded algebra (see Section 2.1 for definition). These algebras were introduced by Mazorchuk and Turowska [18] and they are generalizations of the much studied generalized Weyl algebras, defined independently by Bavula [3], Jordan [14], and Rosenberg [21] (called there *hyperbolic rings*).

Simple weight modules over twisted generalized Weyl algebras have been studied in [18,17,10]. In [19] bounded and unbounded $*$-representations over twisted generalized Weyl algebras were classified. Interesting examples of twisted generalized Weyl algebras were given in [17]. In [11] new examples of twisted generalized Weyl algebras were constructed from symmetric Cartan matrices.

In this paper we define a family of twisted generalized Weyl algebras, called *multiparameter twisted Weyl algebras*. They are unital associative algebras generated by $u_i^{\pm 1}$, $v_i^{\pm 1}$, X_i, Y_i $(i=1,\dots,n)$ subject to certain relations depending on some parameters (see Section 5.1 for details). This definition was inspired by an unpublished note by Benkart [5] where *multiparameter Weyl algebras* were introduced. These algebras are a particular case of the algebras of our class.

Multiparameter quantized Weyl algebras $A_n^{\bar{q},\Lambda}$ were introduced in [16] as a generalization of the quantized Weyl algebras obtained by Pusz and Woronowicz [20] in the context of quantum group covariant differential calculus. They are examples of twisted generalized Weyl algebras. Contrary to the usual Weyl algebras the algebra, $A_n^{\bar{q},\Lambda}$ is in general not simple, even for generic parameters. Jordan [15] found a certain natural simple localization of $A_n^{\bar{q},\Lambda}$.

The first main theorem of the paper parametrizes simple quotients of multiparameter twisted Weyl algebras in terms of maximal ideals of certain Laurent polynomial rings. Jordan's localization of $A_n^{\bar{q},\Lambda}$ is an example in this family, as well as Hayashi's q-deformed Weyl algebras [13].

Theorem A. *Let $A=A_n^k(r,s,\Lambda)$ be a multiparameter twisted Weyl algebra.*

(a) *The assignment*

$$\mathfrak{n}\mapsto A/\langle\mathfrak{n}\rangle \tag{1.1}$$

where $\langle\mathfrak{n}\rangle$ denotes the ideal in A generated by $\mathfrak{n}$, is a bijection between the set of maximal ideals in the invariant subring $R^{\mathbb{Z}^n}$ and the set of simple quotients of A in which the generators X_i, Y_i $(i = 1, \dots, n)$ are regular.

(b) *For any $\mathfrak{n} \in \operatorname{Specm}(R^{\mathbb{Z}^n})$, the quotient $A/\langle\mathfrak{n}\rangle$ is isomorphic to the twisted generalized Weyl algebra $\mathcal{A}_\mu(R/R\mathfrak{n}, \bar\sigma, \bar t)$, where $\bar\sigma_g(r + R\mathfrak{n}) = \sigma_g(r) + R\mathfrak{n}$, $\forall g \in \mathbb{Z}^n$, $r \in R$ and $\bar t_i = t_i + R\mathfrak{n}$, $\forall i$.*

(c) *$A/\langle\mathfrak{n}\rangle$ is a domain for all $\mathfrak{n} \in \operatorname{Specm}(R^{\mathbb{Z}^n})$ if and only if $\mathbb{Z}^{2n}/G$ is torsion-free, where G is the gradation group of $R^{\mathbb{Z}^n}$.*

The second main theorem of the paper gives the explicit relation between four twisted generalized Weyl algebras, namely the multiparameter quantized Weyl algebra $A_n^{\bar q,\Lambda}$, Jordan's localization $B_n^{\bar q,\Lambda}$, a specific multiparameter twisted Weyl algebra $A_n^k(r,s,\Lambda)$ that we define, and a certain quotient $\widehat{A_n^k(r,s,\Lambda)}$ of it which is simple and isomorphic to $B_n^{\bar q,\Lambda}$.

Theorem B. *We have a commutative diagram in the category of $\mathbb{Z}^n$-graded algebras:*

$$\begin{tikzcd}
A_n^k(r,s,\Lambda) \arrow[d, two heads] \arrow[dr, two heads] & \\
\widehat{A_n^k(r,s,\Lambda)} \arrow[r, shift left, "\simeq"] & B_n^{\bar q,\Lambda} \arrow[l, shift left] \\
& A_n^{\bar q,\Lambda} \arrow[u, hook] \arrow[ul, hook]
\end{tikzcd}$$

We end the introduction with an overview of the content of this paper. In Sections 3 and 4, we first consider certain families of twisted generalized Weyl algebras. Section 5 is devoted to the definition and structural results for multiparameter twisted Weyl algebras, with a proof of Theorem A in Section 5.3. Examples and relations to existing algebras are given in Sections 6 and 7, where Theorem B is proved. Representations of multiparameter twisted Weyl algebras are studied in Sections 8 and 9.

1.1. Notation and conventions

By "ring" ("algebra") we mean unital associative ring (algebra). All ring and algebra morphisms are required to be unital. By "ideal" we mean two-sided ideal unless otherwise stated. An element x of a ring R is said to be *regular in R* if for all nonzero $y \in R$ we have $xy \neq 0$ and $yx \neq 0$. The set of invertible elements in a ring R will be denoted by $R^\times$.

Let R be a ring. Recall that an R-ring is a ring A together with a ring morphism $R \to A$. Let X be a set. Let RXR be the free R-bimodule on X. The free R-ring $F_R(X)$ on X is defined as the tensor algebra of the free R-bimodule on X: $F_R(X) = \bigoplus_{n\geqslant 0}(RXR)^{\otimes_R n}$ where $(RXR)^{\otimes_R 0} = R$ by convention and the ring morphism $R \to F_R(X)$ is the inclusion into the degree zero component.

2. Twisted generalized Weyl algebras

Throughout this section we fix a commutative ring $\Bbbk$.

2.1. Definition

We recall the definition of twisted generalized Weyl algebras [18,17].

Definition 2.1 *(TGW datum).* Let n a positive integer. A *twisted generalized Weyl datum* (*over* $\Bbbk$ *of degree* n) is a triple (R, σ, t) where

- R is a unital associative $\Bbbk$-algebra,
- σ is a group homomorphism $\sigma : \mathbb{Z}^n \to \mathrm{Aut}_{\Bbbk}(R)$, $g \mapsto \sigma_g$,
- t is a function $t : \{1, \dots, n\} \to Z(R)$, $i \mapsto t_i$.

A *morphism* between TGW data over $\Bbbk$ of degree n,

$$\varphi : (R, \sigma, t) \to \bigl(R', \sigma', t'\bigr)$$

is a $\Bbbk$-algebra morphism $\varphi : R \to R'$ such that $\varphi\sigma_i = \sigma_i'\varphi$ and $\varphi(t_i) = t_i'$ for all $i \in \{1, \dots, n\}$. We let $\mathrm{TGW}_n(\Bbbk)$ denote the category whose objects are the TGW data over $\Bbbk$ of degree n and morphisms are as above.

For $i \in \{1, \dots, n\}$ we put $\sigma_i = \sigma_{e_i}$, where $\{e_i\}_{i=1}^n$ is the standard $\mathbb{Z}$-basis for $\mathbb{Z}^n$. A *parameter matrix* (*over* $\Bbbk^\times$ *of size* n) is an $n \times n$ matrix $\mu = (\mu_{ij})_{i \neq j}$ without diagonal where $\mu_{ij} \in \Bbbk^\times$ for all $i \neq j$. The set of all parameter matrices over $\Bbbk^\times$ of size n will be denoted by $\mathrm{PM}_n(\Bbbk)$.

Definition 2.2 *(TGW construction).* Let $n \in \mathbb{Z}_{>0}$, (R, σ, t) be an object in $\mathrm{TGW}_n(\Bbbk)$, and $\mu \in \mathrm{PM}_n(\Bbbk)$. The *twisted generalized Weyl construction with parameter matrix* μ *associated to the TGW datum* (R, σ, t) is denoted by $\mathcal{C}_\mu(R, \sigma, t)$ and is defined as the free R-ring on the set $\{x_i, y_i \mid i = 1, \dots, n\}$ modulo two-sided ideal generated by the following set of elements:

$$x_i r - \sigma_i(r) x_i, \qquad y_i r - \sigma_i^{-1}(r) y_i, \quad \forall r \in R,\ i \in \{1, \dots, n\}, \tag{2.1a}$$

$$y_i x_i - t_i, \qquad x_i y_i - \sigma_i(t_i), \quad \forall i \in \{1, \dots, n\}, \tag{2.1b}$$

$$x_i y_j - \mu_{ij} y_j x_i, \quad \forall i, j \in \{1, \dots, n\},\ i \neq j. \tag{2.1c}$$

The images in $\mathcal{C}_\mu(R, \sigma, t)$ of the elements x_i, y_i will be denoted by $\widehat{X}_i$, $\widehat{Y}_i$ respectively. The ring $\mathcal{C}_\mu(R, \sigma, t)$ has a $\mathbb{Z}^n$-gradation given by requiring $\deg \widehat{X}_i = e_i$, $\deg \widehat{Y}_i = -e_i$, $\deg r = 0$, $\forall r \in R$. Let $\mathcal{I}_\mu(R, \sigma, t) \subseteq \mathcal{C}_\mu(R, \sigma, t)$ be the sum of all graded ideals $J \subseteq \mathcal{C}_\mu(R, \sigma, t)$ having zero intersection with the degree zero component, i.e. such that $\mathcal{C}_\mu(R, \sigma, t)_0 \cap J = \{0\}$. It is easy to see that $\mathcal{I}_\mu(R, \sigma, t)$ is the unique maximal graded ideal having zero intersection with the degree zero component.

Definition 2.3 *(TGW algebra).* The *twisted generalized Weyl algebra with parameter matrix* μ *associated to the TGW datum* (R, σ, t) is denoted $\mathcal{A}_\mu(R, \sigma, t)$ and is defined as the quotient $\mathcal{A}_\mu(R, \sigma, t) := \mathcal{C}_\mu(R, \sigma, t)/\mathcal{I}_\mu(R, \sigma, t)$.

Since $\mathcal{I}_\mu(R, \sigma, t)$ is graded, $\mathcal{A}_\mu(R, \sigma, t)$ inherits a $\mathbb{Z}^n$-gradation from $\mathcal{C}_\mu(R, \sigma, t)$. The images in $\mathcal{A}_\mu(R, \sigma, t)$ of the elements $\widehat{X}_i$, $\widehat{Y}_i$ will be denoted by X_i, Y_i. By a *monic monomial* in a TGW construction $\mathcal{C}_\mu(R, \sigma, t)$ (respectively TGW algebra $\mathcal{A}_\mu(R, \sigma, t)$) we will mean a product of elements from $\{\widehat{X}_i, \widehat{Y}_i \mid i = 1, \dots, n\}$ (respectively $\{X_i, Y_i \mid i = 1, \dots, n\}$).

The following statements are easy to check.

Lemma 2.4.

(a) $\mathcal{A}_\mu(R, \sigma, t)$ (*respectively* $\mathcal{C}_\mu(R, \sigma, t)$) *is generated as a left and as a right* R*-module by the monic monomials in* X_i, Y_i $(i = 1, \dots, n)$ (*respectively* $\widehat{X}_i$, $\widehat{Y}_i$ $(i = 1, \dots, n)$).
(b) *The degree zero component of* $\mathcal{A}_\mu(R, \sigma, t)$ *is equal to the image of* R *under the natural map* $\rho : R \to \mathcal{A}_\mu(R, \sigma, t)$.
(c) *Any nonzero graded ideal of* $\mathcal{A}_\mu(R, \sigma, t)$ *has nonzero intersection with the degree zero component.*

Definition 2.5 *(μ-Consistency).* Let (R, σ, t) be a TGW datum over $\Bbbk$ of degree n and μ be a parameter matrix over $\Bbbk^\times$ of size n. We say that (R, σ, t) is *μ-consistent* if the canonical map $\rho : R \to \mathcal{A}_\mu(R, \sigma, t)$ is injective.

Since $\mathcal{I}_\mu(R, \sigma, t)$ has zero intersection with the zero-component, (R, σ, t) is μ-consistent iff the canonical map $R \to \mathcal{C}_\mu(R, \sigma, t)$ is injective. Even in the cases when ρ is not injective, we will often view $\mathcal{A}_\mu(R, \sigma, t)$ as a left R-module and write for example rX_i instead of $\rho(r)X_i$.

Definition 2.6 *(Regularity).* A TGW datum (R, σ, t) is called *regular* if t_i is regular in R for all i.

The following result was proved in [9, Theorem 6.2].

Theorem 2.7. *Let $\Bbbk$ be a commutative unital ring, R be an associative $\Bbbk$-algebra, n a positive integer, $t = (t_1, \dots, t_n)$ be an n-tuple of regular central elements of R, $\sigma : \mathbb{Z}^n \to \mathrm{Aut}_\Bbbk(R)$ a group homomorphism, μ_{ij} $(i, j = 1, \dots, n,\ i \neq j)$ invertible elements from $\Bbbk$, and $\mathcal{A}_\mu(R, \sigma, t)$ the corresponding twisted generalized Weyl algebra, equipped with the canonical homomorphism of R-rings $\rho : R \to \mathcal{A}_\mu(R, \sigma, t)$. Then the following two statements are equivalent*:

(a) *ρ is injective,*
(b) *the following two sets of relations are satisfied in R:*

$$\sigma_i\sigma_j(t_it_j) = \mu_{ij}\mu_{ji}\sigma_i(t_i)\sigma_j(t_j), \quad \forall i, j = 1, \dots, n,\ i \neq j, \tag{2.2}$$

$$t_j\sigma_i\sigma_k(t_j) = \sigma_i(t_j)\sigma_k(t_j), \quad \forall i, j, k = 1, \dots, n,\ i \neq j \neq k \neq i. \tag{2.3}$$

In particular, if (2.2) *and* (2.3) *are satisfied, then $\mathcal{A}_\mu(R, \sigma, t)$ is nontrivial iff R is nontrivial. Moreover, neither of the two conditions* (2.2) *and* (2.3) *imply the other.*

Lemma 2.8. *If $t_i \in R^\times$ for all i, then the canonical projection $\mathcal{C}_\mu(R, \sigma, t) \to \mathcal{A}_\mu(R, \sigma, t)$ is an isomorphism.*

Proof. The algebra $\mathcal{C}_\mu(R, \sigma, t)$ is a $\mathbb{Z}^n$-crossed product algebra over its degree zero subalgebra, since each homogenous component contains an invertible element. Indeed since $t_i \in R^\times$, each X_i is invertible and thus $X_1^{g_1} \cdots X_n^{g_n}$ has degree g and is invertible. Therefore any nonzero graded ideal in $\mathcal{C}_\mu(R, \sigma, t)$ has nonzero intersection with the degree zero component, a property which holds for any strongly graded ring, in particular for crossed product algebras. Thus $\mathcal{I}_\mu(R, \sigma, t) = 0$, which proves the claim. □

2.2. TGW algebras which are domains

The following condition for a TGW algebra to be a domain will be used.

Proposition 2.9. *Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a twisted generalized Weyl algebra where (R, σ, t) is μ-consistent. Then A is a domain if and only if R is a domain.*

Proof. Clearly R must be a domain if A is a domain. For the converse, assume R is a domain and suppose $a, b \in A$ are nonzero but $ab = 0$. Let a_g and b_h be the leading terms in a and b respectively, with respect to some order (we use here that the group $\mathbb{Z}^n$ is orderable). Then $a_gb_h = 0$. As in the proof of [11, Proposition 3.1] this forces $a_g = 0$ or $b_h = 0$ which is a contradiction. □

3. Finitistic TGW algebras

Throughout the rest of the paper we assume that $\Bbbk$ is a field.

In [11] the following notion (there called "locally finite" TGW algebra) was defined.

Definition 3.1. A TGW algebra $A = \mathcal{A}_\mu(R,\sigma,t)$ is called $\Bbbk$-*finitistic* if $\dim_\Bbbk V_{ij} < \infty$ for all $i, j \in \{1,\dots,n\}$, where

$$V_{ij} = \operatorname{Span}_\Bbbk\{\sigma_i^k(t_j) \mid k \in \mathbb{Z}\}. \tag{3.1}$$

For each i, j, we denote by $p_{ij} \in \Bbbk[x]$ be the minimal polynomial for σ_i acting on the finite-dimensional space V_{ij}. The following result was proved in [11] (for the case $\mu_{ij} = \mu_{ji}$ and R a commutative domain, but these restrictions are unnecessary).

Theorem 3.2. *Let $A = \mathcal{A}_\mu(R,\sigma,t)$ be a $\Bbbk$-finitistic TGW algebra where (R,σ,t) is μ-consistent.*

(a) *Define the matrix $C_A = (a_{ij})$ with integer entries as follows*

$$a_{ij} = \begin{cases} 2, & i = j, \\ 1 - \deg p_{ij}, & i \neq j. \end{cases} \tag{3.2}$$

Then C_A is a generalized Cartan matrix.

(b) *Assume t_i is regular in R for all i. Writing*

$$p_{ij}(x) = x^{m_{ij}} + \lambda_{ij}^{(1)} x^{m_{ij}-1} + \cdots + \lambda_{ij}^{(m_{ij})},$$

where all $\lambda_{ij}^{(k)} \in \Bbbk$, the following identities hold in A, for any $i \neq j$:

$$X_i^{m_{ij}} X_j + \lambda_{ij}^{(1)} \mu_{ij}^{-1} X_i^{m_{ij}-1} X_j X_i + \cdots + \lambda_{ij}^{(m_{ij})} \mu_{ij}^{-m_{ij}} X_j X_i^{m_{ij}} = 0 \tag{3.3}$$

and

$$Y_j Y_i^{m_{ij}} + \lambda_{ij}^{(1)} \mu_{ji}^{-1} Y_i Y_j Y_i^{m_{ij}-1} + \cdots + \lambda_{ij}^{(m_{ij})} \mu_{ji}^{-m_{ij}} Y_i^{m_{ij}} Y_j = 0. \tag{3.4}$$

Moreover, for any $i \neq j$ and $m < m_{ij}$, the sets $\{X_i^{m-k} X_j X_i^k\}_{k=0}^m$ and $\{Y_i^{m-k} Y_j Y_i^k\}_{k=0}^m$ are linearly independent in A over $\Bbbk$.

This gives an interpretation of the minimal polynomials p_{ij} for $i \neq j$ in terms of identities in the algebra A. If C_A is of type Z (for example A_n, B_n, C_n, D_n, E_6, E_7, E_8 etc.) we say that A is of Lie type Z.

Here we note that the polynomials p_{ii} also give rise to identities in A.

Theorem 3.3. *Let $A = \mathcal{A}_\mu(R,\sigma,t)$ be a $\Bbbk$-finitistic TGW algebra, where (R,σ,t) is regular and μ-consistent. Writing*

$$p_{ii}(x) = x^{m_{ii}} + \lambda_{ii}^{(1)} x^{m_{ii}-1} + \cdots + \lambda_{ii}^{(m_{ii})},$$

where all $\lambda_{ii}^{(k)} \in \Bbbk$, the following identities hold in A, for any i:

$$X_i^{m_{ii}} Y_i + \lambda_{ii}^{(1)} X_i^{m_{ii}-1} Y_i X_i + \cdots + \lambda_{ii}^{(m_{ii})} Y_i X_i^{m_{ii}} = 0 \tag{3.5}$$

and

$$X_i Y_i^{m_{ii}} + \lambda_{ii}^{(1)} Y_i X_i Y_i^{m_{ii}-1} + \cdots + \lambda_{ii}^{(m_{ii})} Y_i^{m_{ii}} X_i = 0. \tag{3.6}$$

Moreover, for any i and $m < m_{ii}$, the sets $\{X_i^{m-k} Y_i X_i^k\}_{k=0}^m$ and $\{Y_i^{m-k} X_i Y_i^k\}_{k=0}^m$ are linearly independent in A over $\Bbbk$.

Proof. The proof is similar to the proof of Theorem 3.2. □

Example 3.4. Let $R = \Bbbk[t_1, \dots, t_n]$ be the polynomial algebra, $\sigma_i(t_j) = t_j - \delta_{ij}$, $\mu_{ij} \in \Bbbk\backslash\{0\}$ such that $\mu_{ij}\mu_{ji} = 1$. Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be the associated TGW algebra. It is easy to see that it is $\Bbbk$-finitistic. For $i = j$, the minimal polynomials are $p_{ii}(x) = (x-1)^2$. For $i \neq j$ we have $p_{ij}(x) = x - 1$. The matrix C_A, defined in (3.2), is the Cartan matrix of type $(A_1)^n = A_1 \times \cdots \times A_1$ (just a diagonal matrix with 2 on the diagonal). Thus A is of Lie type $(A_1)^n$. By (3.3) we have $X_i X_j = \mu_{ij}^{-1} X_j X_i$ for $i \neq j$. If all $\mu_{ij} = 1$ then A is isomorphic to the nth Weyl algebra.

Example 3.5. The following TGW algebra was first mentioned as an example in [18], but a complete presentation by generators and relations was only given in [11]. Let $n = 2$, $R = \Bbbk[H]$, $\sigma_1(H) = H + 1$, $\sigma_2(H) = H - 1$, $t_1 = H$, $t_2 = H + 1$, $\mu_{12} = \mu_{21} = 1$ and let $A = \mathcal{A}_\mu(R, \sigma, t)$ be the associated TGW algebra. Clearly A is locally finite with $V_{ij} = \mathbb{C}H \oplus \mathbb{C}1$ for $i, j = 1, 2$. Observing that $\sigma_2(t_1)$ and t_1 are linearly independent and that

$$\sigma_2^2(t_1) - 2\sigma_2(t_1) + t_1 = H - 2 - 2(H-1) + H = 0$$

we see that the minimal polynomial p_{21} for σ_2 acting on V_{21} is given by $p_{21}(x) = x^2 - 2x + 1 = (x-1)^2$. Similarly one checks that in fact $p_{ij}(x) = (x-1)^2$ for all $i, j = 1, 2$. Thus $C_A = \left[\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right]$, the Cartan matrix of type A_2, so A is of Lie type A_2. By Theorem 3.2(b), we have for example $X_1^2 X_2 - 2X_1 X_2 X_2 + X_2 X_1^2 = 0$ in A, which is precisely one of the Serre relations in the enveloping algebra of $\mathfrak{sl}_3(\Bbbk)$, the simple Lie algebra of type A_2. It was shown in [11, Example 6.3] that in fact A is isomorphic to the $\Bbbk$-algebra with generators X_1, X_2, Y_1, Y_2, H and defining relations

$$\begin{aligned}
X_1 H &= (H+1)X_1, & X_2 H &= (H-1)X_2, & X_1^2 X_2 - 2X_1 X_2 X_1 + X_2 X_1^2 &= 0,\\
Y_1 H &= (H-1)Y_1, & Y_2 H &= (H+1)Y_2, & X_2^2 X_1 - 2X_2 X_1 X_2 + X_1 X_2^2 &= 0,\\
Y_1 X_1 &= X_2 Y_2 = H, & Y_2 X_2 &= X_1 Y_1 = H+1, & Y_1^2 Y_2 - 2Y_1 Y_2 Y_1 + Y_2 Y_1^2 &= 0,\\
X_1 Y_2 &= Y_2 X_1, & X_2 Y_1 &= Y_1 X_2, & Y_2^2 Y_1 - 2Y_2 Y_1 Y_2 + Y_1 Y_2^2 &= 0.
\end{aligned}$$

In [11], this TGW algebra was also generalized to arbitrary symmetric generalized Cartan matrices, although explicit presentation was only given in type A_2.

4. TGW algebras of Lie type $(A_1)^n$

4.1. Presentation by generators and relations

Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a $\Bbbk$-finitistic TGW algebra of Lie type $(A_1)^n = A_1 \times \cdots \times A_1$, with (R, σ, t) being μ-consistent. Thus C_A has all zeros outside the main diagonal. That is, $\deg p_{ij} = 1$ for all $i \neq j$. Equivalently (since p_{ij} are monic by definition), for $i \neq j$ we have $p_{ij}(x) = x - \gamma_{ij}$ for some $\gamma_{ij} \in \Bbbk\backslash\{0\}$. By Theorem 3.2, this means that in A we have

$$X_iX_j = \gamma_{ij}\mu_{ij}^{-1}X_jX_i, \quad \forall i \neq j, \tag{4.1a}$$

$$Y_jY_i = \gamma_{ij}\mu_{ji}^{-1}Y_iY_j, \quad \forall i \neq j. \tag{4.1b}$$

It also means that

$$\sigma_i(t_j) = \gamma_{ij}t_j \quad \text{for all } i \neq j. \tag{4.2}$$

By Theorem 2.7, (R, σ, t) is μ-consistent if and only if

$$\mu_{ij}\mu_{ji} = \gamma_{ij}\gamma_{ji}, \quad \forall i \neq j. \tag{4.3}$$

We can now prove that (4.1) generate all relations in the ideal $\mathcal{I}_\mu(R, \sigma, t)$.

Theorem 4.1. *Let $A = \mathcal{A}_\mu(R, \sigma, t)$ is a $\Bbbk$-finitistic TGW algebra of type $(A_1)^n$, where (R, σ, t) is regular and μ-consistent (i.e. (4.3) holds). Then A is isomorphic to the R-ring generated by $X_1, \dots, X_n, Y_1, \dots, Y_n$ modulo the relations*

$$X_ir = \sigma_i(r)X_i, \qquad Y_ir = \sigma_i^{-1}(r)Y_i, \quad \forall r \in R, \ \forall i,$$

$$Y_iX_i = t_i, \qquad X_iY_i = \sigma_i(t_i), \quad \forall i, \tag{4.4}$$

$$X_iY_j = \mu_{ij}Y_jX_i, \qquad X_iX_j = \gamma_{ij}\mu_{ij}^{-1}X_jX_i, \qquad Y_jY_i = \gamma_{ij}\mu_{ji}^{-1}Y_iY_j, \quad \forall i \neq j. \tag{4.5}$$

Proof. The statement is equivalent to that the ideal $I := \mathcal{I}_\mu(R, \sigma, t)$ of the TGW construction $A' = \mathcal{C}_\mu(R, \sigma, t)$ used to construct A is generated by the set

$$\{X_iX_j - \gamma_{ij}\mu_{ij}^{-1}X_jX_i, \ Y_jY_i - \gamma_{ij}\mu_{ji}^{-1}Y_iY_j \mid i \neq j\}. \tag{4.6}$$

Let J denote the ideal in A' generated by (4.6). Assume $a \in I$. We will prove that $a \in J$. Since I is graded, we can without loss of generality assume that a is homogenous. Let $g = \deg a \in \mathbb{Z}^n$. Without loss of generality we can also add to a any element of J, and thus can rearrange the Y's and the X's in the terms in a so to obtain

$$a = rZ_1^{(g_1)} \cdots Z_n^{(g_n)}$$

for some $r \in R$, where $Z_i^{(k)}$ equals X_i^k if $k \geqslant 0$ and Y_i^{-k} otherwise. Put $b = Z_n^{(-g_n)} \cdots Z_1^{(-g_1)}$. Then ab has degree zero so that $ab \in I \cap R = \{0\}$, forcing $ab = 0$. On the other hand, using (2.1a), (2.1b) repeatedly we have $ab = rs$ where $s \in R$ is a product of elements of the form $\sigma_g(t_i)$ ($g \in \mathbb{Z}^n$, $i \in \{1, \dots, n\}$). Since the t_i are assumed to be regular in R we must have $r = 0$, i.e. $a = 0 \in J$. $\square$

4.2. Centralizer intersections and a simplicity criterion

Let G be a group acting by automorphisms on a ring R. An ideal J of R is called G-invariant if $g(J) \subseteq J$ for all $g \in G$. The ring R is called G-simple if the only G-invariant ideals are R and $\{0\}$. If $A \subseteq B$ are rings, then we let $C_B(A)$ denote the centralizer of A in B: $C_B(A) = \{b \in B \mid ba = ab, \ \forall a \in A\}$. In [12, Theorem 3.6, Theorem 7.18] the following statements are proved.

Theorem 4.2.

(a) *Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a TGW algebra where R is commutative and (R, σ, t) is μ-consistent. Let J be any nonzero ideal of A. Then $J \cap C_A(R) \neq 0$.*

(b) *Let $A = \mathcal{A}_\mu(R,\sigma,t)$ be a TGW algebra, and assume (R,σ,t) is μ-consistent. Suppose A is $\Bbbk$-finitistic of Lie type $(A_1)^n$. Then A is simple if and only if the following conditions hold*:
   (1) *R is $\mathbb{Z}^n$-simple*;
   (2) *$Z(A) \subseteq R$*;
   (3) *$Rt_i + R\sigma_i^d(t_i) = R$ for all $d \in \mathbb{Z}_{>0}$ and $i = 1,\dots,n$.*

Part (b) is proved in [12, Theorem 7.18] in the more general context of so called R-finitistic TGW algebras. The result is a generalization of D. Jordan's simplicity criterion for generalized Weyl algebras [14, Theorem 6.1].

5. Multiparameter twisted Weyl algebras

Now we define a special class of twisted generalized Weyl algebras. The definition of these algebras was inspired by a class of *multiparameter Weyl algebras* introduced by Benkart [5].

5.1. Definition

Let $n \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}\backslash\{0\}$ and let $\Lambda = (\lambda_{ij})$, $r = (r_{ij})$ and $s = (s_{ij})$ be three $n \times n$ matrices with entries from $\Bbbk\backslash\{0\}$, such that

$$\lambda_{ii} = 1, \quad \forall i \quad \text{and} \quad \lambda_{ij}\lambda_{ji} = 1, \quad \forall i \neq j, \tag{5.1a}$$

$$r_{ii}/s_{ii} \text{ is a nonroot of unity } \forall i, \tag{5.1b}$$

$$r_{ij}^k = s_{ij}^k, \quad \forall i \neq j. \tag{5.1c}$$

Let

$$R = \Bbbk\big[u_1^{\pm 1},\dots,u_n^{\pm 1}, v_1^{\pm 1},\dots,v_n^{\pm 1}\big] \tag{5.2}$$

be the Laurent polynomial ring over $\Bbbk$ in $2n$ indeterminates, define $\sigma_1,\dots,\sigma_n \in \operatorname{Aut}_\Bbbk(R)$ by

$$\sigma_i(u_j) = r_{ij}^{-1}u_j, \qquad \sigma_i(v_j) = s_{ij}^{-1}v_j, \tag{5.3}$$

for all $i, j \in \{1,\dots,n\}$, and define $t_1,\dots,t_n \in R$ by

$$t_i = \frac{(r_{ii}u_i)^k - (s_{ii}v_i)^k}{r_{ii}^k - s_{ii}^k}. \tag{5.4}$$

Finally, put

$$\mu_{ij} = r_{ji}^{-k}\lambda_{ji} \tag{5.5}$$

for all $i \neq j$. Then one easily checks that the consistency relations (2.2), (2.3) hold. Thus, by Theorem 2.7, the TGW datum (R,σ,t) is μ-consistent, that is, the natural map $\rho : R \to \mathcal{A}_\mu(R,\sigma,t)$ is injective. We denote the TGW algebra $\mathcal{A}_\mu(R,\sigma,t)$ by $A_n^k(r,s,\Lambda)$ and call it a *multiparameter twisted Weyl algebra*. It is easy to see that it is $\Bbbk$-finitistic of Lie type $(A_1)^n$ and thus, by Theorem 4.1, $A_n^k(r,s,\Lambda)$ is isomorphic to the unital associative $\Bbbk$-algebra generated by $u_i^{\pm 1}$, $v_i^{\pm 1}$, X_i, Y_i $(i = 1,\dots,n)$ modulo the relations

$$\text{the } u_i^{\pm 1}, v_j^{\pm 1} \text{ all commute and } u_i u_i^{-1} = v_i v_i^{-1} = 1, \ \forall i, \tag{5.6a}$$

$$X_i X_j = \left(\frac{r_{ji}}{r_{ij}}\right)^k \lambda_{ij} X_j X_i, \quad \forall i, j, \tag{5.6b}$$

$$Y_i Y_j = \lambda_{ij} Y_j Y_i, \quad \forall i, j, \tag{5.6c}$$

$$X_i Y_j = r_{ji}^{-k} \lambda_{ji} Y_j X_i, \quad \forall i \neq j, \tag{5.6d}$$

$$Y_i X_i = \frac{(r_{ii} u_i)^k - (s_{ii} v_i)^k}{r_{ii}^k - s_{ii}^k}, \qquad X_i Y_i = \frac{u_i^k - v_i^k}{r_{ii}^k - s_{ii}^k}, \quad \forall i, \tag{5.6e}$$

$$X_i u_j = r_{ij}^{-1} u_j X_i, \qquad X_i v_j = s_{ij}^{-1} v_j X_i,$$
$$Y_i u_j = r_{ij} u_j Y_i, \qquad Y_i v_j = s_{ij} v_j Y_i, \quad \forall i, j. \tag{5.6f}$$

Remark 5.1. One can also consider the larger class of algebras in which (5.1b) in the definition of $A_n^k(r, s, \Lambda)$ is replaced by the weaker condition that $r_{ii}^k \neq s_{ii}^k$ for all i. However in this paper we will always assume (5.1b), which in examples corresponds to that "q is not a root of unity".

5.2. Properties of multiparameter twisted Weyl algebras

Let

$$R^{\mathbb{Z}^n} = \{ r \in R \mid \sigma_i(r) = r, \ \forall i = 1, \dots, n \}$$

be the invariant subring of R under $\mathbb{Z}^n$. For $d \in \mathbb{Z}^{2n}$, put $u^d = u_1^{d_1} \cdots u_n^{d_n} v_1^{d_{n+1}} \cdots v_n^{d_{2n}}$. Let

$$G = \{ d \in \mathbb{Z}^{2n} \mid u^d \in R^{\mathbb{Z}^n} \}. \tag{5.7}$$

We have $R^{\mathbb{Z}^n} = \bigoplus_{d \in G} \Bbbk u^d$.

Proposition 5.2.

(a) *If J is a proper $\mathbb{Z}^n$-invariant ideal of R, then the group homomorphism $\mathbb{Z}^n \to \mathrm{Aut}_{\Bbbk}(R/J)$, induced by the $\mathbb{Z}^n$-action on R, is injective.*

(b) *If J is a proper $\mathbb{Z}^n$-invariant ideal of R, such that R/J is $\mathbb{Z}^n$-simple or a domain, then R/J is maximal commutative in $\bar{A} := \mathcal{A}_\mu(R/J, \bar{\sigma}, \bar{t})$.*

Proof. (a) Assume $g = (g_1, \dots, g_n) \in \mathbb{Z}^n$ is such that $\sigma_g(p + J) = p + J$ for all $p + J \in R/J$. Suppose that $g_i \neq 0$ for some i. Then, taking $p = u_i$ we have $u_i + J = \sigma_{kg}(u_i) + J = r_{1i}^{kg_1} \cdots r_{ni}^{kg_n} u_i + J$, giving $(r_{1i}^{kg_1} \cdots r_{ni}^{kg_n} - 1) u_i \in J$. Since J is proper and u_i invertible we must have $r_{1i}^{kg_1} \cdots r_{ni}^{kg_n} = 1$. Similarly taking $p = v_i$ gives that $s_{1i}^{kg_1} \cdots s_{ni}^{kg_n} = 1$. But $r_{ij}^k = s_{ij}^k$ for $i \neq j$ and thus we get $r_{ii}^{kg_i} / s_{ii}^{kg_i} = 1$, contradicting the fact that r_{ii}/s_{ii} is not a root of unity. Thus $g_i = 0$ for all i.

(b) Follows from part (a) and [12, Corollary 5.2]. $\square$

Proposition 5.3. *Any ideal of A is graded.*

Proof. Let J be any ideal in A and let $a \in J$. Write $a = \sum_{g \in \mathbb{Z}^n} a_g$, where $a_g \in A_g$ for each g. Pick any $h \in \mathbb{Z}^n$. We will show that $a_h \in J$. By Proposition 5.2(a), the group morphism $\mathbb{Z}^n \to \mathrm{Aut}_{\Bbbk}(R)$ is injective. So if $g \in \mathbb{Z}^n$, $g \neq h$, then there is a $d \in \mathbb{Z}^{2n}$ such that $\sigma_h(u^d) \neq \sigma_g(u^d)$. By the definition of the automorphisms σ_i we have $\sigma_h(u^d) = \xi_h u^d$ and $\sigma_g(u^d) = \xi_g u^d$, for some nonzero $\xi_g, \xi_h \in \Bbbk$. Put $b = \xi_g a - u^{-d} a u^d$. Then $b \in J$ and writing $b = \sum_{f \in \mathbb{Z}^n} b_f$ where $b_f \in A_f$ we have $b_g = \xi_g a_g - u^{-d} a_g u^d = (\xi_g - u^{-d}\sigma_g(u^d))a_g = 0$, and $b_h = \xi_g a_h - u^{-d} a_h u^d = (\xi_g - \xi_h)a_h$. So, replacing a by $(\xi_g - \xi_h)^{-1}b$, we have an element in J with the same degree h component but with the g component eliminated. Repeating this we can eliminate all components except a_h and thus we obtain that $a_h \in J$. $\square$

Proposition 5.4. *Let $\mathfrak{I}(R^{\mathbb{Z}^n})$ denote the set of ideals of $R^{\mathbb{Z}^n}$ and $\mathfrak{I}(R)^{\mathbb{Z}^n}$ denote the set of $\mathbb{Z}^n$-invariant ideals of R. Consider the maps*

$$\varepsilon : \mathfrak{I}\big(R^{\mathbb{Z}^n}\big) \to \mathfrak{I}(R)^{\mathbb{Z}^n}, \quad \mathfrak{n} \mapsto R\mathfrak{n},$$
$$\rho : \mathfrak{I}(R)^{\mathbb{Z}^n} \to \mathfrak{I}\big(R^{\mathbb{Z}^n}\big), \quad J \mapsto R^{\mathbb{Z}^n} \cap J.$$

Then ε and ρ are inverse to eachother and set up an order-preserving bijection between the two sets. In particular, for each $\mathfrak{n} \in \operatorname{Specm}(R^{\mathbb{Z}^n})$, $R\mathfrak{n}$ is maximal among $\mathbb{Z}^n$-invariant ideals of R and conversely, every maximal $\mathbb{Z}^n$-invariant ideal of R equals $R\mathfrak{n}$ for some $\mathfrak{n} \in \operatorname{Specm}(R^{\mathbb{Z}^n})$.

Proof. We can view R as a module over $\Bbbk[\mathbb{Z}^n]$ by linearly extending the $\mathbb{Z}^n$-action on R. Let $\Bbbk[\mathbb{Z}^n]^*$ be the group of characters (i.e. algebra morphisms $\Bbbk[\mathbb{Z}^n] \to \Bbbk$). The product is given by $\chi_1\chi_2(g) = \chi_1(g)\chi_2(g)$ for $\chi_1, \chi_2 \in \Bbbk[\mathbb{Z}^n]^*$, $g \in \mathbb{Z}^n$. By definition of σ_i, there is for each $d \in \mathbb{Z}^{2n}$ a $\chi \in \Bbbk[\mathbb{Z}^n]^*$ such that $\sigma_g(u^d) = \chi(g)u^d$ for all $g \in \mathbb{Z}^n$. Thus

$$R = \bigoplus_{\chi \in \Bbbk[\mathbb{Z}^n]^*} R[\chi], \quad R[\chi] = \big\{ r \in R \;\big|\; a.r = \chi(a)r,\ \forall a \in \Bbbk\big[\mathbb{Z}^n\big]\big\}. \tag{5.8}$$

In particular, R is semisimple as a module over $\Bbbk[\mathbb{Z}^n]$. Using that each $R[\chi]$ is spanned by certain u^d, one verifies that the decomposition (5.8) turns R into a *strongly* $\Bbbk[\mathbb{Z}^n]^*$*-graded ring*, that is,

$$R[\chi_1]R[\chi_2] = R[\chi_1\chi_2], \quad \text{for all } \chi_1, \chi_2 \in \Bbbk\big[\mathbb{Z}^n\big]^*. \tag{5.9}$$

Moreover $R^{\mathbb{Z}^n} = R[\mathbf{1}]$ where $\mathbf{1}$ is the unit in the character group $\Bbbk[\mathbb{Z}^n]^*$, given by $\mathbf{1}(g) = 1$ for all $g \in \mathbb{Z}^n$.

We now show the maps ε and ρ are inverses to eachother. Let J be any $\mathbb{Z}^n$-invariant ideal of R. Thus it is a $\Bbbk[\mathbb{Z}^n]$-submodule of R, and therefore $J = \bigoplus_{\chi \in \Bbbk[\mathbb{Z}^n]^*} J[\chi]$, where $J[\chi] = R[\chi] \cap J$. Using the strong gradation property (5.9) we have $J[\chi] = R[\chi]R[\chi^{-1}]J[\chi] \subseteq R[\chi]J[\mathbf{1}] \subseteq J[\chi]$ which proves that $J[\chi] = R[\chi]J[\mathbf{1}]$ for all χ. Thus $J = R(J \cap R^{\mathbb{Z}^n})$. This shows that $\varepsilon\rho$ is the identity. Let $\mathfrak{n}$ be an ideal of $R^{\mathbb{Z}^n}$. Then $R\mathfrak{n} = \sum_{\chi \in \Bbbk[\mathbb{Z}^n]^*} R[\chi]\mathfrak{n}$ and $R[\chi]\mathfrak{n} \subseteq R[\chi]$. Thus $(R\mathfrak{n}) \cap R^{\mathbb{Z}^n} = \mathfrak{n}$. This proves $\rho\varepsilon$ is the identity. $\square$

We end this section with some lemmas that will be used in the next section to prove the first main theorem of the paper.

Lemma 5.5. *$Rt_i + R\sigma_i^d(t_i) = R$ for all $i = 1, \ldots, n$ and all $d \in \mathbb{Z}_{>0}$.*

Proof. We have

$$r_{ii}^{-dk} t_i - \sigma_i^d(t_i) = \frac{(-r_{ii}^{-dk} s_{ii}^k + s_{ii}^{k-dk}) v_{ii}^k}{r_{ii}^k - s_{ii}^k} \tag{5.10}$$

which is invertible since r_{ii}/s_{ii} is assumed to not be a root of 1 and since v_i is invertible. This proves the claim. $\square$

Lemma 5.6. *No product of elements of the form $\sigma_g(t_i)$ ($g \in \mathbb{Z}^n$, $i = 1, \dots, n$) can belong to a $\mathbb{Z}^n$-invariant proper ideal of R.*

Proof. Indeed, such a product can be written $a = \xi\sigma_1^{p_1}(t_1)\cdots\sigma_n^{p_n}(t_n)$ for some nonzero $\xi \in \Bbbk$ and some $p_i \in \mathbb{Z}$. But then the proper $\mathbb{Z}^n$-invariant ideal L containing such element would also contain $\sigma_1(a)$. By Lemma 5.5, $Rt_1 + R\sigma_1(t_1) = R$. So for suitable $r_1, r_2 \in R$, $r_1 a + r_2\sigma_1(a) = \xi'\sigma_2^{p_2}(t_2)\cdots\sigma_n^{p_n}(t_n)$ for some nonzero $\xi' \in \Bbbk$. Continuing this way we would obtain that L contains a nonzero scalar hence the $L = R$ contradicting that L was proper. $\square$

Lemma 5.7. *Assume J is a maximal $\mathbb{Z}^n$-invariant ideal of R. Then all $t_i + J$ ($i = 1, \dots, n$) are regular in R/J.*

Proof. Let T denote the multiplicative submonoid of R/J generated by $\sigma_g(t_i + J)$ for all $g \in \mathbb{Z}^n$ and $i = 1, \dots, n$. By Lemma 5.6, we have $0 \notin T$. Observe that the set

$$L = \{\bar{r} \in R/J \mid u\bar{r} = 0 \text{ for some } u \in T\}$$

is a $\mathbb{Z}^n$-invariant ideal in R/J. But $L = R/J$ is impossible since the ring R/J is unital and $0 \notin T$. Therefore, since R/J is $\mathbb{Z}^n$-simple, we have $L = 0$ which proves that in particular all $t_i + J$ ($i = 1, \dots, n$) are regular in R/J. $\square$

5.3. Simple quotients

We come now to the main result on the structure theory of multiparameter twisted Weyl algebras. The following theorem (which is Theorem A from the Introduction) describes all quotients A/Q of $A = A_n^k(r, s, \Lambda)$ such that A/Q is a simple ring and such that the images of X_i, Y_i in A/Q are regular for all i. It also gives a necessary and sufficient condition under which all such quotients are domains.

We would like to emphasize that the subring $R^{\mathbb{Z}^n}$ of invariants of R under $\mathbb{Z}$ is just a Laurent polynomial ring over the field $\Bbbk$. Thus there are plenty of explicitly known maximal ideals. Moreover, when $\Bbbk$ is algebraically closed there is a bijection $\operatorname{Specm}(R^{\mathbb{Z}^n}) \to (\Bbbk\backslash\{0\})^m$ where m is the number of variables in $R^{\mathbb{Z}^n}$, i.e. the rank of the subgroup $G \subseteq \mathbb{Z}^{2n}$ (see (5.7)). It is in this sense we view the following theorem as a parametrization of the stated family of simple quotients.

Theorem 5.8. *Let $A = A_n^k(r, s, \Lambda)$ be a multiparameter twisted Weyl algebra.*

(a) *The assignment*

$$\mathfrak{n} \mapsto A/\langle\mathfrak{n}\rangle \tag{5.11}$$

where $\langle\mathfrak{n}\rangle$ denotes the ideal in A generated by $\mathfrak{n}$, is a bijection between the set of maximal ideals in $R^{\mathbb{Z}^n}$ and the set of simple quotients of A in which all X_i, Y_i ($i = 1, \dots, n$) are regular.

(b) *For any $\mathfrak{n} \in \operatorname{Specm}(R^{\mathbb{Z}^n})$, the quotient $A/\langle\mathfrak{n}\rangle$ is isomorphic to the twisted generalized Weyl algebra $\mathcal{A}_\mu(R/R\mathfrak{n}, \bar{\sigma}, \bar{t})$, where $\bar{\sigma}_g(r + R\mathfrak{n}) = \sigma_g(r) + R\mathfrak{n}$, $\forall g \in \mathbb{Z}^n$, $r \in R$ and $\bar{t}_i = t_i + R\mathfrak{n}$, $\forall i$.*

(c) *$A/\langle\mathfrak{n}\rangle$ is a domain for all $\mathfrak{n} \in \operatorname{Specm}(R^{\mathbb{Z}^n})$ if and only if $\mathbb{Z}^{2n}/G$ is torsion-free, where G was defined in* (5.7).

Proof. We first prove part (b). Let $\mathfrak{n} \in \operatorname{Specm}(R^{\mathbb{Z}^n})$. Put $J = R\mathfrak{n}$. Trivially $AJA = \langle\mathfrak{n}\rangle$. By Lemma 5.7, $t_i + J$ are regular in R/J. For each $g \in \mathbb{Z}^n$, $A_g = RZ_1^{(g_1)}\cdots Z_n^{(g_n)}$, where $Z_i^{(m)} = X_i^m$ if $m \geqslant 0$ and $Z_i^{(m)} = Y_i^{-m}$ if $m < 0$. We know (R, σ, t) is μ-consistent (see Section 5.1). Thus by [9, Corollary 6.4], $(R/J, \bar{\sigma}, \bar{t})$ is also μ-consistent. Thus the claim follows from [9, Theorem 4.1] using [9, Remark 4.2].

Now we prove part (a). Let $\mathfrak{n} \in \operatorname{Specm}(R^{\mathbb{Z}^n})$. By part (b), $A/\langle \mathfrak{n} \rangle$ isomorphic to $\bar{A} := \mathcal{A}_\mu(R/J, \bar{\sigma}, \bar{t})$. By Proposition 5.4, J is maximal among $\mathbb{Z}^n$-invariant ideals of R, hence R/J is $\mathbb{Z}^n$-simple. By Proposition 5.2(b), R/J is maximal commutative in $\bar{A}$. Hence, in particular, $Z(\bar{A}) \subseteq R/J$. Let $\pi : R \to R/J$ be the canonical projection. We have $(R/J)\bar{t}_i + (R/J)\bar{\sigma}_i^d(\bar{t}_i) = \pi(Rt_i + R\sigma_i^d(t_i)) + J = R/J$ for any $i \in \{1, \dots, n\}$ and $d \in \mathbb{Z}_{>0}$. Thus the requirements in Theorem 4.2(b) are fulfilled and we conclude that $\bar{A}$ is simple. For each $i \in \{1, \dots, n\}$, the elements $\bar{t}_i$, hence also $\bar{\sigma}_i(\bar{t}_i)$, are regular in R/J. By the proof of [9, Theorem 5.2(a)], these elements are also regular in $\bar{A}$. Since $\bar{t}_i = Y_i X_i$ and $\bar{\sigma}_i(\bar{t}_i) = X_i Y_i$, it follows that X_i and Y_i are regular in $\bar{A}$.

Conversely, assume that Q is any nonzero ideal of A such that A/Q is simple and such that $X_i + Q$, $Y_i + Q$ are regular in A/Q for all i. By Proposition 5.2, R is maximal commutative in A, that is $C_A(R) = R$. So by Theorem 4.2(a), $R \cap Q \neq 0$. We claim that $R \cap Q$ is $\mathbb{Z}^n$-invariant. Let $i \in \{1, \dots, n\}$ and $p \in R \cap Q$. Then $X_i p \in Q$ since Q is an ideal. On the other hand, $X_i p = \sigma_i(p) X_i$. Since the image of X_i in A/Q not a zero-divisor we conclude that $\sigma_i(p) \in Q$. Trivially $\sigma_i(p) \in R$. Thus $\sigma_i(R \cap Q) \subseteq R \cap Q$ for all i. Analogously one proves that $\sigma_i^{-1}(R \cap Q) \subseteq R \cap Q$ (or one can use that R is Noetherian). So $R \cap Q$ is indeed $\mathbb{Z}^n$-invariant. Next we show that $R \cap Q$ is maximal among $\mathbb{Z}^n$-invariant ideals in R. Suppose $R \cap Q \subsetneq J \subseteq R$ where J is a $\mathbb{Z}^n$-invariant ideal of R. Since J is $\mathbb{Z}^n$-invariant, AJ is a two-sided ideal of A. Any element of $AJ + Q$ of degree zero has the form $p + a$ where $p \in J$ and a is the degree zero component of an element of Q. But Q is graded by Proposition 5.3 so $a \in Q$. Thus $(AJ + Q) \cap R = J + (Q \cap R) = J$. Thus $AJ + Q$ is an ideal of A which properly contains Q. Since Q was maximal, $AJ + Q = A$ and thus $J = (AJ + Q) \cap R = R$. This shows that $R \cap Q$ is maximal among all $\mathbb{Z}^n$-invariant ideals of R. By Proposition 5.4, we conclude that $R \cap Q$ equals $R\mathfrak{n}$ for some maximal ideal $\mathfrak{n}$ of $R^{\mathbb{Z}^n}$. So for this $\mathfrak{n}$ we have $\langle \mathfrak{n} \rangle \subseteq Q$. But we proved above that $A/\langle \mathfrak{n} \rangle$ is always simple. Thus $\langle \mathfrak{n} \rangle$ is a maximal ideal of A which implies that $\langle \mathfrak{n} \rangle = Q$.

Finally, two different ideals $\mathfrak{n}, \mathfrak{n}'$ in $R^{\mathbb{Z}^n}$ cannot generate the same maximal ideal L in A, since then $1 \in \mathfrak{n} + \mathfrak{n}' \subseteq L$ which is absurd.

(c) By Proposition 2.9 and part (b) we have that $A/\langle \mathfrak{n} \rangle$ is a domain iff $R\mathfrak{n}$ is a prime ideal of R. Assume $R\mathfrak{n}$ is prime for all $\mathfrak{n} \in \operatorname{Specm}(R^{\mathbb{Z}^n})$. Suppose $d \in \mathbb{Z}^{2n}$, $d \notin G$ but that there is a $p \in \mathbb{Z}_{>0}$ such that $pd \in G$. Without loss of generality we can assume p is prime. Then there is a $j \in \{1, \dots, n\}$ such that $\sigma_j(u^d) = \zeta u^d$ where $\zeta \in \Bbbk$, $\zeta \neq 1$, $\zeta^p = 1$. Pick any $\mathbb{Z}$-basis $\{d_1, \dots, d_N\}$ for G and take $\mathfrak{n}$ to be the maximal ideal in $R^{\mathbb{Z}^n}$ generated by $u^{d_i} - 1$ for $i = 1, \dots, N$. Then $u^{pd} - 1 \in \mathfrak{n}$ also, because pd is a $\mathbb{Z}$-linear combination of the d_i. But $u^{pd} - 1 = (u^d - 1)(u^d - \zeta) \cdots (u^d - \zeta^{p-1})$. Since $R\mathfrak{n}$ is prime we conclude that $u^d - \zeta^e \in R\mathfrak{n}$ for some $e \in \{0, \dots, p-1\}$. However $R\mathfrak{n}$ is $\mathbb{Z}^n$-invariant and thus $R\mathfrak{n} \ni u^d - \zeta^e - \zeta^{-1}\sigma_j(u^d - \zeta^e) = (\zeta^{-1} - 1)\zeta^e$ which is invertible. This contradicts that $R\mathfrak{n}$ is a proper ideal of R which we know by Proposition 5.4. Hence $\mathbb{Z}^{2n}/G$ is torsion-free.

Conversely, assume that $\mathbb{Z}^{2n}/G$ is torsion-free. Thus $\mathbb{Z}^{2n} \simeq G \oplus G'$ for some subgroup G' of $\mathbb{Z}^{2n}$. Therefore, viewing R as the group algebra $\Bbbk[\mathbb{Z}^{2n}]$, we have an isomorphism $R = \Bbbk[\mathbb{Z}^{2n}] \simeq \Bbbk[G] \otimes_\Bbbk \Bbbk[G']$. Under this isomorphism, $R\mathfrak{n}$ (where $\mathfrak{n} \in \operatorname{Specm}(R^{\mathbb{Z}^n})$ is arbitrary) is mapped to $\mathfrak{n} \otimes \Bbbk[G']$ which is a prime ideal in $\Bbbk[G] \otimes_\Bbbk \Bbbk[G']$ since

$$\frac{\Bbbk[G] \otimes_\Bbbk \Bbbk[G']}{\mathfrak{n} \otimes \Bbbk[G']} \simeq \big(R^{\mathbb{Z}^n}/\mathfrak{n}\big)\big[G'\big],$$

which is a Laurent polynomial algebra over a field. This proves that $R\mathfrak{n}$ is a prime ideal of R for any $\mathfrak{n} \in \operatorname{Specm}(R^{\mathbb{Z}^n})$. $\square$

6. Multiparameter Weyl algebras and Hayashi's *q*-analog of the Weyl algebras

In this section we consider a class of *multiparameter Weyl algebras* defined in [5], which is a particular case of multiparameter twisted Weyl algebras. For the convenience of the reader we include the definition.

6.1. Definition

Assume $\underline{r} = (r_1, \ldots, r_n)$ and $\underline{s} = (s_1, \ldots, s_n)$ are n-tuples of nonzero scalars in a field $\Bbbk$ such that $(r_i s_i^{-1})^2 \neq 1$ for each i. Let $A_{\underline{r},\underline{s}}(n)$ be the unital associative algebra over the field $\Bbbk$ generated by elements $\rho_i, \rho_i^{-1}, \sigma_i, \sigma_i^{-1}, x_i, y_i$, $i = 1, \ldots, n$, subject to the following relations:

(R1) The $\rho_i^{\pm 1}$, $\sigma_j^{\pm 1}$ all commute with one another and $\rho_i \rho_i^{-1} = \sigma_i \sigma_i^{-1} = 1$;
(R2) $\rho_i x_j = r_i^{\delta_{i,j}} x_j \rho_i$, $\rho_i y_j = r_i^{-\delta_{i,j}} y_j \rho_i$, $1 \leqslant i, j \leqslant n$;
(R3) $\sigma_i x_j = s_i^{\delta_{i,j}} x_j \sigma_i$, $\sigma_i y_j = s_i^{-\delta_{i,j}} y_j \sigma_i$, $1 \leqslant i, j \leqslant n$;
(R4) $x_i x_j = x_j x_i$, $y_i y_j = y_j y_i$, $1 \leqslant i, j \leqslant n$;
$y_i x_j = x_j y_i$, $1 \leqslant i \neq j \leqslant n$;
(R5) $y_i x_i - r_i^2 x_i y_i = \sigma_i^2$ and $y_i x_i - s_i^2 x_i y_i = \rho_i^2$, $1 \leqslant i \leqslant n$,

or equivalently

(R5′) $y_i x_i = \frac{r_i^2 \rho_i^2 - s_i^2 \sigma_i^2}{r_i^2 - s_i^2}$ and $x_i y_i = \frac{\rho_i^2 - \sigma_i^2}{r_i^2 - s_i^2}$, $1 \leqslant i \leqslant n$.

When $r_i = q^{-1}$ and $s_i = q$ for all i, we may quotient by the ideal generated by the elements $\sigma_i \rho_i - 1$, $i = 1, \ldots, n$, to obtain Hayashi's q-analogs of the Weyl algebras $A_q^-(n)$ (see [13]).

6.2. Realization as multiparameter twisted Weyl algebras

Take $k = 2$, and for all i, j put $\lambda_{ij} = 1$, $r_{ij} = r_i^{\delta_{ij}}$, $s_{ij} = s_i^{\delta_{ij}}$, where $r_i, s_i \in \Bbbk \backslash \{0\}$, $i = 1, \ldots, n$. Then $A_n^k(r, s, \Lambda)$ is isomorphic to $A_{\underline{r},\underline{s}}(n)$.

Let us investigate the ring of invariants $R^{\mathbb{Z}^n}$. Consider a monomial

$$u^d := u_1^{d_1} \cdots u_n^{d_n} v_1^{d_{n+1}} \cdots v_n^{d_{2n}},$$

where $d \in \mathbb{Z}^{2n}$. We have

$$\sigma_i\big(u^d\big) = r_i^{d_i} s_i^{d_{n+i}} u^d. \tag{6.1}$$

6.3. Generic case

Assuming that for each $i = 1, \ldots, n$, the only pair $(d, d') \in \mathbb{Z}^2$ such that $r_i^d s_i^{d'} = 1$ is the pair $(0, 0)$ we obtain that $R^{\mathbb{Z}^n} = \Bbbk$ and thus, by Theorem 5.8, $A_{\underline{r},\underline{s}}(n)$ is a simple ring.

6.4. Hayashi's q-analogs of the Weyl algebras $A_q^-(n)$

Assume instead that for all i, $r_i = q^{-1}$ and $s_i = q$, where $q \in \Bbbk$ is nonzero and not a root of 1. Then by (6.1), u^d is fixed by all σ_i iff $d_i = d_{n+i}$ for all i. Thus $R^{\mathbb{Z}^n} = \Bbbk[w_1, \ldots, w_n]$ where $w_i := u_i v_i$. Pick the maximal ideal $\mathfrak{n} := (w_1 - 1, \ldots, w_n - 1)$ of the invariant subring. Then, by Theorem 5.8, we obtain that the quotient of $A_{\underline{r},\underline{s}}(n)$ by the two-sided ideal generated by $w_1 - 1, \ldots, w_n - 1$ is a twisted generalized Weyl algebra which is simple. It is easy to check that this simple algebra is isomorphic to Hayashi's q-analogs of the Weyl algebras $A_q^-(n)$, see [13].

6.5. Connections with generalized Weyl algebras

Assume now that we are in the generic case as in Section 6.3. As it was observed in [5], the multiparameter Weyl algebra $A_{\underline{r},\underline{s}}(n)$ can be realized as a degree n generalized Weyl algebra. For this construction, let D_i be the subalgebra of $A_{\underline{r},\underline{s}}(n)$ generated by the elements ρ_i, ρ_i^{-1}, σ_i, σ_i^{-1}. Thus, D_i is isomorphic to $\Bbbk[\rho_i^{\pm 1}, \sigma_i^{\pm 1}]$. Set $D = D_1 \otimes D_2 \otimes \cdots \otimes D_n$. Let ϕ_i be the automorphism of D_i given by

$$\phi_i(\rho_j) = r_i^{-\delta_{i,j}} \rho_j, \qquad \phi_i(\sigma_i) = s_i^{-\delta_{i,j}} \sigma_i. \tag{6.2}$$

Now set

$$t_i = \frac{r_i^2 \rho_i^2 - s_i^2 \sigma_i^2}{r_i^2 - s_i^2}, \qquad X_i = x_i, \qquad Y_i = y_i, \tag{6.3}$$

and observe that

$$Y_i X_i = t_i, \quad \text{and} \quad X_i Y_i = \frac{\rho_i^2 - \sigma_i^2}{r_i^2 - s_i^2} = \phi_i(t_i)$$

are just the relations in (R5′). The relations in (R1) and (R4) are apparent. The identities in (R2) and (R3) are equivalent to the statements $Y_j d = \phi_j^{-1}(d) Y_j$, $X_j d = \phi_j(d) X_j$ with $d = \rho_i$ and σ_i. Therefore, there is a surjection $W_n := D(\underline{\phi}, \underline{t}) \to A_{\underline{r},\underline{s}}(n)$. But since $A_{\underline{r},\underline{s}}(n)$ has a presentation by (R1)–(R5), there is a surjection $A_{\underline{r},\underline{s}}(n) \to W_n$. Since that map is the inverse of the other one, these algebras are isomorphic. Bavula [4, Proposition 7] has shown that a generalized Weyl algebra $D(\underline{\phi}, \underline{t})$ is left and right Noetherian if D is Noetherian, and it is a domain if D is a domain. Since D is commutative and finitely generated, it is Noetherian, hence so are W_n and $A_{\underline{r},\underline{s}}(n)$. Since D is a domain as it can be identified with the Laurent polynomial algebra $\Bbbk[\rho_i^{\pm 1}, \sigma_i^{\pm 1} \mid i = 1, \ldots, n]$; hence $A_{\underline{r},\underline{s}}(n)$ is a domain also. In summary, we have

Proposition 6.1. *(See [5].) When the parameters r_i, s_i are generic as in Section* 6.3*, the multiparameter Weyl algebra $A_{\underline{r},\underline{s}}(n)$ is isomorphic to the degree n generalized Weyl algebra $W_n = D(\underline{\phi}, \underline{t})$ where D is the $\Bbbk$-algebra generated by the elements ρ_i, ρ_i^{-1}, σ_i, σ_i^{-1}, $i = 1, \ldots, n$, subject to the relations in* (R1)*, ϕ_i is as in* (6.2)*; and the elements t_i are as in* (6.3)*. Thus, $A_{\underline{r},\underline{s}}(n)$ is Noetherian domain.*

7. Jordan's simple localization of the multiparameter quantized Weyl algebra

7.1. Quantized Weyl algebras

Let $\bar{q} = (q_1, \ldots, q_n)$ be an n-tuple of elements of $\Bbbk \backslash \{0\}$. Let $\Lambda = (\lambda_{ij})_{i,j=1}^n$ be an $n \times n$ matrix with $\lambda_{ij} \in \Bbbk \backslash \{0\}$, multiplicatively skewsymmetric: $\lambda_{ij} \lambda_{ji} = 1$ for all i, j. The *multiparameter quantized Weyl algebra of degree n over $\Bbbk$*, denoted $A_n^{\bar{q},\Lambda}(\Bbbk)$, is defined as the unital $\Bbbk$-algebra generated by x_i, y_i, $1 \leqslant i \leqslant n$ subject to the following defining relations:

$$y_i y_j = \lambda_{ij} y_j y_i, \quad \forall i, j, \tag{7.1}$$

$$x_i x_j = q_i \lambda_{ij} x_j x_i, \quad i < j, \tag{7.2}$$

$$x_i y_j = \lambda_{ji} y_j x_i, \quad i < j, \tag{7.3}$$

$$x_i y_j = q_j \lambda_{ji} y_j x_i, \quad i > j, \tag{7.4}$$

$$x_i y_i - q_i y_i x_i = 1 + \sum_{k=1}^{i-1} (q_k - 1) y_k x_k, \quad \forall i. \tag{7.5}$$

This algebra first appeared in [16], and was further studied in [1] and [15] among others. For $\Bbbk = \mathbb{C}$ and $q_1 = \cdots = q_n = \mu^2$, $\lambda_{ji} = \mu$, $\forall j < i$, where $\mu \in \Bbbk\backslash\{0\}$, the algebra $A_n^{\bar{q},\Lambda}(\Bbbk)$ is isomorphic to the quantized Weyl algebra introduced by Pusz and Woronowicz [20].

The quantized Weyl algebra can be realized as a twisted generalized Weyl algebra (first observed in [19]) in the following way. Let $P = \Bbbk[s_1, \ldots, s_n]$ be the polynomial algebra in n variables and τ_i the $\Bbbk$-algebra automorphisms of P defined by

$$\tau_i(s_j) = \begin{cases} s_j, & j < i, \\ 1 + q_i s_i + \sum_{k=1}^{i-1}(q_k - 1)s_k, & j = i, \\ q_i s_j, & j > i. \end{cases} \tag{7.6}$$

One can check that the τ_i commute. Let $\mu = (\mu_{ij})_{i,j=1}^n$ be defined by

$$\mu_{ij} = \begin{cases} \lambda_{ji}, & i < j, \\ q_j \lambda_{ji}, & i > j. \end{cases} \tag{7.7}$$

Put $\tau = (\tau_1, \ldots, \tau_n)$ and $s = (s_1, \ldots, s_n)$. Let $\mathcal{A}_\mu(P, \tau, s)$ be the associated twisted generalized Weyl algebra. From (7.6) it is easy to see that $\mathcal{A}_\mu(P, \tau, s)$ is $\Bbbk$-finitistic, and that the minimal polynomials are $p_{ij}(x) = x - 1$ for $i < j$ and $p_{ij}(x) = x - q_i$ for $j > i$, so the algebra is of type $(A_1)^n$. By Theorem 4.1, one checks that $\mathcal{A}_\mu(P, \tau, s)$ is isomorphic to $A_n^{\bar{q},\Lambda}(\Bbbk)$ via $X_i \mapsto x_i$, $Y_i \mapsto y_i$ and $s_i \mapsto y_i x_i$. The representation theory of $A_n^{\bar{q},\Lambda}$ has been studied from the point of view of TGW algebras in [19] and [10].

In the following it will be convenient to identify P with its isomorphic image in $A_n^{\bar{q},\Lambda}$ via $s_i \mapsto y_i x_i$. Consider the following elements in $A_n^{\bar{q},\Lambda}$:

$$z_i = 1 + \sum_{k \leqslant i}(q_k - 1)s_k, \quad i = 1, \ldots, n. \tag{7.8}$$

It was shown in [15] that the set $Z := \{z_1^{k_1} \cdots z_n^{k_n} \mid k_1, \ldots, k_n \in \mathbb{Z}\}$ is an Ore set in $A_n^{\bar{q},\Lambda}$ and that, provided that none of the q_i is a root of unity, the localized algebra

$$B_n^{\bar{q},\Lambda} := Z^{-1} A_n^{\bar{q},\Lambda}$$

is simple.

The algebra $B_n^{\bar{q},\Lambda}$ can also be realized as a twisted generalized Weyl algebra. To see this, consider the following subset of P:

$$S = \left\{\alpha z_1^{k_1} \cdots z_n^{k_n} \;\middle|\; \alpha \in \Bbbk\backslash\{0\},\ k_i \in \mathbb{Z}\right\}, \tag{7.9}$$

where z_i were defined in (7.8). Then $0 \notin S$, $1 \in S$, $a, b \in S \Rightarrow ab \in S$, the elements of S are regular, and moreover S has the virtue of being $\mathbb{Z}^n$-invariant, using the relation

$$\tau_i(z_j) = \begin{cases} z_j, & j < i, \\ q_i z_j, & j \geqslant i, \end{cases} \tag{7.10}$$

which can be proved using (7.8) and (7.6). Thus [9, Theorem 5.2] can be applied to give, together with the isomorphism $A_n^{\bar{q},\Lambda} \simeq \mathcal{A}_\mu(P, \tau, s)$, that

$$S^{-1} A_n^{\bar{q},\Lambda} \simeq S^{-1}\mathcal{A}_\mu(P, \tau, s) \simeq \mathcal{A}_\mu\left(S^{-1}P, \widetilde{\tau}, s\right).$$

But localizing at S is equivalent to localizing at Z, and thus

$$B_n^{\bar{q},\Lambda} \simeq \mathcal{A}_\mu\big(S^{-1}P, \widetilde{\tau}, s\big).$$

7.2. Relation to multiparameter twisted Weyl algebras

We show here how the algebra $B_n^{\bar{q},\Lambda}$ fits into the framework of multiparameter twisted Weyl algebras. We keep all notation from previous section. Let $\bar{q} = (q_1, \dots, q_n) \in (\Bbbk\backslash\{0\})^n$ and let $\Lambda = (\lambda_{ij})_{i,j=1}^n$ be an $n \times n$ matrix with $\lambda_{ij} \in \Bbbk\backslash\{0\}$, $\lambda_{ii} = 1$, $\lambda_{ij}\lambda_{ji} = 1$ for all i, j. We assume that none of the q_i is a root of unity. Let $k = 1$ and put

$$r_{ij} = \begin{cases} 1, & j \leqslant i, \\ q_i^{-1}, & j > i, \end{cases} \qquad s_{ij} = \begin{cases} 1, & j < i, \\ q_i^{-1}, & j \geqslant i. \end{cases} \tag{7.11}$$

Then conditions (5.1) are satisfied. Let $A_n^k(r, s, \Lambda)$ be the corresponding multiparameter twisted Weyl algebra. Recall that, by definition, this means that $A_n^k(r, s, \Lambda)$ is the twisted generalized Weyl algebra $\mathcal{A}_\mu(R, \sigma, t)$ where

$$R = \Bbbk\big[u_1^{\pm1}, \dots, u_n^{\pm1}, v_1^{\pm1}, \dots, v_n^{\pm1}\big], \tag{7.12}$$

$$\sigma_i(u_j) = r_{ij}^{-1} u_j, \qquad \sigma_i(v_j) = s_{ij}^{-1} v_j, \tag{7.13}$$

$$t_i = \frac{u_i - q_i^{-1} v_i}{1 - q_i^{-1}}, \tag{7.14}$$

$$\mu_{ij} = r_{ji}^{-k} \lambda_{ji}, \tag{7.15}$$

for all $i, j \in \{1, \dots, n\}$. Note that the μ_{ij} in (7.15) coincide with the ones defined in (7.7). The goal now is to explain the following diagram, which proves Theorem B stated in the introduction.

$$\begin{tikzcd}
(R,\sigma,t) \arrow[d, "\pi"'] \arrow[dr, two heads, "\psi"] & \\
(R/J,\bar{\sigma},\bar{t}) \arrow[r, shift left, two heads, "\Psi"] & (S^{-1}P,\widetilde{\tau},s) \arrow[l, shift left, "\Phi"] \\
 & (P,\tau,s) \arrow[u, hook, "\iota"'] \arrow[ul, "\varphi"]
\end{tikzcd}$$

$$\big\downarrow \mathcal{A}_\mu$$

$$\begin{tikzcd}
A_n^k(r,s,\Lambda) = \mathcal{A}_\mu(R,\sigma,t) \arrow[d] \arrow[dr] & \\
\frac{A_n^k(r,s,\Lambda)}{\langle J\rangle} \simeq \mathcal{A}_\mu(R/J,\bar{\sigma},\bar{t}) \arrow[r, shift left, "\simeq"] & \mathcal{A}_\mu(S^{-1}P,\widetilde{\tau},s) \simeq B_n^{\bar{q},\Lambda} \arrow[l, shift left] \\
 & \mathcal{A}_\mu(P,\tau,s) \simeq A_n^{\bar{q},\Lambda} \arrow[u, hook] \arrow[ul, hook]
\end{tikzcd}$$

7.2.1. The map ψ

Define a $\Bbbk$-algebra morphism

$$\psi : R \to S^{-1}P, \quad \psi(u_i) = -q_i^{-1} z_{i-1}, \qquad \psi(v_i) = -z_i, \quad i = 1, \dots, n$$

where $z_0 := 1$. We claim that ψ is $\mathbb{Z}^n$-equivariant. Indeed,

$$\psi\big(\sigma_i(u_j)\big) = \psi\big(r_{ij}^{-1} u_j\big) = -r_{ij}^{-1} q_i^{-1} z_{i-1},$$

while, using (7.10) and (7.11) in the last step,

$$\widetilde{\tau}_i\big(\psi(u_j)\big) = \widetilde{\tau}_i\big(-q_j^{-1} z_{j-1}\big) = -r_{ij}^{-1} q_i^{-1} z_{i-1}.$$

Similarly $\psi(\sigma_i(v_j)) = \widetilde{\tau}_i(\psi(v_j))$. This proves that $\psi\sigma_i = \widetilde{\tau}_i\psi$ for each i, so in other words, that ψ is $\mathbb{Z}^n$-equivariant. Also, for any $i \in \{1, \dots, n\}$,

$$\psi(t_i) = \psi\left(\frac{u_i - q_i^{-1} v_i}{1 - q_i^{-1}}\right) = \frac{-q_i^{-1} z_{i-1} + q_i^{-1} z_i}{1 - q_i^{-1}} = \frac{z_i - z_{i-1}}{q_i - 1} = s_i$$

by (7.8). Recall the category $\mathsf{TGW}_n(\Bbbk)$ from Definition 2.1. We have just proved that ψ is a morphism in the category $\mathsf{TGW}_n(\Bbbk)$ from (R, σ, t) to $(S^{-1}P, \widetilde{\tau}, s)$.

It is easy to see that ψ is surjective because the image contains both $s_1, \dots, s_n$ since $\psi(t_i) = s_i$, and the inverses of the z_i: $z_i^{-1} = \psi(-v_i^{-1})$. Applying the functor $\mathcal{A}_\mu$ from [9, Theorem 3.1] to ψ gives a surjective $\Bbbk$-algebra morphism $\mathcal{A}_\mu(\psi) : \mathcal{A}_\mu(R, \sigma, t) \to \mathcal{A}_\mu(S^{-1}P, \tau, s)$.

7.2.2. The map π

We determine the invariant subring $R^{\mathbb{Z}^n}$. For any $i \in \{1, \dots, n\}$ and $d \in \mathbb{Z}^{2n}$ we have

$$\sigma_i\big(u^d\big) = q_i^{-\sum_{j>i} d_j - \sum_{j\geqslant i} d_{n+j}} u^d.$$

Thus $u^d \in R^{\mathbb{Z}^n}$ iff for each $i = 1, \dots, n$ we have $d_{n+i} + \sum_{j=i+1}^{n} (d_j + d_{n+j}) = 0$. This system of equations is equivalent to that $d_{2n} = 0$, $d_{2n-1} + d_n = 0$, $d_{2n-2} + d_{n-1} = 0, \dots, d_{n+1} + d_2 = 0$. Thus $R^{\mathbb{Z}^n} = \Bbbk[w_1, \dots, w_n]$ where $w_1 = -u_1$, $w_2 = u_2 v_1^{-1}, \dots, w_n = u_n v_{n-1}^{-1}$. Pick

$$\mathfrak{n} := \big(w_1 - q_1^{-1}, \dots, w_n - q_n^{-1}\big) \in \operatorname{Specm}\big(R^{\mathbb{Z}^n}\big).$$

Let $J = R\mathfrak{n}$ be the ideal in R generated by $\mathfrak{n}$. The canonical map $\pi : R \to R/J$ is $\mathbb{Z}^n$-equivariant and maps t_i to $\bar{t}_i = t_i + J$.

7.2.3. The map Ψ

We have $\psi(w_i) = q_i^{-1}$ for $i = 1, \dots, n$ which shows that $J = R\mathfrak{n} \subseteq \ker\psi$. Thus ψ induces a map $\Psi : R/J \to S^{-1}P$, also $\mathbb{Z}^n$-equivariant and $\Psi(\bar{t}_i) = s_i$. Since ψ is surjective, so is Ψ. Applying the functor from [9, Theorem 3.1] we get a surjective homomorphism $\mathcal{A}_\mu(\Psi) : \mathcal{A}_\mu(R/J, \sigma, t) \to \mathcal{A}_\mu(S^{-1}P, \widetilde{\tau}, s)$. However, by Theorem 5.8, the algebra $\mathcal{A}_\mu(R/J, \sigma, t)$ is simple, and thus Ψ is an isomorphism.

7.2.4. The maps φ, ι, Φ

Similarly one can show that the map $\varphi : P \to R/J$ defined by $\varphi(s_i) = \bar{t}_j$ is $\mathbb{Z}^n$-equivariant and that the elements of S are mapped to invertible elements of R/J, showing that φ factorizes through the canonical map $\iota : P \to S^{-1}P$, inducing a map Φ. Applying the functor $\mathcal{A}_\mu$ gives corresponding homomorphisms of twisted generalized Weyl algebras.

8. Simple weight modules

In this section we describe the simple weight modules over the simple algebras $A = A_n^k(r, s, \Lambda)/\langle J\rangle$ from Theorem 5.8. We also assume that the ground field $\Bbbk$ is algebraically closed. We will use notation from Section 5.1.

8.1. Dynamics of orbits and their breaks

The group $\mathbb{Z}^n$ acts on R via the automorphisms σ_i. Explicitly, $g(r) = (\sigma_1^{g_1} \cdots \sigma_n^{g_n})(r)$ for $g = (g_1, \ldots, g_n) \in \mathbb{Z}^n$ and $r \in R$. Using this action and (2.1a) we have $a \cdot r = (\deg a)(r) \cdot a$ for any homogenous $a \in A$ and any $r \in R$. The group $\mathbb{Z}^n$ also acts on $\operatorname{Max}(R)$, the set of maximal ideals of R. Let Ω denote the set of orbits of this action. An element $\mathfrak{m} \in \operatorname{Max}(R)$ is called an *i-break* if $t_i \in \mathfrak{m}$. An orbit $\mathcal{O} \in \Omega$ is called *degenerate* if it contains an i-break for some i. A break $\mathfrak{m}$ in an orbit $\mathcal{O}$ is called *maximal* if $\mathfrak{m}$ is an i-break for all i for which $\mathcal{O}$ contains an i-break.

Proposition 8.1. *Let $\mathfrak{m} \in \operatorname{Specm}(R)$. Then the stabilizer $\operatorname{Stab}_{\mathbb{Z}^n}(\mathfrak{m})$ is trivial.*

Proof. Write

$$\mathfrak{m} = (\bar{u}_i - \alpha_i, \bar{v}_i - \beta_i \mid i = 1, \ldots, n),$$

where $\bar{u}_i = u_i + J$, $\bar{v}_i = v_i + J$ and $\alpha_i, \beta_i \in \Bbbk\backslash\{0\}$. Suppose $g \in \operatorname{Stab}_{\mathbb{Z}^n}(\mathfrak{m})$. Then

$$\begin{aligned} \sigma_g(\mathfrak{m}) &= \bigl(\sigma_g(\bar{u}_i) - \alpha_i, \sigma_g(\bar{v}_i) - \beta_i \bigm| i = 1, \ldots, n\bigr) \\ &= \bigl(\bigl(r_{1i}^{g_1} \cdots r_{ni}^{g_n}\bigr)^{-1} \bar{u}_i - \alpha_i, \bigl(s_{1i}^{g_1} \cdots s_{ni}^{g_n}\bigr)^{-1} \bar{v}_i - \beta_i \bigm| i = 1, \ldots, n\bigr). \end{aligned} \tag{8.1}$$

Thus

$$r_{1i}^{g_1} \cdots r_{ni}^{g_n} = s_{1i}^{g_1} \cdots s_{ni}^{g_n} = 1.$$

Raising all sides to the kth power and using that $r_{ij}^k = s_{ij}^k$ for all $i \neq j$, we obtain that $r_{ii}^{kg_i} = s_{ii}^{kg_i} = 1$ for all i which, since r_{ii}/s_{ii} is not a root of unity, implies that $g_i = 0$ for all i. $\square$

Proposition 8.2. *Consider the maximal ideal*

$$\mathfrak{m} = (u_1 - \alpha_1, \ldots, u_n - \alpha_n, v_1 - \beta_1, \ldots, v_n - \beta_n) \in \operatorname{Specm}(R).$$

Then, for all $i \in \{1, \ldots, n\}$ and all $g = (g_1, \ldots, g_n) \in \mathbb{Z}^n$,

$$t_i \in \sigma_g(\mathfrak{m}) \quad \Leftrightarrow \quad (\alpha_i/\beta_i)^k = (s_{ii}/r_{ii})^{(g_i+1)k}. \tag{8.2}$$

Proof. By the calculation (8.1) and the definition (5.4) of t_i we have $t_i \in \sigma_g(\mathfrak{m})$ iff

$$\left(\frac{r_{1i}^{g_1} \ldots r_{ni}^{g_n} \alpha_i}{s_{1i}^{g_1} \ldots s_{ni}^{g_n} \beta_i}\right)^k = (s_{ii}/r_{ii})^k.$$

Using that $r_{ij}^k = s_{ij}^k$ for $i \neq j$ and simplifying, the claim follows. $\square$

Corollary 8.3. *If $t_i \in \mathfrak{m}$, then for $g = (g_1, \ldots, g_n) \in \mathbb{Z}^n$,*

$$t_i \in \sigma_g(\mathfrak{m}) \quad \Leftrightarrow \quad g_i = 0.$$

Corollary 8.4. *Every degenerate orbit contains a maximal break.*

Remark 8.5. Corollary 8.4 holds for any TGW algebra of Lie type $(A_1)^n$ using the fact that $\sigma_j(t_i) = \gamma_{ji} t_i$ for any $j \neq i$.

8.2. General results on simple weight modules with no proper inner breaks

We collect here some notation and results from [10].

Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a twisted generalized Weyl algebra. Let V be a simple weight module over A.

Definition 8.6. (See [10].) V has *no proper inner breaks* if for any $\mathfrak{m} \in \operatorname{Supp}(V)$ and any homogenous a with $aM_\mathfrak{m} \neq 0$ we have $a'a \notin \mathfrak{m}$ for some homogenous a' with $\deg(a') = -\deg(a)$.

This definition is slightly different than the one given in [10] but can be proved to be equivalent. Consider the following sets (also equivalent to the definitions in [10]), defined for any $\mathfrak{m} \in \operatorname{Specm}(R)$.

$$\widetilde{G}_\mathfrak{m} := \{ g \in \mathbb{Z}^n \mid A_{-g} A_g \text{ is not contained in } \mathfrak{m} \}, \tag{8.3}$$

$$G_\mathfrak{m} := \widetilde{G}_\mathfrak{m} \cap \operatorname{Stab}_{\mathbb{Z}^n}(\mathfrak{m}). \tag{8.4}$$

One can show that $G_\mathfrak{m}$ is a subgroup of $\mathbb{Z}^n$ and $\widetilde{G}_\mathfrak{m}$ is a union of cosets from $\mathbb{Z}^n / G_\mathfrak{m}$.

Fix now $\mathfrak{m} \in \operatorname{Supp}(V)$. One checks that the subalgebra $B(\mathfrak{m}) := \bigoplus_{g \in \operatorname{Stab}_{\mathbb{Z}^n}(\mathfrak{m})} A_g$ of A preserves the weight space $V_\mathfrak{m}$. For any $g \in \widetilde{G}_\mathfrak{m}$ we pick elements $a_g \in A_g$ and $a'_g \in A_{-g}$ such that $a'_g a_g \notin \mathfrak{m}$. The following theorem describes the simple weight modules with no proper inner breaks up to the structure of $V_\mathfrak{m}$ as a $B(\mathfrak{m})$-module.

Theorem 8.7. *(See [10].) Suppose V has no proper inner breaks. If $\{v_i\}_{i \in J}$ is a $\Bbbk$-basis for $V_\mathfrak{m}$ (J some index set), then the following is a $\Bbbk$-basis for V:*

$$C := \{ a_g v_i \mid g \in S,\ i \in J \} \tag{8.5}$$

where $S \subseteq \widetilde{G}_\mathfrak{m}$ is a set of representatives for $\widetilde{G}_\mathfrak{m}$ modulo $G_\mathfrak{m}$. Moreover, for any $v \in V_\mathfrak{m}$, any $i \in \{1, \ldots, n\}$ and $g \in S$ we have

$$X_i a_g v = \begin{cases} a_h b_{g,i} v, & g + e_i \in \widetilde{G}_\mathfrak{m}, \\ 0, & \textit{otherwise}, \end{cases} \qquad Y_i a_g v = \begin{cases} a_k c_{g,i} v, & g - e_i \in \widetilde{G}_\mathfrak{m}, \\ 0, & \textit{otherwise}, \end{cases} \tag{8.6}$$

where $h, k \in S$ with $h \in (g + e_i) + G_\mathfrak{m}$ and $k \in (g - e_i) + G_\mathfrak{m}$ and $b_{g,i}, c_{g,i} \in B(\mathfrak{m})$ are given by

$$b_{g,i} = \sigma_{-h}\big(X_i a_g a'_{g+e_i-h} a'_h \big) a_{g+e_i-h}, \qquad c_{g,i} = \sigma_{-k}\big(Y_i a_g a'_{g-e_i-k} a'_k \big) a_{g-e_i-k}. \tag{8.7}$$

8.3. The case of trivial stabilizer

We prove here a theorem which implies that all simple weight modules over $A_n^k(r,s,\Lambda)/\langle\mathfrak{n}\rangle$ have no proper inner breaks.

Theorem 8.8. *If V is a simple weight module over a twisted generalized Weyl algebra $\mathcal{A}_\mu(R,\sigma,t)$ such that the stabilizer $\mathrm{Stab}_{\mathbb{Z}^n}(\mathfrak{m})$ is trivial for some (hence all) weight $\mathfrak{m}\in\mathrm{Supp}(V)$, then V has no proper inner breaks.*

Proof. Suppose $\mathfrak{m}\in\mathrm{Supp}(V)$ has trivial stabilizer. Let $g\in\mathbb{Z}^n$ and assume $a\in A_g$ is such that $aV_\mathfrak{m}\neq 0$. Since V is simple, $V_\mathfrak{m}\cap AaV_\mathfrak{m}\neq 0$. But $V_\mathfrak{m}\cap AaV_\mathfrak{m}\subseteq A_{-g}aV_\mathfrak{m}$ since $\mathfrak{m}$ has trivial stabilizer. This shows that there exists an element $b\in A_{-g}$ such that $baV_\mathfrak{m}\neq 0$. Since $\deg(ba)=0$ we have $ba\in R$. Then $baV_\mathfrak{m}\neq 0$ implies $ba\notin\mathfrak{m}$. □

8.4. Abstract description of the simple weight modules in case of trivial stabilizer

Let $A=\mathcal{A}_\mu(R,\sigma,t)$ be a TGWA where μ is symmetric. In [17] a description of all simple weight modules with support in an orbit with trivial stabilizer is given in terms of a Shapovalov type form. The form used in [17] requires the matrix μ to be symmetric (due to its formulation in terms of a certain involution on the TGWA). In [12] it was observed that there is another way to define a bilinear form which works for general μ. It is given as follows. Let $\mathfrak{p}_0: A\to A_0=R$ be the graded projection onto the degree zero component of A with respect to the standard $\mathbb{Z}^n$-gradation on A. Then put

$$F: A\times A\to R,\quad F(a,b)=\mathfrak{p}_0(ab).\tag{8.8}$$

Such forms have been studied for arbitrary group graded rings [7].

We have the following result.

Theorem 8.9. *Let $A=\mathcal{A}_\mu(R,\sigma,t)$ be any twisted generalized Weyl algebra. Let V be any simple weight module over A such that $\mathrm{Stab}_{\mathbb{Z}^n}(\mathfrak{m})=\{0\}$ for $\mathfrak{m}\in\mathrm{Supp}(V)$. Then $V\simeq A/N(\mathfrak{m})$ where A is considered as a left module over itself and $N(\mathfrak{m})$ is the left ideal given by*

$$N(\mathfrak{m})=\bigl\{a\in A\bigm| F(b,a)\in\mathfrak{m},\ \forall b\in A\bigr\}.\tag{8.9}$$

Proof. Similar to the case of symmetric μ proved in [17, Lemma 6.1 and Corollary 6.2]. □

8.5. Bases and explicit action on the simple weight modules over $A_n^k(r,s,\Lambda)$

Let n, k, r, s, Λ be as in Section 5.1. Assume that for each $i=1,\dots,n$, the scalar r_{ii}/s_{ii} is not a root of unity. Let R, σ, t, μ be as in Section 5.1.

Let J be any $\mathbb{Z}^n$-invariant ideal of R. Let $A=\mathcal{A}_\mu(R/J,\bar\sigma,\bar t)$. Thus for $J=0$, A equals the multiparameter twisted Weyl algebra $A_n^k(r,s,\Lambda)$, and for $J=R\mathfrak{n}$ where $\mathfrak{n}\in\mathrm{Specm}(R^{\mathbb{Z}^n})$, A equals a simple quotient of the algebra in the former case.

We will describe the simple weight modules over A, using Theorem 8.7.

Let V be a simple weight module over A. Let $\mathfrak{m}\in\mathrm{Supp}(V)$. Since $\Bbbk$ is algebraically closed we have

$$\mathfrak{m}=(\bar u_i-\alpha_i,\bar v_i-\beta_i\mid i=1,\dots,n)$$

where $\bar u_i=u_i+J$, $\bar v_i=v_i+J$ and $\alpha_i,\beta_i\in\Bbbk\backslash\{0\}$ for $i=1,\dots,n$.

We determine the set $\widetilde{G}_\mathfrak{m}$. Let $g\in\mathbb{Z}^n$. Since $A_g=\bar RZ^{(g)}$ (where $Z^{(g)}=Z_1^{(g)}\cdots Z_n^{(g_n)}$ where $Z_i^{(j)}$ equals X_i^j if $j\geqslant 0$ and Y_i^{-j} otherwise) and $\forall i\neq j$: $\sigma_i(t_j)=\gamma_{ij}t_i$ for some $\gamma_{ij}\in\Bbbk\backslash\{0\}$ it is clear that

$$\begin{aligned}\widetilde{G}_{\mathfrak{m}} &= \{g \in \mathbb{Z}^n \mid Z^{(-g)} Z^{(g)} \notin \mathfrak{m}\} \\ &= \{g \in \mathbb{Z}^n \mid Z_1^{(-g_1)} Z_1^{(g_1)} \cdots Z_n^{(-g_n)} Z_n^{(g_n)} \notin \mathfrak{m}\} \\ &= \{g \in \mathbb{Z}^n \mid Z_i^{(-g_i)} Z_i^{(g_i)} \notin \mathfrak{m},\ \forall i\} \\ &= \widetilde{G}_{\mathfrak{m}}^{(1)} \times \cdots \times \widetilde{G}_{\mathfrak{m}}^{(n)},\end{aligned}$$

where

$$\widetilde{G}_{\mathfrak{m}}^{(i)} := \{g \in \mathbb{Z}^n \mid Z_i^{(-g_i)} Z_i^{(g_i)} \notin \mathfrak{m}\}. \tag{8.10}$$

For $j > 0$ we have

$$Z_i^{(-j)} Z_i^{(j)} = Y_i^j X_i^j = t_i \sigma_i^{-1}(t_i) \cdots \sigma_i^{-j+1}(t_i) \tag{8.11}$$

while for $j < 0$,

$$Z_i^{(-j)} Z_i^{(j)} = X_i^{-j} Y_i^{-j} = \sigma_i(t_i) \sigma_i^2(t_i) \cdots \sigma_i^{-j}(t_i). \tag{8.12}$$

So, since $\mathfrak{m}$ is maximal, hence prime, we see that if $j > 0$ and $j \in \widetilde{G}_{\mathfrak{m}}^{(i)}$ then $\{0, 1, \dots, j\} \subseteq \widetilde{G}_{\mathfrak{m}}^{(i)}$. Similarly if $j < 0$ and $j \in \widetilde{G}_{\mathfrak{m}}^{(i)}$ then $\{j, j+1, \dots, 0\} \subseteq \widetilde{G}_{\mathfrak{m}}^{(i)}$.

We distinguish between three possibilities. The first case is that $\widetilde{G}_{\mathfrak{m}}^{(i)} = \mathbb{Z}$. Then we say that (the support of) V is *generic* in the ith direction. The second case is $j \notin \widetilde{G}_{\mathfrak{m}}^{(i)}$ for some positive integer j. Assuming j is the smallest such integer, by (8.10) and (8.11) we get $\sigma_i^{-j+1}(t_i) \in \mathfrak{m}$. By Corollary 8.3 it follows that $\sigma_i^m(t_i) \notin \mathfrak{m}$ for all integers $m \neq j$. Thus $\widetilde{G}_{\mathfrak{m}}^{(i)} = \{m \in \mathbb{Z} \mid m \leqslant j-1\}$. By Theorem 8.7, $\operatorname{Supp}(V) = \{\sigma_g(\mathfrak{m}) \mid g \in G_{\mathfrak{m}}\}$ and thus we can replace $\mathfrak{m}$ by $\sigma_i^{k-1}(\mathfrak{m})$. Doing this, the new j just equals 1 and $G_{\mathfrak{m}}^{(i)} = \mathbb{Z}_{\leqslant 0}$. We say that $\mathfrak{m}$ is a *highest weight* for V in the ith direction. The final case is that $j \notin \widetilde{G}_{\mathfrak{m}}^{(i)}$ for some negative integer j. This is analogous to the previous case and leads to that, without loss of generality, $\widetilde{G}_{\mathfrak{m}}^{(i)} = \mathbb{Z}_{\geqslant 0}$ in which case we say that $\mathfrak{m}$ is a *lowest weight* for V in the ith direction.

In other words, there is an $\mathfrak{m} \in \operatorname{Supp}(V)$ such that the shape of the support of V is characterized by a vector

$$\tau \in \{-1, 0, 1\}^n \tag{8.13}$$

via the relation

$$\widetilde{G}_{\mathfrak{m}}^{(i)} = \{j \in \mathbb{Z} \mid j \cdot \tau_i \geqslant 0\}, \quad \forall i \in \{1, \dots, n\}. \tag{8.14}$$

Since the stabilizer of $\mathfrak{m}$ is trivial by Proposition 8.1, the subalgebra $B(\mathfrak{m})$ in Theorem 8.7 is just R. From well-known results [8] (see [17, Proposition 7.2] for a proof in the TGW algebra case), it follows that $V_{\mathfrak{m}}$ is simple as a $B(\mathfrak{m})$-module since V is simple as an A-module. Thus, since $R/\mathfrak{m} = \Bbbk$, we have $\dim_{\Bbbk} V_{\mathfrak{m}} = 1$. Pick $v_0 \in V_{\mathfrak{m}}$, $v_0 \neq 0$. Then Theorem 8.7 implies that the set

$$C = \{v_g := Z_1^{(g_1)} \cdots Z_n^{(g_n)} v_0 \mid g = (g_1, \dots, g_n) \in \widetilde{G}_{\mathfrak{m}}\} \tag{8.15}$$

is a $\Bbbk$-basis for V, where $Z_i^{(j)} = X_i^j$ if $j \geqslant 0$ and Y_i^{-j} otherwise. Furthermore, the action of X_i, Y_i on the elements of C is given by

$$X_i v_g = \begin{cases} b_{g,i} v_{g+e_i}, & \text{if } (g_i+1)\tau_i \geqslant 0, \\ 0, & \text{otherwise}, \end{cases} \qquad Y_i v_g = \begin{cases} c_{g,i} v_{g-e_i}, & \text{if } (g_i-1)\tau_i \geqslant 0, \\ 0, & \text{otherwise}, \end{cases} \tag{8.16}$$

for certain $b_{g,i}, c_{g,i} \in \Bbbk$. Although the formulas (8.7) can be used to calculate these scalars, one can also use a more direct approach which is available due to our knowledge of the commutation relations (5.6) among the generators X_i, Y_i in A. Straightforward calculation gives the following

$$b_{g,i} = \gamma_{i1}^{(g_1)} \cdots \gamma_{i,i-1}^{(g_{i-1})} \cdot r_{i+1,i}^{kg_i} \cdots r_{ni}^{kg_n} \cdot \begin{cases} 1 & \text{if } g_i \geqslant 0, \\ \frac{r_{ii}^{(1-g_i)k}\alpha_i^k - s_{ii}^{(1-g_i)k}\beta_i^k}{r_{ii}^k - s_{ii}^k} & \text{if } g_i < 0, \end{cases} \tag{8.17}$$

$$c_{g,i} = \varepsilon_{i1}^{(g_1)} \cdots \varepsilon_{i,i-1}^{(g_{i-1})} \cdot r_{i+1,i}^{kg_i} \cdots r_{ni}^{kg_n} \cdot \begin{cases} \frac{r_{ii}^{kg_i}\alpha_i^k - s_{ii}^{kg_i}\beta_i^k}{r_{ii}^k - s_{ii}^k} & \text{if } g_i \geqslant 0, \\ 1 & \text{if } g_i < 0, \end{cases} \tag{8.18}$$

where

$$\gamma_{ij}^{(l)} = \begin{cases} ((r_{ji}/r_{ij})^k \lambda_{ij})^l, & l \geqslant 0, \\ (r_{ji}^k \lambda_{ij})^l, & l < 0, \end{cases} \qquad \varepsilon_{ij}^{(l)} = \begin{cases} (r_{ij}^k \lambda_{ji})^l, & l \geqslant 0, \\ \lambda_{ji}^l, & l < 0. \end{cases} \tag{8.19}$$

9. Whittaker modules

Definition 9.1. Let A be a twisted generalized Weyl algebra of degree n. A module V over A is called a *Whittaker module* if there exists a vector $v_0 \in V$ (called *Whittaker vector*) and nonzero scalars $\zeta_1, \dots, \zeta_n \in \Bbbk \setminus \{0\}$ such that the following conditions hold:

- $V = A v_0$,
- $X_i v_0 = \zeta_i v_0$ for each $i = 1, \dots, n$.

The pair (V, v_0) is called a *Whittaker pair of type* $(\zeta_1, \dots, \zeta_n)$. A morphism of Whittaker pairs $(V, v_0) \to (W, w_0)$ is an A-module morphism $V \to W$ mapping v_0 to w_0.

The term “Whittaker pair” also occurs in the literature with a completely different meaning (e.g. [2]).

The reader may wonder why one requires ζ_i to be nonzero for all i. To see this, note that in this case we have $Y_i v_0 = \zeta_i^{-1} Y_i X_i v_0 = \zeta_i^{-1} t_i v_0$. Thus the A-module V is completely determined by its R-module structure together with the parameters ζ_i. The same argument fails if we would allow ζ_i to be zero for some i. This indicates that the case when some ζ_i is zero requires a different analysis than the one below.

The following theorem describes Whittaker pairs over a family of TGWAs which properly includes all generalized Weyl algebras in which the t_i are regular. It is a generalization of [6, Theorem 3.12]. We use the notation from Section 4.1.

Theorem 9.2. *Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a $\Bbbk$-finitistic TGW algebra of Lie type $(A_1)^n$. Assume that (R, σ, t) is μ-consistent and that R is Noetherian.*

(a) *If A has a Whittaker module, then*

$$\gamma_{ij} = \mu_{ij} \quad \text{for all } i \neq j. \tag{9.1}$$

(b) *Conversely, if* (9.1) *holds, then for each $\zeta \in (\Bbbk \setminus \{0\})^n$, there is a bijection*

$$\left\{\begin{array}{l}\textit{Isomorphism classes } [(V, v_0)] \textit{ of}\\ \textit{Whittaker pairs of type } \zeta\end{array}\right\} \xrightarrow{\Psi} \left\{\textit{Proper } \mathbb{Z}^n\textit{-invariant left ideals } Q \textit{ of } R\right\},$$

$$\big[(V, v_0)\big] \mapsto \operatorname{Ann}_R v_0,$$

$$\big[(R/Q, 1+Q)\big] \leftarrow Q$$

where R/Q is given an A-module structure by

$$\begin{aligned} s.\bar{r} &= \overline{sr}, \quad \forall s \in R,\\ X_i.\bar{r} &= \zeta_i \overline{\sigma_i(r)},\\ Y_i.\bar{r} &= \zeta_i^{-1}\overline{\sigma_i^{-1}(r)t_i}, \end{aligned} \tag{9.2}$$

for all $\bar{r} \in R/Q$, where $\bar{r} := r + Q \in R/Q$ for $r \in R$.

(c) *Furthermore, there is a morphism of Whittaker pairs $(V, v_0) \to (W, w_0)$ iff $\Psi([(V, v_0)]) \subseteq \Psi([(W, w_0)])$.*

Proof. (a) Suppose (V, v_0) is a Whittaker pair with respect to $\zeta \in (\Bbbk\backslash\{0\})^n$. Then for $i \neq j$, $X_i X_j v_0 = \zeta_i \zeta_j v_0$. On the other hand, by relation (4.5), $X_i X_j = \gamma_{ij}\mu_{ij}^{-1} X_j X_i$ and thus $X_i X_j v_0 = \gamma_{ij}\mu_{ij}^{-1} X_j X_i v_0 = \gamma_{ij}\mu_{ij}^{-1}\zeta_i\zeta_j v_0$. Thus, since v_0 and all ζ_i are nonzero by definition, we conclude that (9.1) must hold.

(b) Suppose (V, v_0) is a Whittaker pair with respect to $\zeta \in (\Bbbk\backslash\{0\})^n$. Let $Q = \operatorname{Ann}_R v_0$. Clearly Q is a proper left ideal of R. For any $r \in Q$ we have $0 = X_i r v_0 = \sigma_i(r) X_i v_0 = \zeta_i \sigma_i(r) v_0$ which shows that $\sigma_i(Q) \subseteq Q$ for any $i \in \{1, \dots, n\}$. Since R is Noetherian, $\sigma_i^{-1}(Q) \subseteq Q$ as well, which proves that Q is $\mathbb{Z}^n$-invariant. In addition, if (V, v_0) and (W, w_0) are two isomorphic Whittaker pairs, then clearly $\operatorname{Ann}_R v_0 = \operatorname{Ann}_R w_0$. This shows that the map Ψ is well defined.

To prove that Ψ is surjective, suppose that Q is a proper $\mathbb{Z}^n$-invariant left ideal of R. We show that (9.2) extends the natural R-module structure on R/Q to an A-module structure. We only prove that the following relations are preserved: $X_i Y_j = \mu_{ij} Y_j X_i$ $(i \neq j)$ and $Y_j Y_i = \gamma_{ij}\mu_{ji}^{-1} Y_i Y_j$ $(i \neq j)$. The other cases are identical to the generalized Weyl algebra case considered in [6, Section 3]. We have

$$X_i Y_j.\bar{r} = X_i.\zeta_j^{-1}\overline{\sigma_j^{-1}(r)t_j} = \zeta_i\zeta_j^{-1}\overline{\sigma_i\sigma_j^{-1}(r)\sigma_i(t_j)}.$$

Using that $\sigma_i(t_j) = \gamma_{ij} t_j$ (see (4.2)) and condition (9.1) we see that $X_i Y_j.\bar{r} = \mu_{ij} Y_j X_i.\bar{r}$ for any $\bar{r} \in R/Q$. Similarly $Y_j Y_i.\bar{r} = \zeta_i^{-1}\zeta_j^{-1}\overline{\sigma_i^{-1}\sigma_j^{-1}(r)\sigma_j^{-1}(t_i)t_j}$ so using $\sigma_j^{-1}(t_i)t_j = \gamma_{ji}^{-1}\gamma_{ij} t_i \sigma_i^{-1}(t_j)$ and (9.1) again, we see that $Y_j Y_i.\bar{r} = \gamma_{ij}\mu_{ji}^{-1} Y_i Y_j.\bar{r}$, $\forall i \neq j$. Thus R/Q becomes an A-module which is a Whittaker module of type ζ with Whittaker vector $1 + Q$.

To prove that Ψ is injective we may, as in [6], construct a universal Whittaker module V_{u} of type ζ by putting $V_{\mathrm{u}} = A \otimes_{A_+} \Bbbk_\zeta$ where A_+ is the subalgebra of A generated over $\Bbbk$ by $X_1, \dots, X_n$, and $\Bbbk_\zeta$ is the 1-dimensional module over A_+ given by $X_i.1 := \zeta_i$. The map $\iota : R \to V_{\mathrm{u}}$, $r \mapsto r \otimes 1$ is an R-module isomorphism. Then there is a unique morphism of Whittaker pairs from $(V_{\mathrm{u}}, 1 \otimes 1)$ to any other Whittaker pair (V, v_0) of type ζ. And, identifying V_{u} with R via ι, the kernel of the map $V_{\mathrm{u}} \to V$, is precisely $\operatorname{Ann}_R v_0$. So if (V, v_0) and (W, w_0) are two Whittaker pairs with $\operatorname{Ann}_R v_0 = \operatorname{Ann}_R w_0$, it means that they are isomorphic to the same quotient of the universal Whittaker pair of type ζ, hence are isomorphic to eachother.

(c) If $\varphi : (V, v_0) \to (W, w_0)$ is a morphism of Whittaker pairs, then $\varphi(r v_0) = r\varphi(v_0) = r w_0$ so clearly $\operatorname{Ann}_R v_0 \subseteq \operatorname{Ann}_R w_0$. Conversely, if $Q_1 \subseteq Q_2$ are proper $\mathbb{Z}^n$-invariant left ideals, then there is an R-module morphism $\pi : R/Q_1 \to R/Q_2$ mapping $1 + Q_1$ to $1 + Q_2$. Since π commutes with the $\mathbb{Z}^n$-action, one verifies that π is automatically an A-module morphism. $\square$

Corollary 9.3. *Let $A = A_n^k(r, s, \Lambda)/\langle \mathfrak{n} \rangle$ be a simple quotient of a multiparameter twisted Weyl algebra as obtained in Theorem* 5.8. *Then A has a Whittaker module iff*

$$\lambda_{ij} = (r_{ij}/r_{ji})^k, \quad \forall i, j. \tag{9.3}$$

Moreover, if (9.3) *holds, then for each $\zeta \in (\Bbbk\backslash\{0\})^n$ there is a unique Whittaker module over A of type ζ, namely the universal one, and it is a simple module.*

Acknowledgments

This work was carried out during the second author's postdoc at IME-USP, funded by FAPESP, processo 2008/10688-1. The first author is supported in part by the CNPq grant (301743/2007-0) and by the Fapesp grant (2005/60337-2).

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