ELSEVIER
Available online at www.sciencedirect.com
SciVerse ScienceDirect
Procedia Engineering 29 (2012) 2313 – 2321
Procedia Engineering
www.elsevier.com/locate/procedia

2012 International Workshop on Information and Electronics Engineering (IWIEE)

Applying Minimum-Risk Criterion to Stochastic Hub Location Problems

Hao Zhai, Yankui Liu*, Weili Chen

College of Mathematics and Computer Science, Hebei University, Baoding 071002, Hebei, China

Abstract

This paper presents a new class of two-stage stochastic hub location (HL) programming problems with minimum-risk criterion, in which uncertain demands are characterized by random vector. Meanwhile we demonstrate that the two-stage programming problem is equivalent to a single-stage stochastic P-model. Under mild assumptions, we develop a deterministic binary programming problem by using standardization, which is equivalent to a binary fractional programming problem. Moreover, we show that the relaxation problem of the binary fractional programming problem is a convex programming problem. Taking advantage of branch-and-bound method, we provide a number of experiments to illustrate the efficiency of the proposed modeling idea.

Keywords: stochastic hub location ; two-stage stochastic programming ; minimum-risk criterion ; multivariate normal distribution ; binary fractional programming

1. Introduction

Hub location problems arise in transportation, telecommunication and computer networks, where hub-and-spoke architectures are used to route commodities between many origin and destination pairs. Several variants of these problems have been developed in the literature, such as uncapacitated hub location, *p*-hub location, and hub covering. The location of hub facilities corresponds to long-term strategic decisions which are typically made under an uncertain environment. There exist basically two streams of research dealing with optimization under uncertainty: stochastic optimization and robust optimization. Based on probability theory, some of stochastic programming models (see [1][2]) considered capacities on the

* Corresponding author.
E-mail address: zhaihlab@gmail.com

doi:10.1016/j.proeng.2012.01.307

facilities, and facility size was considered as a first-stage decision. Ravi and Sinha [3] proposed a stochastic problem in which facilities may be open in either the first or second stage; Louveaux [4] presented reviews on modeling approaches for stochastic facility location in which the location of the facilities is considered as the first-stage decision and the distribution pattern is the second-stage decision; Marianov and Serra [5] focused on stochasticity at the hub nodes by representing hub airports as M/D/c queues and limiting through chance constraints the number of airplanes that can queue at an airport; Sim et al. [6] introduced the stochastic *p*-hub center problem and employed a chance-constrained formulation to model the minimum service-level requirement, and Yang [7] presented a two-stage stochastic programming model for air freight hub location and flight route planning under seasonal demand variations. Particularly, one of the problems that have received most attention is the uncapacitated hub location problem with multiple assignments (see [8][9][10]). Under random environment, Contreras et al. [11] modeled this problem as a two-stage integer stochastic programming with recourse in the presence of uncertainty on demands and transportation costs, and introduced three different stochastic models.

In the literature, the expected value models are often used for modeling stochastic hub location problems. On the one hand, having an equivalent linear programming problem is an attractive feature from the numerical point of view. On the other hand, however, replacing the probability distribution by a one-point distribution leads to a very crude approximation of the original distribution in general. This is usually an indication of a modeling or data error: the corresponding stochastic model is not truly stochastic. When doing this, extreme care is needed, since the solution obtained this way may turn out to be quite risky when evaluated by expectation. To overcome the imprecise measurement, this paper is devoted to pursuing the idea of using probability function as a risk criterion to measure the uncertainty. Probability has a very general meaning, including virtually all aspects of randomness. It is known that stochastic programming with probability objective function is usually called P-model [12], and applied to many real-life decision problems, especially decision problems involving risk.

The remainder of this paper is organized as follows. Section 2 presents a new class of two-stage stochastic hub location programming and provides its equivalent P-model. In Section 3, we focus on the issue about stochastic demands with multivariate normal distribution, and turn the P-model with probability objective function into a binary fractional programming problem. In Section 4 we discuss the solution methods for the fractional programming model. In Section 5 we perform some numerical experiments to illustrate the developed new modeling idea. Section 6 gives the conclusions.

2. Formulation of Two-Stage Staochastic HL Problem

For stochastic uncapacitated hub location problems, it consists in locating a set of hubs and in determining the routing of commodities through the hub nodes, with the objective of minimizing the total set-up and transportation costs. We will adopt the following notations for our model:

- **Notations**
- Q the set of nodes including the origin $o(k)$ and the destination $d(k)$;
- H the set of potential hub locations, $H \subseteq Q$, index $i, j \in H$;
- f_i the fixed set-up cost for locating a hub at node i ;
- z_i location variable equals to 1 if a hub is located at node i and equals to 0 otherwise;
- K the set of commodities whose origin and destination points belong to Q, index $k \in K$;
- $W_k(\xi)$ the demands of commodity $k \in K$ about random variable ξ ;
- x_{ijk} routing variable, it equals to 1 if commodity k transits via a first hub i and a second Hub j and equals to 0 otherwise;
- F_{ijk} the unit transportation cost for commodity k transits via a first hub i and a second hub j ;
- Ξ the support of ξ.

Under uncertain environment, the decision-maker may face various uncertain factors in the whole location process. That is, demands may change after location decisions characterized by $\mathbf{z}=(z_1,z_2,\ldots,z_{|H|})^T$ have been made. To minimize the total cost of location and transportation, routing decisions x_{ijk} or $x_{ijk}(\xi)$ may also change after location decisions have been made. In this paper, we assume that these uncertain parameter *demands* are characterized by independent random variables $W_k(\xi)$ with known probability distribution. Given the uncertain surroundings after location decisions, we model this class problem as a two-stage stochastic programming with recourse. The first-stage decisions correspond to the location of hub facilities, and the second-stage decisions correspond to the optimal routing of the commodities. Location decisions $\mathbf{z}=(z_1,z_2,\ldots,z_{|H|})^T$, called the first-stage decisions, must be taken before knowing the particular values taken by the random variables $W_k(\xi)$. Routing decisions $\mathbf{X}=\{x_{ijk}\mid i,j\in H,k\in K\}$, called the second-stage decisions, can be taken after the realizations of random variable $W_k(\xi)$ are known. In order to emphasis the dependence of $\mathbf{X}$ on $\mathbf{z}$ and ξ, we can denote $\mathbf{X}$ as $\mathbf{X}(\mathbf{z},\xi)$ or $x_{ijk}(\xi)$. The dependence of x_{ijk} on ξ is not functional but simply indicates that the decision x_{ijk} are typically not the same under different realizations of ξ.

In hub location problems, there is a single path connecting the origin and destination nodes of every commodity $k\in K$. Constraints (c1) impose the single connection of the origin to the destination:

$$\sum_{i\in \mathrm{H}}\sum_{j\in \mathrm{H}} x_{ijk}(\xi)=1,\ k\in \mathrm{K},\xi\in\Xi. \qquad (c1)$$

In order to prohibit commodity $k\in K$ from being routed via a non-hub node for every commodity $k\in K$ and every hub $i\in \mathrm{H}$, constraints (c2) can be stated as follows:

$$\sum_{j\in \mathrm{H}} x_{ijk}(\xi)+\sum_{j\in \mathrm{H}\backslash\{i\}} x_{ijk}(\xi)\le z_i,\ i\in \mathrm{H},k\in \mathrm{K},\xi\in\Xi. \qquad (c2)$$

Constraints (c3) are the standard non-negativity constraints:

$$x_{ijk}(\xi)\ge 0, i\in \mathrm{H},k\in \mathrm{K},\xi\in\Xi. \qquad (c3)$$

If the first-stage decision vector $\mathbf{z}$ is given, and a realization $W_k(\hat{\xi})$ of random variable $W_k(\xi)$ is known for every commodity $k\in K$, then the second-stage programming can be built as:

$$\begin{cases} \min\limits_{\boldsymbol{X}} & \sum\limits_{i\in \mathrm{H}}\sum\limits_{j\in \mathrm{H}}\sum\limits_{k\in \mathrm{K}} (W_k(\xi)\cdot F_{ijk})\cdot x_{ijk}(\xi) \\ \text{subject to :} & \sum\limits_{i\in \mathrm{H}}\sum\limits_{j\in \mathrm{H}} x_{ijk}(\xi)=1,\ k\in \mathrm{K},\xi\in\Xi \\ & \sum\limits_{j\in \mathrm{H}} x_{ijk}(\xi)+\sum\limits_{j\in \mathrm{H}\backslash\{i\}} x_{ijk}(\xi)\le z_i,\ i\in \mathrm{H},k\in \mathrm{K},\xi\in\Xi \\ & x_{ijk}(\xi)\ge 0, i\in \mathrm{H},k\in \mathrm{K},\xi\in\Xi. \end{cases} \qquad (1)$$

The principle of the second-stage programming (1) is to minimize the routing costs in the second stage for a fixed first-stage decisions $\mathbf{z}$ and a known realization $W_k(\hat{\xi})$ of stochastic demands $W_k(\xi)$ for every commodity $k\in K$.

In the first stage, we use the probability of occurrence of random event

$$\{\omega \mid \sum_{i\in \mathrm{H}} f_i z_i + \sum_{i\in \mathrm{H}} \sum_{j\in \mathrm{H}} \sum_{k\in \mathrm{K}} (W_k(\xi(\omega)) \cdot F_{ijk}) \cdot x_{ijk}(\xi(\omega)) \le C_0\},$$

called risk measure, to denote the probability that the total costs does not exceed the level C_0, where C_0 denotes the prescribed upper bound of total costs. For a given C_0, the higher probability corresponds to lower risk for the decision-maker. In order to minimize the total costs over the two stages, the objective function of model can be built as follows:

$$\mathbf{Pr}\{\sum_{i\in \mathrm{H}} f_i z_i + \sum_{i\in \mathrm{H}} \sum_{j\in \mathrm{H}} \sum_{k\in \mathrm{K}} (W_k(\xi) \cdot F_{ijk}) \cdot x_{ijk}(\xi) \le C_0\},$$

and we are required to find a feasible decision $\mathbf{z}$ such that the probability of the random total costs less than C_0 is maximum. As a consequence, the first-stage programming can be formulated as:

$$\begin{cases} \max\limits_{z} & \mathbf{Pr}\{\sum_{i\in \mathrm{H}} f_i z_i + \sum_{i\in \mathrm{H}} \sum_{j\in \mathrm{H}} \sum_{k\in \mathrm{K}} (W_k(\xi) \cdot F_{ijk}) \cdot x_{ijk}(\xi) \le C_0\} \\ \text{subject to}: & \mathbf{z} \in \{0,1\}^{|\mathrm{H}|}. \end{cases} \tag{2}$$

Combining models (1) and (2), we present the following new two-stage stochastic hub location model:

$$\begin{cases} \max\limits_{z} & \mathbf{Pr}\{\sum_{i\in \mathrm{H}} f_i z_i + Q(\mathbf{z},\xi) \le C_0\} \\ \text{subject to}: & \mathbf{z} \in \{0,1\}^{|\mathrm{H}|}, \end{cases} \tag{3}$$

where $Q(\mathbf{z},\xi)$ is the optimal value of the second-stage programming problem:

$$\begin{cases} Q(\mathbf{z},\xi) = & \min\limits_{\boldsymbol{X}} \sum_{i\in \mathrm{H}} \sum_{j\in \mathrm{H}} \sum_{k\in \mathrm{K}} (W_k(\xi) \cdot F_{ijk}) \cdot x_{ijk}(\xi) \\ \text{subject to}: & \sum_{i\in \mathrm{H}} \sum_{j\in \mathrm{H}} x_{ijk}(\xi) = 1,\ k \in \mathrm{K}, \xi \in \Xi \\ & \sum_{j\in \mathrm{H}} x_{ijk}(\xi) + \sum_{j\in \mathrm{H}\setminus\{i\}} x_{ijk}(\xi) \le z_i,\ i \in \mathrm{H}, k \in \mathrm{K}, \xi \in \Xi \\ & x_{ijk}(\xi) \ge 0, i \in \mathrm{H}, k \in \mathrm{K}, \xi \in \Xi. \end{cases} \tag{4}$$

In the following, we prove that the two-stage stochastic programming is equivalent to a single-stage stochastic P-model, which facilitates us to solve our hub location problem.

Theorem 1. *The two-stage stochastic programming problem (3)-(4) is equivalent to the following minimum-risk P-model (5).*

Proof. Observe that, given a first-stage vector $\mathbf{z}$, the second-stage term of the objective function can be separated into $|K|$ independent subproblems, and each commodity $k \in K$ corresponds to one subproblem. For each of these subproblems, the optimal solution does not depend on the particular realization of random variable ξ. That is, the optimal route called second-stage decisions is the same for sending each commodity regardless of the actual value of the demand $W_k(\xi)$. Let $x_{ijk}(z)$ be the optimal solution vector associated to a first-stage solution $\mathbf{z}$. Then, we have

$$x_{ijk}(\xi) = x_{ijk}(z) = x_{ijk}, \forall k \in \mathrm{K}, \forall i, j \in \mathrm{H}, \forall \xi \in \Xi.$$

$$\begin{cases} \max\limits_{z,X} & \mathbf{Pr}\{\sum\limits_{i\in \mathrm{H}} f_i z_i + \sum\limits_{i\in \mathrm{H}} \sum\limits_{j\in \mathrm{H}} \sum\limits_{k\in \mathrm{K}} (W_k(\xi)\cdot F_{ijk})\cdot x_{ijk} \le C_0\} \\ \text{subject to :} & \sum\limits_{i\in \mathrm{H}} \sum\limits_{j\in \mathrm{H}} x_{ijk} = 1,\ k\in \mathrm{K} \\ & \sum\limits_{j\in \mathrm{H}} x_{ijk} + \sum\limits_{j\in \mathrm{H}\setminus\{i\}} x_{ijk} \le z_i,\ i\in \mathrm{H}, k\in \mathrm{K} \\ & x_{ijk} \ge 0, i\in \mathrm{H}, k\in \mathrm{K} \\ & \mathbf{z}\in\{0,1\}^{|\mathrm{H}|}. \end{cases} \tag{5}$$

Therefore, the two-stage stochastic programming problem (3)-(4) is equivalent to the single-stage stochastic programming problem (5). □

Although P-model (5) is a single-stage programming problem with deterministic constraints, it is also very hard to be solved due to the probability objective function. That is, it is difficult to handle numerically for general stochastic demands. Therefore, it is desirable to obtain an explicit representation of objective function. Under appropriate assumptions in next section, we will give the equivalent formulation for the probability objective function, and turn the stochastic programming model (5) into a deterministic equivalent programming problem.

3. Equivalent Binary Fractional Programming Problem

In this section, we assume that the $|K|$-dimensional random vector

$$\boldsymbol{W} = (W_1(\xi), W_2(\xi), \ldots, W_{|K|}(\xi))^{\mathrm{T}}$$

has a multivariate normal distribution, then there exist an $|K|\times S$ matrix $\boldsymbol{D}$ and a vector $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_{|K|}) \in \boldsymbol{R}^{|\mathrm{K}|}$, such that:

$$\boldsymbol{W} = \boldsymbol{D}\cdot\boldsymbol{\zeta}' + \boldsymbol{\mu}$$

holds, where $\boldsymbol{\zeta}'$ is an S-dimensional random vector with $\boldsymbol{\zeta}_i'$ being stochastically independent and having a standard normal distribution [12]. Therefore, the mathematical expectation vector of stochastic demands $\boldsymbol{W}$ is $\boldsymbol{\mu}$ and the covariance matrix of $\boldsymbol{W}$ is $\boldsymbol{\Sigma} = \boldsymbol{D}\cdot\boldsymbol{D}^T$.

If we define

$$\zeta(z,x,\xi) \overset{\text{def}}{=} \sum_{i\in \mathrm{H}} f_i\cdot z_i + \sum_{i\in \mathrm{H}}\sum_{j\in \mathrm{H}}\sum_{k\in \mathrm{K}} (W_k(\xi)\cdot F_{ijk})\cdot x_{ijk},$$

then $\zeta(z,x,\xi)$ can be represented as follows:

$$\boldsymbol{\zeta}(z,x,\xi) = \boldsymbol{f}^{\mathrm{T}}\cdot\boldsymbol{z} + \boldsymbol{W}^{\mathrm{T}}\cdot\boldsymbol{F}\cdot\boldsymbol{X} = \boldsymbol{\zeta}'^{\mathrm{T}}(\boldsymbol{D}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{X}) + \boldsymbol{f}^{\mathrm{T}}\boldsymbol{z} + \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{X},$$

where

$$\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{|H|} \end{pmatrix},\ z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{|H|} \end{pmatrix},\ \boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{|K|} \end{pmatrix},$$

$$\boldsymbol{F} = \begin{pmatrix} F_1 & 0 & \cdots & 0 \\ 0 & F_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & F_{|K|} \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{W} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{D} \end{pmatrix} \cdot \zeta' + \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{\mu} \end{pmatrix},$$

$$X_k = (\overbrace{x_{11k}, x_{12k}, \ldots, x_{1|H|k}}^{|H|}, \overbrace{x_{21k}, x_{22k}, \ldots, x_{2|H|k}}^{|H|}, \ldots, \overbrace{x_{|H|1k}, x_{|H|2k}, \ldots, x_{|H||H|k}}^{|H|})^{\mathrm{T}},$$

$$F_k = (\overbrace{F_{11k}, F_{12k}, \ldots, F_{1|H|k}}^{|H|}, \overbrace{F_{21k}, F_{22k}, \ldots, F_{2|H|k}}^{|H|}, \ldots, \overbrace{F_{|H|1k}, x_{|H|2k}, \ldots, x_{|H||H|k}}^{|H|}).$$

According to the definition of multivariate normal random variable [12][13], we get:

$$\mathbf{E}(\zeta(z, x, \xi)) = \boldsymbol{f}^{\mathrm{T}} \boldsymbol{z} + \boldsymbol{\mu}^{\mathrm{T}} \boldsymbol{FX}, \ \ \mathbf{Var}(\zeta(z, x, \xi)) = \| \boldsymbol{D}^{\mathrm{T}} \boldsymbol{FX} \|^2 .$$

As a consequence, the objective function can be reformulated as follows:

$$\begin{aligned} \mathbf{Pr}\{\zeta(z, x, \xi) \le C_0\} &= \mathbf{Pr}\{\frac{\zeta(z, x, \xi) - \mathbf{E}(\zeta(z, x, \xi))}{\sqrt{\mathbf{Var}(\zeta(z, x, \xi))}} \le \frac{C_0 - \mathbf{E}(\zeta(z, x, \xi))}{\sqrt{\mathbf{Var}(\zeta(z, x, \xi))}}\} \\ &= \mathbf{\Phi}(\frac{C_0 - \mathbf{E}(\zeta(z, x, \xi))}{\sqrt{\mathbf{Var}(\zeta(z, x, \xi))}}) = \mathbf{\Phi}(\frac{C_0 - \boldsymbol{f}^{\mathrm{T}} \boldsymbol{z} - \boldsymbol{\mu}^{\mathrm{T}} \boldsymbol{FX}}{\| \boldsymbol{D}^{\mathrm{T}} \boldsymbol{FX} \|}). \end{aligned}$$

Therefore, model (5) can be reformulated as follows:

$$\begin{cases} \max\limits_{z, \boldsymbol{X}} & \mathbf{\Phi}(\frac{C_0 - \boldsymbol{f}^{\mathrm{T}} \boldsymbol{z} - \boldsymbol{\mu}^{\mathrm{T}} \boldsymbol{FX}}{\| \boldsymbol{D}^{\mathrm{T}} \boldsymbol{FX} \|}) \\ \text{subject to :} & \sum\limits_{i \in \mathrm{H}} \sum\limits_{j \in \mathrm{H}} x_{ijk} = 1, \ k \in \mathrm{K} \\ & \sum\limits_{j \in \mathrm{H}} x_{ijk} + \sum\limits_{j \in \mathrm{H} \setminus \{i\}} x_{ijk} \le z_i, \ i \in \mathrm{H}, k \in \mathrm{K} \\ & x_{ijk} \in \{0, 1\}, \ i \in \mathrm{H}, k \in \mathrm{K} \\ & \mathbf{z} \in \{0, 1\}^{|\mathrm{H}|}. \end{cases} \tag{6}$$

The deterministic programming is a maximization problem with non-linear objective function and linear constraints. Because of the computational complexity of normal distribution function $\Phi(\cdot,\cdot)$, and taking into account of the strict monotonicity of $\Phi(\cdot,\cdot)$, model (6) is equivalent to the following binary fractional programming substitute problem:

$$\begin{cases} \max\limits_{z,X} & \dfrac{C_0 - f^{\mathrm{T}} z - \mu^{\mathrm{T}} FX}{\|D^{\mathrm{T}} FX\|} \\ \text{subject to}: & \sum\limits_{i\in \mathrm{H}} \sum\limits_{j\in \mathrm{H}} x_{ijk} = 1, \ k \in \mathrm{K} \\ & \sum\limits_{j\in \mathrm{H}} x_{ijk} + \sum\limits_{j\in \mathrm{H}\backslash\{i\}} x_{ijk} \le z_i, \ i \in \mathrm{H}, k \in \mathrm{K} \\ & x_{ijk} \in \{0,1\}, \ i \in \mathrm{H}, k \in \mathrm{K} \\ & \mathbf{z} \in \{0,1\}^{|\mathrm{H}|}. \end{cases} \tag{7}$$

The binary fractional programming problem (7) is an integer fractional programming problem. From the point of view of efficient numerical solution, the most desirable property of problem (7) is that its relaxation problem should be a convex programming problem. We will proceed with a property concerning the convexity of its relaxation problem.

Theorem 2. *The relaxation problem (8) of binary fractional programming (7) is a convex programming problem on open half-space* $\{(z, X) \mid f^{\mathrm{T}} z + \mu^{\mathrm{T}} FX < C_0\}$.

$$\begin{cases} \max\limits_{z,X} & \dfrac{C_0 - f^{\mathrm{T}} z - \mu^{\mathrm{T}} FX}{\|D^{\mathrm{T}} FX\|} \\ \text{subject to}: & \sum\limits_{i\in \mathrm{H}} \sum\limits_{j\in \mathrm{H}} x_{ijk} = 1, \ k \in \mathrm{K} \\ & \sum\limits_{j\in \mathrm{H}} x_{ijk} + \sum\limits_{j\in \mathrm{H}\backslash\{i\}} x_{ijk} \le z_i, \ i \in \mathrm{H}, k \in \mathrm{K} \\ & x_{ijk} \ge 0, i \in \mathrm{H}, k \in \mathrm{K} \\ & z_i \ge 0, i \in \mathrm{H}. \end{cases} \tag{8}$$

Proof. Since the constraint functions are linear in model (8), the feasible domain is convex set. Next, we will discuss the concavity of the objective function. For this purpose, we denote

$$g(z,x) = \frac{C_0 - f^{\mathrm{T}} z - \mu^{\mathrm{T}} FX}{\|D^{\mathrm{T}} FX\|}.$$

When $f^{\mathrm{T}} z + \mu^{\mathrm{T}} FX < C_0$ holds, i.e., $C_0 - f^{\mathrm{T}} z - \mu^{\mathrm{T}} FX > 0$, the function $C_0 - f^{\mathrm{T}} z - \mu^{\mathrm{T}} FX$ is a positive linear function, and the Euclid norm $\| D^{\mathrm{T}} FX \|$ is convex. Therefore, the fractional function $g(z,x)$ is a pseudo-concave function on the open half-space $\{(z, X) \mid f^{\mathrm{T}} z + \mu^{\mathrm{T}} FX < C_0\}$. Thus, the relaxation problem (8) is a fractional convex programming problem. The proof of the theorem is complete. □

4. Solution Methods

Since binary programming problem (7) belongs to the class of binary integer programming problems, one possibility for solving it is to use branch-and-bound method [14]. The code LINGO is a state-of-the-art commercial general branch-and-bound IP-code, which works in conjunction with the linear, nonlinear, and quadratic solvers [15]. The structure of the constraints in the problem makes the use of modeling language particularly appropriate. This yields a rather efficient solution method for this kind of problems. In the next section, we will rely on LINGO to solve the problem.

5. Numerical Experiments

In this section, we present the computational results of numerical experiments preformed to assess the behavior of the proposed model for hub location problems with three commodities. All experiments are coded in LINGO 11.0 and run on a personal computer. For instances $|H| = 10, 12$ and 15, we choose the set-up cost f_i randomly from the interval $[10, 30]$. All hub nodes are generated randomly in the region $[1, 25] \times [1, 25]$. Let $\chi = 0.6$, $\tau = 0.3$ and $\delta = 0.6$. The unit transportation cost F_{ijk} is given by $F_{ijk} = \chi d_{o(k)i} + \tau d_{ij} + \delta d_{jd(k)}$, where d_{ij} is the distance between nodes i and j. The mathematical expectation and the covariance matrix of the random vector $\boldsymbol{W} = (W_1(\xi), W_2(\xi), W_3(\xi))^{\mathrm{T}}$ are given as follows:

$$\boldsymbol{\mu} = (13.4950, 8.8800, 19.8300)^{\mathrm{T}}, \ \boldsymbol{\Sigma} = \begin{pmatrix} 2.4939 & 1.3934 & 3.1018 \\ 1.3934 & 0.9396 & 1.5102 \\ 3.1018 & 1.5102 & 5.2283 \end{pmatrix}.$$

Table 1. Computational Results for Routing Three Commodities

Commodity	Hub	C_0	v^*	$\Phi(v^*)$	Opt. Hub	Time CPU (sec)
3	10	820	0.006	50.4%	2,7,10	136
3	10	850	0.381	64.8%	6,7,10	148
3	10	900	1.502	93.3%	2,3,5,8	169
3	10	1000	2.765	99.7%	2,3,5,8	269
3	12	830	0.246	59.9%	2,7,12	621
3	12	900	1.131	87.1%	6,7,12	695
3	12	910	1.595	94.5%	3,5,6,12	912
3	12	940	1.853	96.8%	2,3,5,12	1028
3	15	850	0.796	78.5%	2,11,12	1297
3	15	860	1.377	91.5%	1,4,5,11,12	1501
3	15	900	1.447	92.6%	6,11,12	1535
3	15	910	1.990	97.7%	1,4,5,10,11	1759

In Table 1, the first three columns provide the number of commodities, the number of hubs and the given upper bound of total costs. The fourth column and the fifth column denote the optimal value and confidence-level corresponding to the optimal value. The sixth column shows the optimal hubs. The results presented in Table 1 show that, for the considered instance with the same number of hubs, the optimal hubs may change when the given upper bound of total costs is increased. The CPU time grows fast when instance size and variability increase.

6. Conclusions

This paper proposed a two-stage stochastic programming model with probability objective function to optimize location and transportation, and showed that the model is equivalent to a single-stage minimum-risk P-model. For random demands with multivariate normal distribution, we turned the P-model into its equivalent deterministic binary fractional programming problem. Finally, we employed LINGO to solve this binary programming problem. The computational results showed that the developed new modeling idea is feasible and effective. An interesting avenue of research for our future work consists in stochastic demands with arbitrary distribution for hub location problems.

Acknowledgements

This work is supported by National Natural Science Foundation of China (No. 60974134), and Natural Science Foundation of Hebei Province (A2011201007).

References

[1] Louveaux FV, Peeters D. A dual-based procedure for stochastic facility location. *Operations Research*; 1992, 40, 564-573.

[2] Laporte G, Louveaux FV, Van hamme L. Exact solution to a location problem with stochastic demands. *Transportation Science*; 1994, 28, 95-103.

[3] Ravi R, Sinha A. Hedging uncertainty: Approximation algorithms for stochastic optimization problems. *Mathematical Programming*; 2006, 108, 97-114.

[4] Louveaux FV. Stochastic location analysis. *Location Science*; 1993, 1, 127-154.

[5] Marianov V, Serra D. Location models for airline hubs behaving as M/D/c queues. *Computers & Operations Research*; 2003, 30, 983-1003.

[6] Sim T, Lowe TJ, Thomas BW. The stochastic *p*-hub center problem with service-level constraints. *Computers & Operations Research*; 2009, 36, 3166-3177.

[7] Yang TH. Stochastic air freight hub location and flight routes planning. *Applied Mathematical Modelling*; 2009, 33(12), 4424-4430.

[8] Hamacher HW, Labbe M, Nickel S, Sonneborn T. Adapting polyhedral properties from facility to hub location problems. *Discrete Applied Mathematics*; 2004, 145, 104-116.

[9] Marin A, Canovas L, Landete M. New formulations for the uncapacitated multiple allocation hub location problem. *European Journal of Operational Research*; 2006, 172, 274-292.

[10] Camargo RS, Miranda Jr. G, Luna HP. Benders decomposition for the uncapacitated multiple allocation hub location problem. *Computers & Operations Research*; 2008, 35, 1047-1064.

[11] Contreras I, Cordeau JF, Laporte G. Stochastic uncapacitated hub location. *European Journal of Operational Research*; 2011, 212, 518-528.

[12] Kall P, Mayer J. *Stochastic Linear Programming: Models, Theory, and Computation*. Springer-Verlag, New York; 2005.

[13] Gao HX. *Applied Multivariate Statistical Analysis*. Peking University Press, Beijing; 2005.

[14] Walker RC. *Introduction to Mathematical Programming*. Pearson Education Inc., London; 1999.

[15] Atamturk A, Savelsbergh MWP. Integer-programming software systems. *Annual of Operations Research*; 2005, 140, 67-124.